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RESEARCH ARTICLE

Abelian doppelsemigroups*

Anatolii V. Zhuchok and Kolja Knauer

Abstract. A doppelsemigroup is an algebraic system consisting of a set with two binary associative operations satisfying certain identities. Doppelsemigroups are a generalization of semigroups and they have relationships with such algebraic structures as doppelalgebras, duplexes, interassociative semigroups, restrictive bisemigroups, dimonoids and trioids. This paper is devoted to the study of abelian doppelsemigroups. We show that every abelian doppelsemigroup can be constructed from a left and right commutative semigroup and describe the free abelian doppelsemigroup. We also characterize the least abelian congruence on the free doppelsemigroup, give examples of abelian doppelsemigroups and find conditions under which the operations of an abelian doppelsemigroup coincide.

1. Introduction

For semigroups, Drouzy introduced the notion of interassociativity [3]: two semigroups are interassociative if certain axioms relating operations of these semigroups are satisfied. Interassociativity of semigroups was studied in [1–9, 31]. In this paper, we consider doppelsemigroups which are sets with two binary associative operations satisfying axioms of interassociativity. Doppelalgebras introduced by Richter [15] in the context of algebraic K-theory are linear analogs of doppelsemigroups and

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commutative dimonoids in the sense of Loday [10, 18] are examples of doppelsemigroups. Therefore, doppelsemigroup theory has connections to doppelalgebra theory and dimonoid theory. Trioids were introduced in the paper of Loday and Ronco [11] and were studied in different papers (see, e.g., [28, 29]). These algebras were used in [11] to describe classes of trialgebras. The system of axioms of a trioid includes some axioms of a doppelsemigroup. Bisemigroups were considered in the work of Schein [16] and, in particularly, restrictive bisemigroups were investigated in [17]. The latter algebras have applications in the theory of binary relations [17]. It turns out that the varieties of commutative doppelsemigroups and of commutative bisemigroups coincide. A doppelsemigroup can also be determined by using the notion of a duplex [14]. Free duplexes were constructed in [14]. It should be noted that doppelsemigroups satisfy the hyperidentity of associativity [12, 13]. If operations of a doppelsemigroup coincide, we obtain the notion of a semigroup.

The class of all doppelsemigroups forms a variety. Relatively free doppelsemigroups [25] play a crucial role in studying the variety of doppelsemigroups. This motivates us to investigate explicit constructions of free algebras in the variety of doppelsemigroups. It should be noted that some relatively free doppelsemigroups were studied recently: the constructions of the free (strong) doppelsemigroup, of the free commutative (strong) doppelsemigroup and of the free n-nilpotent (strong) doppelsemigroup were presented in [23, 26]. The free n-dinilpotent (strong) doppelsemigroup was constructed in [22, 26]. In [21], the first author described the free left n-dinilpotent doppelsemigroup. The problem is to study abelian doppelsemigroups and construct the free object in the variety of abelian doppelsemigroups. This is the main focus of the present paper.

The paper is organized as follows. In Section 2, we present various notions and results used in the paper, and show that every abelian doppelsemigroup can be constructed from a left and right commutative semigroup. In Section 3, examples of abelian doppelsemigroups are given. In Section 4, we construct the free abelian doppelsemigroup of arbitrary rank and, as consequences, obtain the free abelian doppelsemigroup of rank 1 and the free left and right commutative semigroup. We also establish that the automorphism group of the free abelian doppelsemigroup is isomorphic to the symmetric group. In Section 5, we characterize the least abelian congruence on the free doppelsemigroup. In the final section, some properties of (abelian) doppelsemigroups are established.

