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Representations of strongly algebraically closed algebras

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Abstract. We introduce the notion of q'-compactness for MV-algebras. One of the main results of the paper is a characterization of a class of orthomodular lattices that are horizontal sums of strongly algebraically closed algebras.

1. Introduction

This paper deals with MV-algebras (multi-valued-algebras), Hajek's basic algebras, and the algebras of Hajek's basic logic. MV-algebras are models of an equational theory in universal algebra. In order to prove the completeness theorem of Łukasiewicz infinite-valued logic, Chang introduced MV-algebras in [8]. In recent years, MV-algebras and pseudo-MV-algebras have been extensively studied and investigated by W. Chen, W.A. Dudek, B. Davvaz, R.A. Borzooei and others (see [4], [8]-[13], [20]-[29], [40], and [33]). Recall that pseudo-MV-algebras were introduced by Dvurecenskij [24]. It is noted that MV-algebras and their generalizations have been rapidly developed into the so called quasi-pseudo-MV-algebras with board applications in quantum computational logics and theoretical computer science. An interesting problem in mathematics is to construct relative complicated objects from simple ones or conversely. For this purpose, I. Chajda in [5] and [6] studied the structure of basic algebras which are horizontal sums of chain basic algebras or, in particular, the

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cases when the components are chains or MV-algebras. Similarly, we can study the case when the components are strongly algebraically closed algebras.

In [49] J. Schmid proved that a distributive lattice is a algebraically closed lattice if and only if it is a Boolean lattice. Also, he shows that any strongly algebraically closed lattice is a complete Boolean lattice. Later, it is proved by the author in [39] that if a complete Boolean lattice is q'-compact, then it is a strongly algebraically closed lattice. We recall from [39] that an algebra A is strongly algebraically closed in a class of algebras, if every set of equations (finite or infinite) with coefficients from A, which is solvable in some algebras of the class of algebras containing A, already has a solution in A.

In this paper we study the q'-compactness of MV-algebras. In particular, we prove in Theorem 3.5 that if every subalgebra of an orthomodular lattice L is compact Hausdorff MV-algebra and q'-compact, then L is a horizontal sum of strongly algebraically closed algebras, where L satisfies (H1) and (H2). We recall that the notion of algebraically compactness for a general algebraic systems was introduced and studied in early 50ties of XX century by Łoś in [34] and [35] , later developed by Balcerzyk [3] for abelian groups and by Simson [45] for Grothendieck categories.

2. Preliminaries

We recall from [32] that an *orthomodular lattice* $(L, \wedge, \vee, ', \mathbf{0}, \mathbf{1})$ is an ortholattice L which satisfies the orthomodular law: if $x \leqslant y$ then $y = x \vee (x' \wedge y)$, for all x, $y \in L$. It is known (see [6]), that a subalgebra of an orthomodular lattice L is a structure $(L^*, \wedge_*, \vee_*, '*, \mathbf{0}^*, \mathbf{1}*)$ where: (i) $L^* \in L$, (ii) $\wedge_*$, $\vee_*$, $'*$ are the restrictions of L, $\wedge$, $\vee$, $'$ to L^*, (iii) $\mathbf{0}^* = \mathbf{0}$ and $\mathbf{1}* = \mathbf{1}$, (iv) L^* is an orthomodular lattice. A *Boolean algebra* is an ortholattice satisfying the distributive law: $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for all x, y, $z \in L$. Sachs [48] showed that a Boolean algebra is determined by its lattice of subalgebras. In [30], it is shown that an orthomodular lattice L is determined by its lattice of subalgebras, as well as by its poset of Boolean subalgebras.

In [2], an orthomodular L is called the *horizontal sum* of a family $(L_i)_{i\in I}$ of at least two subalgebras of L, if $\cup L_i = L$ and $L_i \cap L_j = \{0, 1\}$ whenever $i \neq j$ and one of the following equivalent conditions is satisfied:

(1) if $x \in L_i \backslash L_j$ and $y \in L_j \backslash L_i$, then $x \vee y = 1$,

(2) every block of L belongs to some L_i,

(3) if S_i is a subalgebra of L_i, then $\cup S_i$ is a subalgebra of L.

