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RESEARCH ARTICLE

On the anticommutativity in Leibniz algebras

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Abstract. Lie algebras are exactly the anticommutative Leibniz algebras. In this article, we conduct a brief analysis of the approach to Leibniz algebras which based on the concept of the anti-center (Lie-center) and antinilpotency (Lie nilpotentency).

Let L be an algebra over a field F with the binary operations $+$ and $[\cdot,\cdot]$. Then L is called a *Leibniz algebra* (more precisely a *left Leibniz algebra*) if it satisfies the Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] \quad \text{for all } a,b,c \in A.$$

If L is a Lie algebra, then L is a Leibniz algebra. Conversely, if L is a Leibniz algebra such that $[a,a] = 0$ for each element $a \in L$, then L is a Lie algebra. Therefore, Lie algebras can be characterized as the Leibniz algebras in which $[a,a] = 0$ for every element a. In other words, Lie algebras can be described as anticommutative Leibniz algebras.

The following analogy comes up:

$$\{\text{abelian groups}\} \Longleftrightarrow \{\text{Lie algebras}\}$$

and

$$\{\text{non-abelian groups}\} \Longleftrightarrow \{\text{Leibniz algebras}\}.$$

It is immediately understandable that such an analogy can not be sufficiently deep, since the properties of commutativity and anticommutativity

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differ significantly (they coincide in the case of algebras over a field of characteristic 2). In this we convince ourselves by looking at cyclic subgroups in groups and cyclic subalgebras in Leibniz algebras. Cyclic subgroups in an arbitrary group are commutative, while cyclic subalgebras in Leibniz algebras do not generally possess anticommutativity, as can be seen from their description given in [1].

A Leibniz algebra L has one specific ideal. Denote by $\mathrm{Leib}(L)$ the subspace generated by the elements $[a, a], a \in L$. It is possible to prove that $\mathrm{Leib}(L)$ is an ideal of L. Moreover, $L/\mathrm{Leib}(L)$ is a Lie algebra. Conversely, if H is an ideal of L such that L/H is a Lie algebra, then $\mathrm{Leib}(L) \leqslant H$. The ideal $\mathrm{Leib}(L)$ is called the *Leibniz kernel* of algebra L.

We can consider the Leibniz kernel as an analog of the derived subgroup in a group. To this analogy, we will come again later. So, the difference between Lie algebras and Leibniz algebras is that they have a non-zero Leibniz kernel, just as the difference between abelian groups and non-abelian groups is in the presence of non-trivial derived subgroups in the latter.

Let us try to continue the analogy with the theory of groups. Along with the derived subgroup in G, there is another characteristic subgroup-its center $\zeta(G)$, that is, the set of all elements z such that $zg = gz$ for each element $g \in G$. Taking into account the fact that the difference between Leibniz algebras and Lie algebras consists in the absence of anticommutativity, we naturally come to the next object in Leibniz algebras.

Let L be a Leibniz algebra. Put

$$\alpha(L) = \{z \in L \mid [a, z] = -[z, a] \text{ for every elements } a \in L\}.$$

This subset is called the *anticenter* of a Leibniz algebra L.

Clearly the anticenter is a subspace of L. It is also a subalgebra of L. Indeed, let $z, y \in \alpha(L)$ and a be an arbitrary element of L. Then

$$\begin{aligned}[[z, y], a] &= [z, [y, a]] - [y, [z, a]] = -[z, [a, y]] + [y, [a, z]] \\ &= -[z, [a, y]] - [[a, z], y] = -([[a, z], y] + [z, [a, y]]) = -[a, [z, y]].\end{aligned}$$

Moreover, the anticenter is an ideal of L. In fact, let $z \in (L)$ and a be an arbitrary element of A. For every element $b \in A$ we have

$$\begin{aligned}[[z, a], b] &= [z, [a, b]] - [a, [z, b] = -[[a, b], z] + [a, [b, z]] \\ &= -[[a, b], z] + [[a, b], z] + [b, [a, z]] = [b, [a, z]] = -[b, [z, a]], \\ [[a, z], b] &= [a, [z, b]] - [z, [a, b]] = -[a, [b, z]] + [[a, b], z] \\ &= -[a, [b, z]] + [a, [b, z]] - [b, [a, z]] = -[b, [a, z]].\end{aligned}$$

Note that in [2] the term Lie-center of a Leibniz algebra is used. However, the property of anticommutativity is inherent not only to Lie algebras, therefore, instead of the term Lie-center, it seems to us preferable to use a more general term anticenter.

Note also that if $\mathrm{char}(F) = 2$, then the anticenter of Leibniz algebra coincides with the set

$$\{z \in L \mid [a, z] = [z, a] \text{ for every element } a \in L\}.$$

This set, in general, is not an ideal. Therefore, it is worthwhile to use the considerations related to the anticentre over the field F such that $\mathrm{char}(F) \neq 2$. In this paper, we shall further assume that $\mathrm{char}(F) \neq 2$.