2. Preliminaries

Recall that a nonempty set D equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms

$$(x \dashv y) \vdash z = x \dashv (y \vdash z), \tag{D1}$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z) \tag{D2}$$

for all $x, y, z \in D$ is called a *doppelsemigroup* [23]. A doppelsemigroup $(D, \dashv, \vdash)$ is called *strong* [26] if it satisfies the axiom

$$x \dashv (y \vdash z) = x \vdash (y \dashv z).$$

A doppelsemigroup is called *commutative* [23] if both its operations are commutative. A nonempty set equipped with two binary associative operations $\dashv$ and $\vdash$ satisfying the axioms (D2) and

$$(x \dashv y) \dashv z = x \dashv (y \vdash z), \tag{D3}$$

$$(x \dashv y) \vdash z = x \vdash (y \vdash z) \tag{D4}$$

is called a *dimonoid* [10]. For more information on dimonoids see [19,20,27]. A dimonoid $(D, \dashv, \vdash)$ is called *abelian* [30] if

$$x \dashv y = y \vdash x \tag{1}$$

for all $x, y \in D$. Examples of abelian dimonoids and, in particulary, free abelian dimonoids were presented in [30].

A doppelsemigroup will be called *abelian* if it satisfies the identity (1). The class of all abelian doppelsemigroups forms a subvariety of the variety of doppelsemigroups which does not coincide with the variety of commutative doppelsemigroups. A doppelsemigroup which is free in the variety of abelian doppelsemigroups will be called *the free abelian doppelsemigroup*. If ρ is a congruence on a doppelsemigroup $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is an abelian doppelsemigroup, we say that ρ is an *abelian congruence*.

Recall that a semigroup S is called *left (respectively, right) commutative* if it satisfies the identity $xya = yxa$ (respectively, $axy = ayx$). It is clear that a left commutative semigroup satisfies any identity of the form

$$x_1 x_2 \dots x_n a = x_{1\pi} x_{2\pi} \dots x_{n\pi} a, \tag{2}$$

where π is a permutation of $\{1, 2, \dots, n\}$. Dually, a right commutative semigroup satisfies any identity of the form

$$a x_1 x_2 \dots x_n = a x_{1\pi} x_{2\pi} \dots x_{n\pi}, \tag{3}$$

where π is a permutation of $\{1, 2, \ldots, n\}$. The class of all left and right commutative semigroups forms a variety. A semigroup which is free in the variety of left and right commutative semigroups will be called *the free left and right commutative semigroup*.

If $(D, \dashv)$ is a semigroup, then D with an operation $\vdash$, defined by (1), is a semigroup. The semigroup $(D, \vdash)$ is called *dual* to the semigroup $(D, \dashv)$. If $(D, \vdash)$ is dual to $(D, \dashv)$, then it is clear that $(D, \dashv)$ is anti-isomorphic to $(D, \vdash)$. Obviously, a binary relation ϱ on a semigroup $(D, \dashv)$ is a congruence if and only if ϱ is a congruence on the semigroup $(D, \vdash)$ dual to $(D, \dashv)$. From here, for an arbitrary abelian doppelsemigroup $(D, \dashv, \vdash)$, the set of all congruences on $(D, \dashv, \vdash)$ coincides with the set of all congruences on $(D, \dashv)$.

The following lemma establishes necessary and sufficient conditions under which two dual semigroups give rise to an abelian doppelsemigroup.

Lemma 2.1. Let $(D, \dashv)$ be an arbitrary semigroup and $(D, \vdash)$ the dual semigroup to $(D, \dashv)$. Then $(D, \dashv, \vdash)$ is an abelian doppelsemigroup if and only if $(D, \dashv)$ or $(D, \vdash)$ is a left and right commutative semigroup.

Proof. Let $(D, \dashv, \vdash)$ be a doppelsemigroup and $(D, \vdash)$ dual to $(D, \dashv)$. Then for all $x, y, z \in D$,

$$x \dashv y \dashv z = x \dashv (z \vdash y) = (x \dashv z) \vdash y = y \dashv x \dashv z,$$
$$x \dashv y \dashv z = (y \vdash x) \dashv z = y \vdash (x \dashv z) = x \dashv z \dashv y$$

according to (1) and the axioms of a doppelsemigroup. Hence $(D, \dashv)$ is a left and right commutative semigroup.