System of equations we mean an arbitrary set of equations. Recall from [46] that Boolean algebras B_1, B_2 are geometrically equivalent if for any system of equations the coordinate algebras over B_1 and B_2 are isomorphic. This means that a description of coordinate algebras over an algebra B_1 automatically implies the corresponding description over any algebra B_2 which is geometrically equivalent to B_1 [18]. We recall that the problem "*When two geometrically equivalent extensions L_1 and L_2 of a field P have different elementary theories in the logic?*" of the study of geometric equivalence was posed in [43]. In [37] this problem was solved for equationally Noetherian groups. Theorem 7.2 of [46] contains a criterion for a pair of boolean algebras to be geometrically equivalent. Following this, in the present paper we define geometrically equivalence for MV-algebras, and we try to establish a relationship between MV-algebras and the strongly algebraically closed algebras.

We recall from [5, 6] the following definitions. A *basic algebra* is called an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following four identities:

(1) $x \oplus 0 = x$,

(2) $\neg\neg x = x$,

(3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

(4) $\neg(\neg(\neg(x \oplus y)\oplus) \oplus z) \oplus (x \oplus z) = 1$,

for all x, y, $z \in A$.

Basic algebras are to some extent similar to MV-algebras; especially, the stipulation

$$x \leqslant y \Longleftrightarrow \neg x \oplus y = 1.$$

defines a bounded lattice in which the join $x \vee y$ and the meet $x \wedge y$ are given by

$$x \vee y = \neg(\neg x \oplus y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y).$$

It is well known that $(A, \leqslant)$ is a bounded lattice, where 0 is the least and 1 the greatest element. The lattice $(A, \vee, \wedge, 0, 1)$ is said the *assigned lattice* of A.

For more information the theory of lattices, the reader is referred to Birkhoff [1]. In order to make this preliminary section not too long we give only a quick review of MV-algebras, referring to [15] for further details.

An *MV-algebra* is a structure $(A, \oplus, \neg, 0)$ where $\oplus$ is a binary operation, $\neg$ is a unary operation and 0 is a constant such that the following axioms are satisfied for any x, $y \in A$:

(1) $(A, \oplus, 0)$ is an abelian monoid,

(2) $\neg\neg x = x$,

(3) $x \oplus \neg 0 = \neg 0$,

(4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

In an MV-algebra A the constant 1 and the auxiliary operation $\odot$ are defined as follows:

1) $1 := \neg 0$,

2) $x \odot y := \neg(\neg a \oplus \neg b)$.

Let A be an MV-algebra. Then the natural order determines a lattice structure over A. The join $x \vee y$ and the meet $x \wedge y$ of the elements x and y are given by

$$x \vee y = (x \odot \neg y) \oplus y, \quad x \wedge y = \neg(\neg x \vee \neg y).$$

As it was shown in [25], an MV-algebra is a basic algebra $(A, \oplus, \neg, 0)$ whose binary operation $\oplus$ is commutative and associative. Recall that an element x of a basic algebra A is called sharp if $x \oplus x = x$. An MV-algebra A is *complete* iff its underlying lattice is complete ([17], p. 129).

As mentioned in [36], an MV-equation in the variables $x_1, \ldots, x_t$ is a pair (τ, σ) of MV-terms in the variables $x_1, \ldots, x_t$. Following a tradition, we write $\tau = \sigma$ instead of (τ, σ). An MV-algebra A satisfies the MV-equation $\tau = \sigma$, in symbols, $A \models (\tau = \sigma)$, if $\tau^A(a_1, \ldots, a_t) = \sigma^A(a_1, \ldots, a_t)$ for any $a_1, \ldots, a_t \in A$.

Notice that a *topological MV-algebra* is an MV-algebra $(A, \oplus, \neg, 0)$ together with a topology τ such that $\oplus$ and $\neg$ (and in particular $\vee$, $\wedge$) are τ-continuous. For any MV-algebra A, its maximal ideal space equipped with the spectral topology is a nonempty compact Hausdorff space [50].

3. Horizontal sums of strongly algebraically closed algebras

The following proposition and lemma of [5] are our main motivation to investigate that an orthomodular lattice or a basic algebra is horizontal sum of strongly algebraically closed algebras.

Recall from [39] that an algebra A is strongly algebraically closed in a class of algebras, if every set of equations (finite or infinite) with coefficients from A, which is solvable in some algebra of the class containing A, already has a solution in A (compare with [3], [35] and Section 4 in [45]).