In a Leibniz algebra L, the concept of a center is introduced as follows: The center $\zeta(L)$ is the set of all elements z such that $[z, x] = [x, z] = 0$ for an arbitrary element $x \in L$. Clearly the center is an ideal of L. The following concept is naturally connected with the center.

Let L be a Leibniz algebra over a field F, M be non-empty subset of L and H be a subalgebra of L. Put

$$\mathrm{Ann}_H(M) = \{a \in H \mid [a, M] = \langle 0 \rangle = [M, a]\}.$$

The subset $\mathrm{Ann}_H(M)$ is called the *annihilator* or the *centralizer* of M in subalgebra H. It is not hard to see that the subset $\mathrm{Ann}_H(M)$ is a subalgebra of H. If M is an ideal of L, then $\mathrm{Ann}_H(M)$ is also an ideal of L. The center of L is the intersection of the annihilators of all elements of L. It leads us to the following concept.

Let L be a Leibniz algebra over a field F, M be non-empty subset of L and H be a subalgebra of L. Put

$$\mathrm{AC}_H(M) = \{a \in H \mid [a, u] = -[u, a] \text{ for all } x \in M\}.$$

The subset $\mathrm{AC}_H(M)$ is called the *anticentralizer* of M in subalgebra H. It is clear that the anticenter of L is the intersection of the anticentralizers of all elements of L. But on this all the good ends. Unlike annihilator, the anticentralizer of subset is not always a subalgebra, so an anticentralizer can no longer be such a good technical tool as a centralizer. This can be seen from the following example.

Example 1. Let F be an arbitrary field, L be a vector space over F having a basis $\{a, b, c, d\}$. Define the operation $[\cdot\,,\,\cdot]$ on the elements of basis by the following rule:

$$\begin{gathered}[d,a]=-c,\quad [d,b]=b+c,\quad [d,c]=-b,\\ [a,d]=b+c,\quad [b,d]=[c,d]=0,\quad [d,d]=b,\\ [a,a]=b,\quad [a,b]=c,\quad [a,c]=-b-c,\\ [c,a]=[b,a]=[b,c]=[c,b]=0,\end{gathered}$$

and expand it in a natural way on all elements of L. It is possible to prove that L becomes a Leibniz algebra over F. Let's find the anticentralizer of an element a. Let $x=\sigma_1 d+\sigma_2 a+\sigma_3 b+\sigma_4 c\in \mathrm{AC}_L(a)$, then

$$\begin{aligned}0&=[a,x]+[x,a]\\ &=[a,\sigma_1 d+\sigma_2 a+\sigma_3 b+\sigma_4 c]+[\sigma_1 d+\sigma_2 a+\sigma_3 b+\sigma_4 c,a]\\ &=\sigma_1(b+c)+\sigma_2 b+\sigma_3 c-\sigma_4(b+c)-\sigma_1 c+\sigma_2 b\\ &=(\sigma_1+\sigma_2+\sigma_2-\sigma_4)b+(\sigma_1+\sigma_3-\sigma_4-\sigma_1)c.\end{aligned}$$

Thus we obtain $\sigma_1+2\sigma_2-\sigma_4=0$, and $\sigma_1+\sigma_3-\sigma_4-\sigma_1=0$, which follows that $\sigma_4=\sigma_3=\gamma$, $\sigma_2=\tau$, $\sigma_1=\gamma-2\tau$. Hence

$$\mathrm{AC}_L(a)=\{(\gamma-2\tau)d+\tau a+\gamma b+\gamma c\mid \gamma,\tau\in F\}.$$

Let $x=(\gamma-2\tau)d+\tau a+\gamma b+\gamma c$, $y=(\beta-2\rho)d+\rho a+\beta b+\beta c$. We will find $[x,y]$. We have

$$\begin{aligned}&[(\gamma-2\tau)d+\tau a+\gamma b+\gamma c,(\beta-2\rho)d+\rho a+\beta b+\beta c]\\ &\quad=(\gamma-2\tau)(\beta-2\rho)b+\tau(\beta-2\rho)(b+c)-\rho(\gamma-2\tau)c+\tau\rho b\\ &\qquad+(\gamma-2\tau)(b+c)+\tau\beta c-\beta(\gamma-2\tau)b-\tau\beta(b+c)\\ &\quad=((\gamma-2\tau)(\beta-2\rho)+\tau(\beta-2\rho)+\tau\rho+(\gamma-2\tau)-\beta(\gamma-2\tau)-\tau\beta)b\\ &\qquad+(\tau(\beta-2\rho)-\rho(\gamma-2\tau)+(\gamma-2\tau)+\tau\beta-\tau\beta)c\\ &\quad=(\gamma\beta-2\tau\beta-2\rho\gamma+4\tau\rho+\tau\beta-2\rho\tau+\tau\rho+\gamma-2\tau-\beta\gamma+2\tau\beta-\tau\beta)b\\ &\qquad+(\tau\beta-2\rho\tau-\rho\gamma-2\tau\rho+\gamma-2\tau)c\\ &\quad=(-2\rho\gamma+3\tau\rho+\gamma-2\tau)b+(\tau\beta-2\rho\tau-\rho\gamma-2\tau\rho+\gamma-2\tau)c\end{aligned}$$