Conversely, let $(D, \dashv)$ be a left and right commutative semigroup, $(D, \vdash)$ dual to $(D, \dashv)$ and $x, y, z \in D$. Then

$$x \dashv z \dashv y = x \dashv (y \vdash z), \qquad z \dashv x \dashv y = (x \dashv y) \vdash z,$$
$$y \dashv x \dashv z = (x \vdash y) \dashv z, \qquad y \dashv z \dashv x = x \vdash (y \dashv z)$$

according to (1). Hence, using left commutativity and right commutativity of $(D, \dashv)$, we obtain (D1) and (D2). So, $(D, \dashv, \vdash)$ is a doppelsemigroup. It is abelian because (1) holds.

The remaining case is considered in a similar way. □

Corollary 2.2. Let $(D, \dashv)$ be a left and right commutative semigroup, and $(D, \vdash)$ the dual semigroup to $(D, \dashv)$. Then $(D, \dashv, \vdash)$ is an abelian doppelsemigroup. Conversely, any abelian doppelsemigroup $(D, \dashv, \vdash)$ can be constructed in this way.

Using Corollary 2.2 and the remark above, we conclude that the set of all congruences on an abelian doppelsemigroup $(D, \dashv, \vdash)$ coincides with the set of all congruences on a left and right commutative semigroup $(D, \dashv)$. So, the problem of the description of congruences on an abelian doppelsemigroup is reduced to the description of congruences on a left and right commutative semigroup.

Corollary 2.3. Let $(D, \vdash)$ be an arbitrary semigroup and $(D, \dashv)$ the dual semigroup to $(D, \vdash)$. Then $(D, \dashv, \vdash)$ is an abelian doppelsemigroup if and only if $(D, \vdash)$ or $(D, \dashv)$ is a left and right commutative semigroup.

Corollary 2.4. Let $(D, \vdash)$ be a left and right commutative semigroup, and $(D, \dashv)$ the dual semigroup to $(D, \vdash)$. Then $(D, \dashv, \vdash)$ is an abelian doppelsemigroup. Conversely, any abelian doppelsemigroup $(D, \dashv, \vdash)$ can be constructed in this way.

As usual, $\mathbb{N}$ denotes the set of all positive integers.

Lemma 2.5. In an abelian doppelsemigroup $(D, \dashv, \vdash)$, for any $n > 1$ with $n \in \mathbb{N}$, and any $x_i \in D$, $1 \leqslant i \leqslant n+1$, and $*_j \in \{\dashv, \vdash\}$, $1 \leqslant j \leqslant n$,

$$\begin{aligned} x_1 *_1 x_2 *_2 \cdots *_n x_{n+1} &= x_{1\pi} *_1 x_{2\pi} *_2 \cdots *_n x_{(n+1)\pi} \\ &= x_1 \dashv x_2 \dashv \cdots \dashv x_{n+1}, \end{aligned}$$

where π is a permutation of $\{1, 2, \ldots, n+1\}$.

Proof. Lemma 3.1 from [23] states that in any doppelsemigroup the product of three and more multipliers does not depend on parenthesizing. The proof follows from Lemmas 3.1 [23], 2.1 and the identities (1)–(3). $\square$

Corollary 2.6. Every abelian doppelsemigroup is strong.

3. Examples of abelian doppelsemigroups

In this section, we give examples of abelian doppelsemigroups.

Let X be an arbitrary set such that $|X| > 4$ and $a, b, x, y, 0$ pairwise distinct elements from X. Define binary operations $\dashv$ and $\vdash$ on X by

$$f \dashv g = \begin{cases} a, & f = x,\ g = y, \\ b, & f = y,\ g = x, \\ 0 & \text{otherwise}, \end{cases}$$

$$f \vdash g = \begin{cases} a, & f = y,\, g = x, \\ b, & f = x,\, g = y, \\ 0 & \text{otherwise} \end{cases}$$

for all $f, g \in X$.