Proposition 3.1. *A basic algebra $(A, \oplus, \neg, 0)$ is a lattice effect algebra if and only if it satisfies the quasi-identity:*

$$x \leqslant \neg y \quad \text{and} \quad x \oplus y \leqslant \neg z \Longrightarrow x \oplus (z \oplus y) = (x \oplus y) \oplus z \qquad (H_1)$$

for all x, y, $z \in A$.

Lemma 3.2. *Let a basic algebra* $(A, \oplus, \neg, 0)$ *be a horizontal sum of MV-algebras. Then A satisfies the condition:*

$$x \oplus y \neq y \oplus x,\ x \oplus z = z \oplus x,\ y \oplus z = z \oplus y \Longrightarrow z = 0 \text{ or } z = 1 \quad (H_2)$$

for all x, y, $z \in A$.

Let S be a system of equations in MV-algebra A and $V_A(S)$ be the set of solutions of S in underlying lattice A ([22]). The set of all logical consequences of S over A is the radical $\mathrm{Rad}_A(S)$. In other words, $\mathrm{Rad}_A(S)$ is the set of all MV-algebra equations $\tau \approx \sigma$ such that $V_A(S) \subseteq V_A(\tau \approx \sigma)$, where τ and σ are MV-terms. It is known that $\mathrm{Rad}_A(S)$ is an ideal in the term algebra [22].

Now, we introduce geometrically equivalent and q'-compactness in MV-algebras.

Definition 3.3. Let A and B be two MV-algebras. Then A and B are meant geometrically equivalent, if $\mathrm{Rad}_A(S) = \mathrm{Rad}_B(S)$ for any system S.

Definition 3.4. An MV-algebra A is called q'-compact if it is a geometrically equivalent to any of its elementary extensions of underlying lattice A.

We are ready to prove our main theorem.

Theorem 3.5. *Let an orthomodular lattice L satisfies* (H_1) *and* (H_2)*. If every subalgebra of L is compact Hausdorff MV-algebra and* q'*-compact, then L is a horizontal sum of strongly algebraically closed algebras.*

Proof. Suppose that L is an orthomodular lattice. By [7], we know that every orthomodular lattice is a basic algebra $(A, \oplus, \neg, 0)$ satisfying the identity $x \oplus (x \wedge y) = x$ and every element of A is sharp. According to the assumption, since L satisfies (H_1) and (H_2) then by ([5], Theorem 2), L is a horizontal sum of MV-algebras. It remains to prove that if an MV-algebra K admits a topology making it a compact and Hausdorff space, then K is complete. Now, we are focusing the Boolean center $B(K) = \{a \in K \mid a \wedge \neg a = 0\} = f^{-1}(\{0\})$ of K equipped with the subspace topology of K, where $f : K \longrightarrow K$ the map defined by $f(a) = a \wedge \neg a$. We have that f is clearly continuous and $\{0\}$ is closed and $B(K)$ is a closed subspace of the compact space K. Therefore, $B(K)$ admits a compact and Hausdorff topological making it a topological Boolean algebra. In conclusion, every compact Hausdorff MV-algebra is complete ([31], Theorem 3.41) and here K is complete. Thus, we observe that

L is a horizontal sum of complete MV-algebras. On the other hand, every element of L is sharp. By ([5], Corollary 2), L is horizontal sum of complete Boolean algebras. Now, we prove that each of these components are strongly algebraically closed algebras. Suppose C is one of these components. We claim that C is a strongly algebraically closed algebra. In [39], we prove that if C is complete Boolean algebra which is q'-compact, then C is strongly algebraically closed algebra. So C is a horizontal sum of strongly algebraically closed algebras. To prove, suppose $\mathcal{F}'$ is the language of boolean algebras and we have that C is a lattice in the language of ortholattices $\mathcal{F}$. Note that, in the same time we can consider C as a boolean algebra. Because if we attach the elements of C as constants to $\mathcal{F}$, then the new language will be denoted by $\mathcal{F}(C)$. Let S be a consistent system in the language $\mathcal{F}(C)$. Clearly, S is also a system in $\mathcal{F}'(C)$. Since C is complete so by [46], it is a weak equational Noetherian boolean algebra. So, there is a finite system T in the language $\mathcal{F}'(C)$ equivalent to S over C. We know that every finite $S_0 \subseteq S$ is consistent, and by the result [49] of Schmid , C is algebraically closed. Hence, every such S_0 has a solution in L. So S has a solution in some ultra-power $D = C^I/\mathcal{U}$. Note that C is also a distributive lattice, and since it is an elementary extension of C, and C is q'-compact then