Suppose now that $[x,y]\in\mathrm{AC}_L(a)$, then must be

$$-2\rho\gamma+3\tau\rho+\gamma-2\tau=\tau\beta-2\rho\tau-\rho\gamma-2\tau\rho+\gamma-2\tau$$

which follows that $-\rho\gamma+5\tau\rho-\tau\beta=0$.

Put now $\gamma=0,\tau=1,x=-2d+a,\rho=0,\beta=1$, then the elements $x=-2d+a$ and $y=d+b+c$ belong to $\mathrm{AC}_L(a)$, but $[x,y]=-2b-c\notin\mathrm{AC}_L(a)$.

In Leibniz algebras the derived ideal $[L, L]$ generated by all elements $[x, y], x, y \in L$ is dual to the center. From our analogy, we can consider $\alpha(L)$ as an analogue of the center, while a subspace $(L, L\,)$, generated by all elements $(x, y) = [x, a] + [a, x], x, a \in L$, can be considered as an analog of the derived subgroup. At once, remark that this subspace is an ideal. Moreover, if $x, y, z \in L$, then the element $[[x, a] + [a, x], y] = 0$ for every element $y \in L$. Indeed

$$\begin{aligned}[][[x, y] + [y, x], z] &= [[x, y], z] + [[y, x], z] \\ &= [x, [y, z]] - [y, [x, z]] + [y, [x, z]] - [x, [y, z]] = 0.\end{aligned}$$

Further,

$$\begin{aligned}[][z, [x, y] + [y, x]] &= [z, [x, y]] + [z, [y, x]] \\ &= [[z, x], y] + [x, [z, y]] + [[z, y], x] + [y, [z, x]] \\ &= ([[z, x], y] + [y, [z, x]]) + ([x, [z, y]] + [[z, y], x]).\end{aligned}$$

On the other hand, $[a, a] + [a, a] = 2[a, a] \in (L, L)$, and $\mathrm{char}(F) \neq 2$ implies that $[a, a] \in (L, L\,)$, so that $\mathrm{Leib}(L) \leqslant (L, L)$. Since $L/\,\mathrm{Leib}(L)$ is a Lie algebra, $[x, a] + [a, x] \in \mathrm{Leib}(L)$, so that $\mathrm{Leib}(L) = (L, L)$. Thus, with this approach, the Leibniz kernel is dual to the anticenter. In this connection, it is useful to recall the presence in Leibniz algebras another important ideal, namely, the left center. If L is a Leibniz algebra, then put

$$\zeta^{\mathrm{left}}(L) = \{x \in L | [x, y] = 0 \text{ for each element } y \in L\}.$$

It is possible to prove that $\zeta^{\mathrm{left}}(L)$ is an ideal of L and $\mathrm{Leib}(L) \leqslant \zeta^{\mathrm{left}}(L)$.

Starting from the anticenter, we define the upper anticentral series

$$\begin{aligned}\langle 0\rangle = \alpha_0(L) \leqslant \alpha_1(L) \leqslant \alpha_2(L) \leqslant \ldots \leqslant \alpha_\lambda(L) \\ \leqslant \alpha_{\lambda+1}(L) \leqslant \ldots \leqslant \alpha_\gamma(L) = \alpha_\infty(L)\end{aligned}$$

of a Leibniz algebra L by the following rule: $\alpha_1(L) = \alpha(L)$ is the anticenter of L, and recursively, $\alpha_{\lambda+1}(L)/\alpha_\lambda(L) = \alpha(L/\alpha_\lambda(L))$ for all ordinals λ, and $\alpha_\mu(L) = \cup_{v<\mu}\alpha_v(L)$ for the limit ordinals μ. By definition, each term of this series is an ideal of L. The last term $\alpha_\infty(L)$ of this series is called the *upper hyperanticenter* of L. A Leibniz algebra L is said to be *hyperanticentral* if it coincides with the upper hypercenter. Denote by $\mathrm{al}(L)$ the length of upper central series of L. If L is hyperanticentral and $\mathrm{al}(L)$ is finite, then L is said to be *antinilpotent*.

Let A, B be the ideals of L such that $B \leqslant A$. The factor A/B is called *anticentral*, if $A/B \leqslant \alpha(L/B)$. By definition, the factor A/B is anticentral if and only if $[x, a] + [a, x] \in B$ for each $a \in A$ and each $x \in L$.