Proposition 3.1. $(X, \dashv, \vdash)$ is an abelian doppelsemigroup.

Proof. It is immediate to check that $(X, \dashv, \vdash)$ is an abelian doppelsemigroup. □

Let X be an arbitrary set, $|X| > 1$ and $F^*[X]$ the free commutative semigroup on X. Fix $x, y \in X$ such that $x \neq y$. Define binary operations $\dashv$ and $\vdash$ on

$$K = (F^*[X] \backslash X) \cup \{x, y\} \cup \{(x, y)\}$$

by

$$w \dashv u = \begin{cases} (x, y), & w = x,\, u = y, \\ wu & \text{otherwise}, \end{cases}$$

$$w \vdash u = \begin{cases} (x, y), & w = y,\, u = x, \\ wu & \text{otherwise}, \end{cases}$$

$$w \dashv (x, y) = w \vdash (x, y) = wxy,$$

$$(x, y) \dashv w = (x, y) \vdash w = xyw,$$

$$(x, y) \dashv (x, y) = (x, y) \vdash (x, y) = xyxy$$

for all $w, u \in K \backslash \{(x, y)\}$.

Proposition 3.2. $(K, \dashv, \vdash)$ is an abelian doppelsemigroup.

Proof. An immediate verification shows that four identities of a doppelsemigroup hold concerning the operations $\dashv$ and $\vdash$ and thus, $(K, \dashv, \vdash)$ is a doppelsemigroup. It is clear that the operations $\dashv$ and $\vdash$ satisfy (1). □

Let X_1 and X_2 be arbitrary nonempty subsets of $\mathbb{N}$ such that $X_1 \cap X_2 = \varnothing$. Define binary operations $\dashv$ and $\vdash$ on $\mathbb{N} \cup (X_1 \times X_2)$ by

$$m \dashv k = \begin{cases} (m, k), & m \in X_1,\, k \in X_2, \\ m + k & \text{otherwise}, \end{cases}$$

$$m \vdash k = \begin{cases} (m, k), & m \in X_2,\, k \in X_1, \\ m + k & \text{otherwise}, \end{cases}$$

$$m \dashv (x, y) = m \vdash (x, y) = m + x + y,$$
$$(x, y) \dashv m = (x, y) \vdash m = x + y + m,$$
$$(x, y) \dashv (a, b) = (x, y) \vdash (a, b) = x + y + a + b$$

for all $m, k \in \mathbb{N}$ and $(x, y), (a, b) \in X_1 \times X_2$.

Proposition 3.3. $(\mathbb{N} \cup (X_1 \times X_2), \dashv, \vdash)$ is an abelian doppelsemigroup.

Proof. The proof follows by a direct verification. □

4. Free objects

In this section, we construct the free abelian doppelsemigroup of arbitrary rank and consider separately free abelian doppelsemigroups of rank 1. We also show that the automorphism group of the free abelian doppelsemigroup is isomorphic to the symmetric group and present the free left and right commutative semigroup.

Let X be an arbitrary nonempty set, $F[X]$ the free semigroup on X and $F^*[X]$ the free commutative semigroup on X. The length of an arbitrary word ω in the alphabet X will be denoted by l_ω. Consider the sets

$$A_1 = \{w \in F[X] \mid l_w \in \{1, 2\}\} \quad \text{and} \quad A_2 = \{w \in F^*[X] \mid l_w > 2\}.$$

Define binary operations $\dashv$ and $\vdash$ on $A_1 \cup A_2$ by

$$w \dashv u = wu \quad \text{and} \quad w \vdash u = uw$$

for all $w, u \in A_1 \cup A_2$. The algebra obtained in this way will be denoted by $\mathrm{FAD}(X)$. Note that the semigroup $(A_1 \cup A_2, \vdash)$ is dual to the semigroup $(A_1 \cup A_2, \dashv)$.

Theorem 4.1. $\mathrm{FAD}(X)$ is the free abelian doppelsemigroup.