$$\mathrm{Rad}_C(S) = \mathrm{Rad}_D(S).$$

On the other hand, we have

$$\mathrm{Rad}_C(S) = \mathrm{Rad}_C(T).$$

Since T is finite, we also have

$$\mathrm{Rad}_C(T) = \mathrm{Rad}_D(T).$$

This shows that S and T are equivalent over D. Therefore, T has a solution in D and consequently in C. Thus S has a solution in C. On the other hand, T is a finite system in the language $\mathcal{F}'(C)$. But by introducing a finite number of new variables and a finite number of new equations we can transform it to a finite system in $\mathcal{F}(C)$. To do this, we perform the following actions:

1. If T contains the boolean constants 0 and 1, then there will be no change, since $0, 1 \in C$.

2. If T contains therm x', then we introduce a new variable y and insert new equations $x \wedge y \approx 0$ and $x \vee y \approx 1$, instead.

3. If there appears a term of the form a', then again there will not be any changes.

Therefore, each of these components are strongly algebraically closed algebras and the proof is complete. □

In [19], orthologic or minimal quantum logic is the logic that is semantically characterized by the class of all algebraic realizations based on ortholattices. Generally, an algebraic realization for a logic has the following form $(\mathcal{B}, \nu)$, where $\mathcal{B}$ is an element of given category of algebraic structures, while ν is a valuation-function that transforms sentences into elements of $\mathcal{B}$, preserving the logical form. Moreover, an algebraic realization for orthologic is a pair $(\mathcal{B}, \nu)$ consisting of an ortholattice $(\mathcal{B}, \leqslant, ', \mathbf{0}, \mathbf{1})$ and a valuation-function ν that associates to any sentence α of the language an element in $\mathcal{B}$, satisfying the following conditions:

(i) $\nu(\sim \beta) = \nu(\beta)'$,

(ii) $\nu(\beta \curlywedge \gamma) = \nu(\beta) \wedge \nu(\gamma)$,

which the notions $\sim$ and $\curlywedge$ of sentence (or formula) of the logic language is defined "*not*" and "*and*", respectively. Note that an orthomodular lattice L is a Boolean algebra iff for any algebraic realization (L, ν) and any i $(1 \leqslant i \leqslant 5)$ and any α, β the following condition is satisfied:

$$\nu(\gamma) \wedge \nu(\alpha) \leqslant \nu(\beta) \quad \Longleftrightarrow \quad \nu(\gamma) \leqslant \nu(\alpha \rightarrow_i \beta), \tag{R}$$

where the notion $\longrightarrow_i$ of the logic language is "*if*". Consequently, we have the following theorem:

Theorem 3.6. *Let an orthomodular lattice L satisfies (H_1) and (H_2) and (R). If every subalgebra of L is q'-compact and complete, then L is a horizontal sum of strongly algebraically closed algebras.*

Definition 3.7. Two elements a and b, in a distributive lattice D with zero, are consonant if there are $x, y \in D$ such that $a \leqslant b \vee x$, $b \leqslant a \vee y$, and $x \wedge y = 0$. A subset X of D is consonant if every pair of elements in X is consonant. A distributive lattice D is called completely normal if it is a consonant subset of itself. We recall from [21] the following definition.

Given A an MV-algebra and $a \in A$, then $\mathrm{Spec}(A)$ and $\langle a \rangle$ denote the set of prime ideals of A and the principal ideal generated by a, respectively. We recall that an MV-algebra A *hypernormal* if and only if $\mathrm{Spec}(A)$ is a cardinal sum of (spectral) chains, i.e., if and only if the algebraic lattice A is a hypernormal lattice in the sense of Monteiro [38] or a perfect lattice in the terminology of [26]. Notice that an MV-algebra A is hypernormal if and only if $\mathrm{Spec}(A)$ is a completely normal lattice (see [16]).