If U, V the ideals of L, then denote by (U, V) a subspace, generated by all elements $[u, v] + [v, u], u \in U, v \in V$. As we have seen above, $[u, v] + [v, u] \in \zeta^{\text{left}}(L)$. Using the above arguments, we can show, that (U, V) is an ideal of L.

Note at once, that a factor A/B is anticentral if and only if $(L, A) \leqslant B$.

Now we can introduce an analog of the lower central series. Define the lower anticentral series of L

$$L = \kappa_1(L) \geqslant \kappa_2(L) \geqslant \ldots \kappa_\alpha(L) \geqslant \kappa_{\alpha+1}(L) \geqslant \ldots \kappa_\delta(L)$$

by the following rule: $\kappa_1(L) = L, \kappa_2(L) = (L, L)$, and recursively $\kappa_{\lambda+1}(L) = (L, \kappa_\lambda(L))$ for all ordinals λ and $\kappa_\mu(L) = \cap_{v<\mu}\kappa_v(L)$ for the limit ordinals μ. The last term $\kappa_\delta(L)$ is called the *lower hypoanticenter* of L. We have $\kappa_\delta(L) = (L, \kappa_\delta(L))$.

As we have seen above $\kappa_2(L) = (L, L) = \text{Leib}(L) = K$. Furthermore, $\kappa_3(L) = (L, \kappa_2(L))$. If $x \in L, a \in K = \kappa_2(L)$, then $(x, a) = [x, a] + [a, x] = [x, a]$, because $\text{Leib}(L) \leqslant \zeta^{\text{left}}(L)$. It follows that $\kappa_3(L) = [L, \kappa_2(L)] = [L, \text{Leib}(L)]$.

If A is an ideal of L, then put $\gamma_{L_1}(A) = A, \gamma_{L_2}(A) = [L, A]$, and recursively $\gamma_{L_{n+1}}(A) = [L, \gamma_{L\, n}(A)]$ for all positive integers n.

Thus we obtain $\kappa_1(L) = L$, $\kappa_2(L) = \text{Leib}(L)$, $\kappa_3(L) = \gamma_{L_2}(\text{Leib}(L))$, $\kappa_{n+1}(L) = \gamma_{L_n}(\text{Leib}(L))$ for all positive integers n.

Suppose now that L has finite series of ideals

$$\langle 0 \rangle = A_0 \leqslant A_1 \leqslant A_2 \leqslant \ldots \leqslant A_n = L.$$

This series is said to be *anticentral*, if every factor A_j/A_{j-1} is anticentral, $1 \leqslant j \leqslant n$.

Proposition 1. *Let L be an Leibniz algebra over a field F and*

$$\langle 0 \rangle = C_0 \leqslant C_1 \leqslant \ldots \leqslant C_n = L$$

be a finite anticentral series of L. Then

(i) $\kappa_j(L) \leqslant C_{n-j+1}$, *so that* $\kappa_{n+1}(L) = \langle 0 \rangle$.

(ii) $C_{j \leqslant \alpha_j}(L)$, *so that* $\alpha_n(L) = L$.

These statements were proved in [2] for right Leibniz algebras; for left Leibniz algebras the proof is similar.

Corollary. *Let L be an antinilpotent Leibniz algebra. Then the length of the lower anticentral series coincides with the length of the upper anticentral series. Moreover, the length of these two series is the smallest among the lengths of all anticentral series of L.*

The length of the upper anticentral series (or lower anticentral series) is called *the class of antinilpotency of a Leibniz algebra L* and denote by $\mathrm{ancl}(L)$. In the article [2] it was called a Lie-nilpotent algebra and the class of Lie-nilpotency. However, the concept of Lie-nilpotency arose much earlier in the theory of associative rings, so in order to avoid confusion it is better to use another term. In addition, as we have already noted, the property of anticommutativity is inherent not only in Lie algebras, therefore, we focus on it.

Note some properties of hyperanticentral Leibniz algebras.

Proposition 2. *Let $\{L_\lambda \mid \lambda \in \Lambda\}$ be a family of Leibniz algebra over a field F.*

(i) *If n is a positive integer, then $\alpha_n(\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda) = \mathrm{Cr}_{\lambda\in\Lambda}\ \alpha_n(L_\lambda)$.*

(ii) *If ω is a first infinite ordinal, then $\alpha_\omega(\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda) \leqslant \mathrm{Cr}_{\lambda\in\Lambda}\ \alpha_\omega(L_\lambda)$.*

(iii) *If μ is an arbitrary ordinal, then $\alpha_\mu(\oplus_{\lambda\in\Lambda} L_\lambda) = \oplus_{\lambda\in\Lambda}\ \alpha_\omega(L_\lambda)$, in particular, if every algebra L_λ is hyperanticentral, then the direct sum $\oplus_{\lambda\in\Lambda} L_\lambda$ also is hyperanticentral.*