Proof. For any $w, u, \omega \in \mathrm{FAD}(X)$ and $*, \circ \in \{\dashv, \vdash\}$, we have

$$(w * u) \circ \omega = w * (u \circ \omega) = wu\omega,$$

hence $\mathrm{FAD}(X)$ is a doppelsemigroup. Moreover, $w \dashv u = wu = u \vdash w$ and so, $\mathrm{FAD}(X)$ is abelian.

Let us show that $\mathrm{FAD}(X)$ is free in the variety of abelian doppelsemigroups.

Obviously, FAD(X) is generated by X. Let $(V, \dashv', \vdash')$ be an arbitrary abelian doppelsemigroup. Let $\beta : X \to V$ be an arbitrary map. Define a map

$$\phi : \mathrm{FAD}(X) \to (V, \dashv', \vdash')$$

by

$$(x_1 x_2 \dots x_n)\phi = x_1\beta \dashv' x_2\beta \dashv' \cdots \dashv' x_n\beta, \quad x_1, x_2, \dots, x_n \in X.$$

According to Lemma 2.5 and (1) ϕ is well-defined. We will now show that ϕ is a homomorphism by using Lemma 2.5 and (1) again.

Let $y_1 y_2 \dots y_s \in \mathrm{FAD}(X)$, where $y_1, y_2, \dots, y_s \in X$. Consider the following two cases. If $n = s = 1$, then

$$(x_1 \vdash y_1)\phi = (y_1 x_1)\phi = y_1\beta \dashv' x_1\beta = x_1\beta \vdash' y_1\beta = x_1\phi \vdash' y_1\phi.$$

In the case $n \neq 1$ or $s \neq 1$ we get

$$\begin{aligned}((x_1 x_2 \dots x_n) \vdash (y_1 y_2 \dots y_s))\phi &= (x_1 x_2 \dots x_n y_1 y_2 \dots y_s)\phi \\ &= x_1\beta \dashv' x_2\beta \dashv' \cdots \dashv' x_n\beta \dashv' y_1\beta \dashv' y_2\beta \dashv' \cdots \dashv' y_s\beta \\ &= x_1\beta \dashv' x_2\beta \dashv' \cdots \dashv' x_n\beta \vdash' y_1\beta \dashv' y_2\beta \dashv' \cdots \dashv' y_s\beta \\ &= (x_1 x_2 \dots x_n)\phi \vdash' (y_1 y_2 \dots y_s)\phi.\end{aligned}$$

A direct verification shows that

$$((x_1 x_2 \dots x_n) \dashv (y_1 y_2 \dots y_s))\phi = (x_1 x_2 \dots x_n)\phi \dashv' (y_1 y_2 \dots y_s)\phi$$

for all $x_1 x_2 \dots x_n, y_1 y_2 \dots y_s \in \mathrm{FAD}(X)$. So, ϕ is a homomorphism.

Clearly, $x\phi = x\beta$ for all $x \in X$. Since X generates FAD(X), the uniqueness of such homomorphism ϕ is obvious. Thus, FAD(X) is free in the variety of abelian doppelsemigroups.

□

Corollary 4.2. The operations of a singly generated free abelian doppelsemigroup coincide and it is the additive semigroup of positive integers.

Corollary 4.3. $(A_1 \cup A_2, \dashv)$ is the free left and right commutative semigroup.

The free abelian doppelsemigroup FAD(X) is determined uniquely up to isomorphism by cardinality of the set X. Hence the automorphism group of FAD(X) is isomorphic to the symmetric group on X.

5. The least abelian congruence on the free doppelsemigroup

In this section, we characterize the least abelian congruence on the free doppelsemigroup.

The free doppelsemigroup is given in [23]. Recall this construction. We will use notations from Section 4.