(iv) *If every algebra L_λ is antinilpotent and there exists a positive integer k such that $\mathrm{ancl}(L_\lambda) \leqslant k$ for all $\lambda \in \Lambda$, then the Cartesian product $\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda$ is alsoantinilpotent and $\mathrm{ancl}(\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda) \leqslant k$.*

(v) *If every algebra L_λ is antinilpotent and the set Λ is finite, then $\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda = \oplus_{\lambda\in\Lambda} L_\lambda$ is antinilpotent, moreover $\mathrm{ancl}(\oplus_{\lambda\in\Lambda} L_\lambda) \leqslant \max\{\mathrm{ancl}\, L_\lambda) \mid \lambda \in \Lambda\}$.*

Proof. Let

$$\langle 0\rangle = A_{\lambda\,0} \leqslant A_{\lambda\,1} \leqslant \dots A_{\lambda\,\mu} \leqslant A_{\lambda\,\mu+1} \leqslant \dots A_{\lambda\,\gamma}(\lambda) = \alpha_{\lambda\,\infty}(L_\lambda)$$

be the upper anticentral series of $L_\lambda, \lambda \in \Lambda$. Put $L = \mathrm{Cr}_{\lambda\in\Lambda} L_\lambda$, $K = \oplus_{\lambda\in\Lambda} L_\lambda$.

(i) Let $x_\lambda \in L_\lambda$, $a_\lambda \in A_{\lambda\,1}$, then $[x_\lambda, a_\lambda] = -[a_\lambda, x_\lambda]$ for every $\lambda \in \Lambda$. It follows that

$$\begin{aligned}[(x_\lambda)_{\lambda\in\Lambda}, (a_\lambda)_{\lambda\in\Lambda}] &= ([x_\lambda, a_\lambda])_{\lambda\in\Lambda} = (-[a_\lambda, x_\lambda])_{\lambda\in\Lambda} \\ &= -([a_\lambda, x_\lambda])_{\lambda\in\Lambda} = -[(a_\lambda)_{\lambda\in\Lambda}, (x_\lambda)_{\lambda\in\Lambda}],\end{aligned}$$

which shows that $\mathrm{Cr}_{\lambda\in\Lambda} A_{\lambda\,1} \leqslant \alpha(\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda)$. Conversely, let $(x_\lambda)_{\lambda\in\Lambda} \in L$, $(b_\lambda)_{\lambda\in\Lambda} \in \alpha(L)$, then

$$\begin{aligned}([x_\lambda, b_\lambda])_{\lambda\in\Lambda} &= [(x_\lambda)_{\lambda\in\Lambda}, (b_\lambda)_{\lambda\in\Lambda}] = -[(b_\lambda)_{\lambda\in\Lambda}, (x_\lambda)_{\lambda\in\Lambda}]\\ &= -([b_\lambda, x_\lambda])_{\lambda\in\Lambda} = (-[b_\lambda, x_\lambda])_{\lambda\in\Lambda}.\end{aligned}$$

In particular, have $[x_\lambda, a_\lambda] = -[b_\lambda, x_\lambda]$ for each $\lambda \in \Lambda$. This means that $b_\lambda \in \alpha(L_\lambda)$ for each $\lambda \in \Lambda$, so that $\mathrm{Cr}_{\lambda\in\Lambda} A_{\lambda\,1} = \alpha(\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda)$.

Using similar arguments and ordinary induction we will prove an equality $\mathrm{Cr}_{\lambda\in\Lambda} A_{\lambda\,n} = \alpha_n(\mathrm{Cr}_{\lambda\in\Lambda} L_\lambda)$.

(ii) Let $(c_\lambda)_{\lambda\in\Lambda} \in \alpha_\omega(L)$. An equality $\alpha_\omega(L) = \cup_{n\in\mathbf{N}}\alpha_n(L)$ shows that there exists a positive integer m such that $(c_\lambda)_{\lambda\in\Lambda} \in \alpha_m(L)$. By above proved, it follows that $c_\lambda \in \alpha_m(L_\lambda) \leqslant \alpha_\omega(L_\lambda)$ for each $\lambda \in \Lambda$. It proves the inclusion $\alpha_\omega(L) \leqslant \mathrm{Cr}_{\lambda\in\Lambda} \alpha_\omega(L_\lambda)$.

(iii) Let $(d_\lambda)_{\lambda\in\Lambda} \in \oplus_{\lambda\in\Lambda}\alpha_\omega(L_\lambda)$. There exists a subset Σ of Λ such that $\Lambda \in \Sigma$ is finite and $d_\lambda = 0$ for all $\lambda \in \Sigma$. Let $\lambda \notin \Sigma$. The fact that $d_\lambda \in \alpha_\omega(L_\lambda)$ implies that there exists a positive integer $t(\lambda)$ such that $d_\lambda \in \alpha_{t(\lambda)}(L_\lambda)$. Let $t = \max\{t(\lambda) \mid \lambda \in \Lambda \in \Sigma\}$, then $d_\lambda \in \alpha_t(L_\lambda)$ for each $\lambda \in \Lambda\backslash\Sigma$ and therefore $d_\lambda \in \alpha_t(L_\lambda)$ for all $\lambda \in \Lambda$. It follows that $(d_\lambda)_{\lambda\in\Lambda} \in \alpha_t(K) \leqslant \alpha_\omega(K)$, which proves the equality $\alpha_\omega(K) = \oplus_{\lambda\in\Lambda} \alpha_\omega(L_\lambda)$.

Using the above arguments and transfinite induction we obtain the equality $\alpha_\mu(K) = \oplus_{\lambda\in\Lambda} \alpha_\mu(L_\lambda)$ for all ordinals μ.

(iv) follows from (i), and (v) follows from (v). □

Proposition 3. *Let L be a Leibniz algebra over a field F. Then*

(i) *If L is hyperanticentral, then every subalgebra of L is hyperanticentral and every factor-algebra of L is hyperanticentral.*

(ii) *If L is antinilpotent, then every subalgebra of L is antinilpotent and every factor-algebra of L is antinilpotent.*

(iii) *If A is a non-zero ideal of L such that $A \cap \alpha_\infty(L) \neq \langle 0\rangle$, then $A \cap \alpha(L) \neq \langle 0\rangle$.*

(iv) *If A, B are antinilpotent ideals of L, then $A + B$ is antinilpotent ideal of L.*

Proof. The assertion (i) and (ii) are obvious. We will prove (iii). Put $B = A\cap\alpha_\infty(L)$. There exists the least ordinal λ such that $B\cap\alpha_\lambda(L) \neq \langle 0\rangle$. Clearly λ is a not limit ordinal, so that $\lambda - 1$ exists. If $\lambda = 1$, all is proved. Therefore assume that $\lambda > 1$. Let $0 \neq b \in B \cap \alpha_\lambda(L)$. Then $[x, b] = -[b, x] + \alpha_{\lambda-1}(L)$, so that $[x, b] = -[b, x] + y$ for some element $y \in \alpha_{\lambda-1}(L)$. Since B is an ideal of L, $[x, b], [b, x] \in B$. It follows that $y \in B$. More precisely, $y \in B \cap \alpha_{\lambda-1}(L) = \langle 0\rangle$. Thus we obtain $[x, b] = r - [b, x]$.

Since it is true for each element $x \in L, b \in \alpha(L)$, and we obtain a contradiction, which proves (iii).

(iv) If $A \cap B = \langle 0 \rangle$, then using Proposition 2(v) we obtain that $A + B = A \oplus B$ is antinilpotent. Assume now that $A \cap B = C \neq \langle 0 \rangle$. By Proposition 2(ii) C has finite upper anticentral A-series

$$\langle 0 \rangle = C_0 \leqslant C_1 \leqslant \ldots \leqslant C_n = C.$$

Clearly every term of this series is an ideal in $A + B$. Consider a factor C_j/C_{j-1}. Using again Proposition 2(ii) we obtain that it has a finite upper B-series, that is, we obtain a finite series

$$C_{j-1} = D_{j-1\,0} \leqslant D_{j-1\,1} \leqslant \ldots \leqslant D_{j-1\,m(j)} = C_j$$

of ideals of $A + B$ between C_{j-1} and C_j, every factor of which is B-anticentral, $1 \leqslant j \leqslant n$. Since a factor C_j/C_{j-1} is A-anticentral, each factor $D_{j-1\,t}/D_{j-1\,t-1}$ is $(A + B)$-anticentral, $1 \leqslant t \leqslant m(j)$. Thus we obtain a finite series

$$\langle 0 \rangle = D_{0\,0} \leqslant D_{0\,1} \leqslant \ldots \leqslant D_{0\,m(0)} = D_{1\,0} \leqslant D_{1\,1} \leqslant \ldots \leqslant D_{n-1\,m(n)} = C_j$$

of ideal of $A + B$, whose factors are $(A + B)$-anticentral. It follows that some term of the upper anticentral series of $A + B$, having finite number, includes $A \cap B$. By above remarked $(A + B)/(A \cap B)$ is antinilpotent. It follows that $A + B$ is also antinilpotent. □

The above properties show a certain analogy between nilpotent and antinilpotent Leibniz algebras. However, this analogy is very shallow. Thus every chief central factor of Leibniz algebra L has dimension 1. On the other hand, every chief factor of Lie algebra is anticentral, but it can have infinite dimension. Further, finitely generated nilpotent Leibniz algebra has finite dimension [7], Corollary 2.2. On the other hand, there are finitely generated Lie algebras, which have infinite dimension.

Note the following analog. In the paper [5], Corollary B1, it was proved that if a center of a Leibniz algebra has finite codimension, then a derived ideal has finite dimension.