Let T be the free monoid on the two-element set $\{a, b\}$ and $\theta \in T$ the empty word. By definition, the length l_θ of θ is equal to 0. Define operations $\dashv$ and $\vdash$ on

$$F = \{(w, u) \in F[X] \times T \,|\, l_w - l_u = 1\}$$

by

$$(w_1, u_1) \dashv (w_2, u_2) = (w_1 w_2, u_1 a u_2)$$

and

$$(w_1, u_1) \vdash (w_2, u_2) = (w_1 w_2, u_1 b u_2)$$

for all $(w_1, u_1), (w_2, u_2) \in F$. The algebra $(F, \dashv, \vdash)$ is denoted by $\mathrm{FDS}(X)$.

Theorem 5.1. ([23], Theorem 3.5) $\mathrm{FDS}(X)$ is the free doppelsemigroup.

If $f : D_1 \to D_2$ is a homomorphism of doppelsemigroups, the corresponding congruence on D_1 will be denoted by Δ_f.

Theorem 5.2. Let $\mathrm{FDS}(X)$ be the free doppelsemigroup, $(x_1 x_2 \dots x_n, u) \in \mathrm{FDS}(X)$, $x_1, x_2, \dots, x_n \in X$ and $\mathrm{FAD}(X)$ the free abelian doppelsemigroup. Then the map

$$\mu : \mathrm{FDS}(X) \to \mathrm{FAD}(X),$$

$$(x_1 x_2 \dots x_n, u) \mapsto (x_1 x_2 \dots x_n, u)\mu = \begin{cases} x_2 x_1, & n = 2,\ u = b, \\ x_1 x_2 \dots x_n & \text{otherwise} \end{cases}$$

is an epimorphism inducing the least abelian congruence on $\mathrm{FDS}(X)$.

Proof. For arbitrary elements $(x_1 x_2 \dots x_n, u)$, $(y_1 y_2 \dots y_s, t) \in \mathrm{FDS}(X)$, where $x_1, x_2, \dots, x_n$, $y_1, y_2, \dots, y_s \in X$, we consider the following two cases.

Case 1: $n = s = 1$. Then

$$((x_1, \theta) \dashv (y_1, \theta))\mu = (x_1 y_1, a)\mu = x_1 y_1 = x_1 \dashv y_1 = (x_1, \theta)\mu \dashv (y_1, \theta)\mu,$$
$$((x_1, \theta) \vdash (y_1, \theta))\mu = (x_1 y_1, b)\mu = y_1 x_1 = x_1 \vdash y_1 = (x_1, \theta)\mu \vdash (y_1, \theta)\mu.$$

Case 2: $n \neq 1$ or $s \neq 1$. Then

$$\begin{aligned}
((x_1x_2\dots x_n, u) \dashv (y_1y_2\dots y_s, t))\mu &= (x_1x_2\dots x_ny_1y_2\dots y_s, uat)\mu \\
&= x_1x_2\dots x_ny_1y_2\dots y_s = (x_1x_2\dots x_n, u)\mu \dashv (y_1y_2\dots y_s, t)\mu, \\
((x_1x_2\dots x_n, u) \vdash (y_1y_2\dots y_s, t))\mu &= (x_1x_2\dots x_ny_1y_2\dots y_s, ubt)\mu \\
&= x_1x_2\dots x_ny_1y_2\dots y_s = (x_1x_2\dots x_n, u)\mu \vdash (y_1y_2\dots y_s, t)\mu.
\end{aligned}$$

Thus, μ is a homomorphism.

For arbitrary $a_1a_2\dots a_n \in \mathrm{FAD}(X)$ there exists $(a_1a_2\dots a_n, a^{n-1}) \in \mathrm{FDS}(X)$ such that

$$(a_1a_2\dots a_n, a^{n-1})\mu = a_1a_2\dots a_n,$$

where $a^0 = \theta$. So, μ is surjective. By Theorem 4.1, FAD(X) is the free abelian doppelsemigroup. Then Δ_μ is the least abelian congruence on FDS(X). □

6. Some properties

In this section, we show that the variety of abelian doppelsemigroups is a subclass of the variety of abelian dimonoids, give conditions under which the operations of an abelian doppelsemigroup coincide and establish that the varieties of commutative doppelsemigroups and of commutative bisemigroups coincide.