Proposition 4. *Let L be a Leibniz algebra over a field F. If the anticenter of L has finite codimension d, then the Leibniz kernel of L has finite dimension at most d^2.*

Proof. Let $A = \alpha(A)$, then $L = A \oplus B$ for some subspace B. Choose a basis $\{a_\lambda | \lambda \in \Lambda\}$ in A and a basis $\{b_1, \dots, b_d\}$ in B. If y is an arbitrary element of L, then $y = a + \Sigma_{1 \leqslant j \leqslant d}\, \beta_j b_j$ where $a \in A, \beta_j \in F, 1 \leqslant j \leqslant d$. We have

$$\begin{aligned}[y, y] &= [a + \Sigma_{1 \leqslant j \leqslant d}\, \beta_j b_j, a + \Sigma_{1 \leqslant k \leqslant d}\, \beta_k b_k] \\ &= [a, a] + [a, \Sigma_{1 \leqslant k \leqslant d}\, \beta_k bk_j] + [\Sigma_{1 \leqslant j \leqslant d}\, \beta_j b_j, a] \\ &\quad + [\Sigma_{1 \leqslant j \leqslant d}\, \beta_j b_j, \Sigma_{1 \leqslant k \leqslant d}\, \beta_k b_k].\end{aligned}$$

Since $a \in \alpha(A)$, then $[a, a] = 0$ and $[a, \Sigma_{1 \leqslant k \leqslant d} \beta_k bk_j] + [\Sigma_{1 \leqslant j \leqslant d} \beta_j b_j, a] = 0$, so we obtain

$$[y, y] = \Sigma_{1 \leqslant j \leqslant d, 1 \leqslant k \leqslant d} \beta_j\, \beta_k [b_j, b_k].$$

It follows that $\mathrm{Leib}(L)$ generates by the following elements $\{[b_j, b_k] | 1 \leqslant j \leqslant d, 1 \leqslant k \leqslant d\}$. In particular, $\dim_F(\mathrm{Leib}(L)) \leqslant d^2$. □

We note at once that converse is not true. The following example justifies this.

Example 2. Let $F = \mathbb{Q}$ be a field of all rational numbers and let L_n be a vector space with a basis $\{a_n, c_n\}, n \in \mathbb{N}$. We can define an operation $[\,\cdot\,,\,\cdot\,]$ on L_n assumed that $[a_n, a_n] = c_n, [a_n, c_n] = [c_n, a_n] = [c_n, c_n] = 0$, and expanding this operation, using the property of bilinearity, to all elements of L_n. It is not difficult to verify that such a particular operation makes L_n a Leibniz algebra over $\mathbb{Q}$, moreover, this algebra is nilpotent and $\mathrm{ncl}(L_n) = 2$.

Put $A = \oplus_{n \in \mathbb{N}} L_n$ and let B be a subspace of A, generated by the elements $c_n - c_{n+1}, n \in \mathbb{N}$. It is not hard to see, that B is a subalgebra of A, moreover $B \leqslant \zeta(A)$, in particular, B is an ideal of A. Put $L = A/B$, $d_n = a_n + B$, $n \in \mathbb{N}$, $e = c_1 + B$. We obtain $[d_n, d_n] = e$, $[d_n, e] = [e, d_n] = [e, e] = 0$, $[d_n, d_k] = 0$ whenever $n \neq k$. It shows that $\zeta(L) = Fe$ and $L/\zeta(L)$ is an abelian Leibniz algebra, so that L is an extraspecial Leibniz algebra. Let x be an arbitrary element of $L\ \zeta(L)$, then $x = \Sigma_{n \in \Lambda} \lambda_n d_n + \mu e$, where Λ is a finite subset of $\mathbb{N}$. Then

$$[x, x] = [\Sigma_{n \in \Lambda} \lambda_n d_n + \mu e, \Sigma_{k \in \Lambda} \lambda_k d_k + \mu e] = (\Sigma_{n \in \Lambda} \lambda_n^2) e.$$

Since $x \notin \zeta(L)$, there exists a positive integer m such that $\lambda_m \neq 0$. It follows that $[x, x] \neq 0$.

The fact that $L/\zeta(L)$ is abelian implies that $L/\zeta(L)$ is a Lie algebra, so that, $\zeta(L) = \mathrm{Leib}(L)$. Clearly, $\zeta(L) \leqslant \alpha(L)$. If we suppose that $\zeta(L) \neq \alpha(L)$, then there exists an element $y \in \alpha(L) \setminus \zeta(L)$. The fact that $y \in \alpha(L)$

implies that $[y, y] = 0$. On the other hand, by above proved the fact that $y \neq \zeta(L)$ implies that $[y, y] \notin 0$, and we obtain a contradiction. This contradiction proves the equality $\zeta(L) = \alpha(L)$. Thus $\mathrm{Leib}(L)$ has a dimension 1, but $\alpha(L) = \mathrm{Leib}(L)$ has infinite codimension.

In conclusion, we give a result arising from another, more familiar analogy. Above we already noted one of the results of the paper [5], Corollary B1, which states that if the center of a Leibniz algebra has finite codimension, then the derived ideal has finite dimension. This result is analogous to the following known group-theoretical result.

If the center of a group G has finite index, then the derived subgroup of G is finite.

This theorem first appeared in the work of B.H. Neumann [8]. Nevertheless, very often it is called the Schur theorem (see [6] on this subject). The inversion of this theorem is false both for groups and for Leibniz algebras. Nevertheless, if the derived subgroup of a group is finite, then the second hypercenter of G has finite index [3]. The same situation holds for the Leibniz algebras, as our following result shows.

Theorem 1. *Let L be a Leibniz algebra over a field F. Suppose that the derived ideal of L has finite dimension d. Then the second hypercenter of L has finite codimension at most $2d^2(1 + 2d)$.*

Proof. Put $D = [G, G]$. Then $L/\mathrm{Ann}^{\mathrm{left}}(D)$ is isomorphic to some subalgebra of $\mathrm{Der}(D)$ (the algebra of all derivations of the subalgebra D), see [5], Proposition 3.2, for example. In particular, $\dim_F(L/\mathrm{Ann}^{\mathrm{left}}(D)) \leqslant d^2$.

For every element $a \in L$ consider the mapping $r_a : D \to D$ defined by the rule $r_a(x) = [x, a], x \in D$. This mapping is linear and $\beta r_a = r_{\beta a}$ and $r_a + r_b = r_{a+b}$ for all $a, b \in L$ and $\beta \in F$ (see, for example, [4], p. 131). It follows that the mapping $f : L \to \mathrm{End}_F(D)$ defined by the rule $f(a) = \mathbf{r}a, a \in L$ is linear. Thus we obtain that $L/\mathrm{Ker}(f) \cong \mathrm{Im}(f) \leqslant \mathrm{End}_F(D)$. Clearly, $\mathrm{Ker}(f) = \mathrm{Ann}^{\mathrm{right}}(D)$, so that $\dim_F(L/\mathrm{Ann}^{\mathrm{right}}(D)) \leqslant d2$. Since $\mathrm{Ann}^{\mathrm{left}}(D) \cap \mathrm{Ann}^{\mathrm{right}}(D) = \mathrm{Ann}(D)$, we obtain that $\dim_F(L/\mathrm{Ann}(D)) = k \leqslant 2d^2$. The fact that D is an ideal implies that $\mathrm{Ann}(D)$ is ideal (see, for example, [4], p. 132).

Put $A = \mathrm{Ann}(D), B = [A, A]$. If a_1, a_2 are arbitrary elements of A and x is an arbitrary element of L, then we have

$$[[a_1, a_2], x] = [a_1, [a_2, x]] - [a_1, [a_1, x]] = 0$$

and

$$[x,[a_1,a_2]] = [[x,a_1],a_2] + [a_1,[x,a_2]] = 0.$$

It shows that the center of L includes B.

If x is an arbitrary element of L, then the mappings $a \to [x,a], a \in A$, and $a \to [a,x], a \in A$, are linear and their kernels are respectively $\mathrm{Ann}^{\mathrm{right}}(x) \cap A$ and $\mathrm{Ann}^{\mathrm{left}}(x) \cap A$. Since $[x,a] \in D$ and $\dim_F(D) = d$, we obtain that $\dim_F(A/(\mathrm{Ann}^{\mathrm{right}}(x) \cap A) \leqslant d$, $\dim_F(A/(\mathrm{Ann}^{\mathrm{left}}(x) \cap A) \leqslant d$. It follows that $\dim_F(A/(\mathrm{Ann}(x) \cap A) \leqslant 2d$.

Choose the elements $x_1, \dots, x_k$ such that a set $\{x_1 + A, \dots, x_k + A\}$ is a basis of L/A. Put $C = \mathrm{Ann}(x_1) \cap \ldots \cap \mathrm{Ann}(x_k) \cap A$, then $\dim_F(A/C) \leqslant 2kd \leqslant 4d^3$. It follows that $\dim_F(L/C) \leqslant 2d^2 + 4d^3 = 2d^2(1+2d)$.

By above proved the factor $(A + \zeta(L))/\zeta(L)$ is abelian, so that and $(C + \zeta(L))/\zeta(L)$ is abelian, in particular, $[A,C],[C,A] \leqslant \zeta(L)$. The choice of C yields that $[x_j, C] = [C, x_j] = \langle 0 \rangle, 1 \leqslant j \leqslant k$. The equality $Fx_1 + \ldots + Fx_k + A = L$ shows that $(C + \zeta(L))/\zeta(L) \leqslant \zeta(L/\zeta(L))$, i.e $C \leqslant \zeta_2(L)$. Thus we obtain that $\dim_F(L/\zeta_2(L))$ is finite, moreover, $\dim_F(L/\zeta_2(L)) \leqslant 2d^2(1+2d)$. □

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