A dimonoid $(D, \dashv, \vdash)$ is a *left zero and right zero dimonoid* [24] provided that $(D, \dashv)$ is a left zero semigroup, and $(D, \vdash)$ is a right zero semigroup.

Proposition 6.1. Every abelian doppelsemigroup is an abelian dimonoid. The converse statement is not true in general.

Proof. We need to prove that in an abelian doppelsemigrooup $(D, \dashv, \vdash)$ the identities (D3) and (D4) hold. For all $x, y, z \in D$, we have

$$(x \dashv y) \dashv z = x \dashv (y \vdash z)$$

and

$$x \vdash (y \vdash z) = (x \dashv y) \vdash z$$

by Lemma 2.5.

Consider a left zero and right zero dimonoid $(D, \dashv, \vdash)$. Obviously, it is abelian. Moreover, if $|D| > 1$, $x, y, z \in D$ and $x \neq z$, then

$$(x \dashv y) \vdash z = z \neq x = x \dashv (y \vdash z). \qquad \square$$

Corollary 6.2. The variety of abelian doppelsemigroups is a subclass of the variety of abelian dimonoids.

A semigroup S is called *separative* if for any $s, t \in S$, $s^2 = st = t^2$ implies $s = t$. A semigroup S is called a *left cancellative* semigroup if for any $a, b, c \in S$, $ca = cb$ implies $a = b$. *Right cancellative* semigroups are defined dually. A semigroup S is called a *two-sided cancellative* semigroup if it is a left cancellative semigroup and a right cancellative semigroup. A semigroup S is called *globally idempotent* if $S^2 = S$. A semigroup, which satisfies two quasiidentities

$$x^2 = xy, \ \ y^2 = yx \quad \Longrightarrow \quad x = y,$$

$$x^2 = yx, \ \ y^2 = xy \quad \Longrightarrow \quad x = y,$$

is called *weakly cancellative.*

The following statement establishes necessary and sufficient conditions under which the operations of two dual semigroups coincide.

Proposition 6.3. Let $(D, \dashv)$ be an arbitrary semigroup and $(D, \vdash)$ the dual semigroup to $(D, \dashv)$. The operations of $(D, \dashv, \vdash)$ coincide if and only if $\dashv$ is commutative.

Proof. Let $\dashv = \vdash$. Then, using (1), we get $x \dashv y = y \vdash x = y \dashv x$ for all $x, y \in D$ and so, $\dashv$ is commutative.

Conversely, let $\dashv$ be commutative. Since $(D, \vdash)$ is the dual semigroup to $(D, \dashv)$, we have $x \dashv y = y \dashv x = x \vdash y$ for all $x, y \in D$. Thus, $\dashv = \vdash$. □

Corollary 6.4. The operations of an abelian doppelsemigroup $(D, \dashv, \vdash)$ coincide if $(D, \dashv)$ or $(D, \vdash)$ is

(i) a commutative semigroup,
(ii) an idempotent semigroup,
(iii) a separative semigroup,
(iv) a globally idempotent semigroup,
(v) a left (right, two-sided) cancellative semigroup,
(vi) a weakly cancellative semigroup, or
(vii) a monoid.

Proof. Let $(D, \dashv, \vdash)$ be an abelian doppelsemigroup. We prove all cases for $(D, \dashv)$. The remaining cases are considered in a similar way.

(i) Obviously, in $(D, \dashv, \vdash)$, the operation $\dashv$ is commutative if and only if $\vdash$ is commutative. Hence, by Proposition 6.3, $\dashv = \vdash$.

(ii) By Lemma 2.5,

$$x \dashv y \dashv z = y \dashv x \dashv z \tag{4}$$

for all $x, y, z \in D$. If $y = z$, then, using the idempotent property of $\dashv$ and Lemma 2.5, we obtain $x \dashv y = y \dashv x$ for all $x, y \in D$. Hence, according to Proposition 6.3, $\dashv = \vdash$.

(iii) Let $x, y \in D$. Assume that $a = x \dashv y$, $b = y \dashv x$. Using Lemma 2.5, we have

$$a^2 = (x \dashv y) \dashv (x \dashv y) = (x \dashv y)^2, \tag{5}$$

$$a \dashv b = (x \dashv y) \dashv (y \dashv x) = (x \dashv y)^2 \tag{6}$$

and

$$b^2 = (y \dashv x) \dashv (y \dashv x) = (x \dashv y)^2. \tag{7}$$

Since the semigroup $(D, \dashv)$ is separative, $a^2 = a \dashv b = b^2$ implies $a = b$. Thus, $\dashv$ is commutative and by Proposition 6.3, $\dashv = \vdash$.

(iv) Let $x, y \in D$. By global idempotentity of $(D, \dashv)$, $y = y_1 \dashv y_2$ for some y_1, $y_2 \in D$. Then

$$x \dashv y = x \dashv (y_1 \dashv y_2) = (y_1 \dashv y_2) \dashv x = y \dashv x$$

according to Lemma 2.5. By virtue of Proposition 6.3, $\dashv = \vdash$.

(v) For all $x, y, z \in D$, using Lemma 2.5, we have

$$z \dashv x \dashv y = z \dashv y \dashv x.$$

Hence, by left cancellativity of $(D, \dashv)$, we obtain $x \dashv y = y \dashv x$ for all $x, y \in D$ and so, $\dashv = \vdash$ by Proposition 6.3.

The case of right cancellativity is proved in a similar way. The proof for the case of two-sided cancellativity follows from the above.

(vi) For arbitrary elements $x, y \in D$ assume that $a = x \dashv y$ and $b = y \dashv x$. By virtue of Lemma 2.5, we obtain (5), (6), (7) and

$$b \dashv a = (y \dashv x) \dashv (x \dashv y) = (x \dashv y)^2.$$

Due to weak cancellativity of $(D, \dashv)$, $a^2 = a \dashv b$, $b^2 = b \dashv a$ implies $a = b$. Thus, $\dashv$ is commutative. Then, by Proposition 6.3, $\dashv = \vdash$.

(vii) Let $(D, \dashv)$ be a monoid with the identity element e. For all $x, y, z \in D$, we have (4) according to Lemma 2.5. If we substitute $z = e$ into (4), we obtain $x \dashv y = y \dashv x$ for all $x, y \in D$. From here $\dashv = \vdash$ by Proposition 6.3. □

At the end of the paper we establish a relation between doppelsemigroups and bisemigroups [16].

Recall that a nonempty set B with the operations $\dashv$ and $\vdash$ is called a *bisemigroup* if $(B, \dashv)$ and $(B, \vdash)$ are semigroups and (D2) holds. Restrictive bisemigroups were studied by Schein (see, e.g., [17]) and they have applications in the theory of binary relations. A bisemigroup $(B, \dashv, \vdash)$ will be called *commutative* if the operations $\dashv$ and $\vdash$ are commutative.

It is not hard to prove the following statement.

Proposition 6.5. The varieties of commutative doppelsemigroups and of commutative bisemigroups coincide.

Observe that doppelalgebras are linear analogs of doppelsemigroups, therefore all results obtained for doppelsemigroups hold for doppelalgebras.

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CONTACT INFORMATION

Anatolii V. Zhuchok
Department of Algebra and System Analysis, Luhansk Taras Shevchenko National University, Gogol square, 1, Starobilsk, 92703, Ukraine
E-Mail(s): zhuchok.av@gmail.com

Kolja Knauer
Laboratory of Computer Science and Systems, Aix-Marseille University, LIS UMR 7020, Case Courrier 901, 163, avenue de Luminy 13288, Marseille Cedex 9, France
E-Mail(s): kolja.knauer@lis-lab.fr

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