Theorem 3.8. *Let an orthomodular lattice L satisfies (H_1) and (H_2). If for each x, y of arbitrary subalgebra K of L, there exists $z \in K$ such that $\langle x\rangle \cap \langle z\rangle \subseteq \langle y\rangle$ and $\langle y\rangle \cap \langle \neg z\rangle \subseteq \langle x\rangle$, then L is a horizontal sum of hypernormals.*

Proof. By [7], every orthomodular lattice is a basic algebra $(L, \oplus, \neg, 0)$ satisfying the identity $x \oplus (x \wedge y) = x$ in which every element is sharp. On the other hand, using ([5], Theorem 2), the basic algebra $(L, \oplus, \neg, 0)$ satisfies (H_1) and (H_2). So L is a horizontal sum of MV-algebras. Now, we prove that the subalgebra K is hypernormal. Suppose that $z \in L$ we will have $\langle z\rangle \vee \langle \neg t\rangle = K$. Therefore, for each x, y of the subalgebra K of L, $x \wedge y = 0$ implies that there exists $z \in K$ such that $x \wedge z = 0$ and $y \wedge \neg z = 0$. By [27], a bounded distributive lattice is hypernormal if and only if it is simultaneously completely normal and dual completely normal. Therefore, an MV-algebra A is hypernormal if and only if $\operatorname{Spec}(A)$ is a completely normal lattice. In this part of the proof we make use of the following facts [16]

$$(x \odot y) \wedge (y \odot x) = 0.$$

By assumption, for each x, y of the subalgebra K of L, there exists $z \in K$ such that $\langle x\rangle \cap \langle z\rangle \subseteq \langle y\rangle$ and $\langle y\rangle \cap \langle \neg z\rangle \subseteq \langle x\rangle$. Therefore,

$$\langle x\rangle \cap \langle z\rangle \subseteq \langle x\rangle, \quad \langle x\rangle \cap \langle z\rangle \subseteq \langle y\rangle,$$

and then

$$\langle x\rangle \cap \langle z\rangle \subseteq \langle x\rangle \cap \langle y\rangle.$$

If $x \wedge y = 0$, then $\langle x\rangle \cap \langle z\rangle \subseteq \langle x\rangle \cap \langle y\rangle = 0 \Rightarrow x \wedge z = 0$. Also,

$$\langle y\rangle \cap \langle \neg z\rangle \subseteq \langle y\rangle, \quad \langle y\rangle \cap \langle \neg z\rangle \subseteq \langle x\rangle,$$

and then

$$\langle y\rangle \cap \langle \neg z\rangle \subseteq \langle y\rangle \cap \langle x\rangle \text{ and } \langle y\rangle \cap \langle \neg z\rangle \subseteq \langle y\rangle \cap \langle x\rangle = 0$$

and we obtain $y \wedge \neg z = 0$.

On the other hand, since $\langle x\rangle \subseteq \langle x \vee y\rangle$ and $\langle z\rangle \subseteq \langle z\rangle$, then

$$\langle x\rangle \cap \langle z\rangle \subseteq \langle y \vee x\rangle \cap \langle z\rangle = \langle y \oplus (x \odot \neg y)\rangle \cap \langle t\rangle =$$

$$(\langle y\rangle \vee \langle (x \odot \neg y)\rangle) \cap \langle t\rangle = \langle y\rangle \cap \langle z\rangle \subseteq \langle y\rangle.$$

Consequently, $\langle y\rangle \cap \langle \neg z\rangle \subseteq \langle y\rangle$. Similarly, we will have $\langle y\rangle \cap \langle \neg z\rangle \subseteq \langle x\rangle$. In the sequel, for any x, $y \in A$ there exists $t \in A$ such that $\langle y\rangle \cap \langle \neg z\rangle \subseteq \langle x\rangle$ and $\langle y\rangle \cap \langle \neg z\rangle \subseteq \langle y\rangle$. As a result, we set $x = y = t$, then $\langle t\rangle \vee \langle \neg t\rangle = A$. So the subalgebra K is hypernormal and then L is a horizontal sum of hypernormals. □

Theorem 3.9. *Let an orthomodular lattice L satisfies* (H_1) *and* (H_2)*. If every subalgebra of L is compact Hausdorff MV-algebra and* q'*-compact, then L is a horizontal sum of products of copies* $[0, 1]$ *and finite Łukasiewicz chains.*

Proof. In [41], MV-algebras admitting compact Hausdorff topologies are product of copies $[0, 1]$ and finite Łukasiewicz chains. Thus, we conclude L is a horizontal sum of MV-algebras that every MV-algebra is product of copies $[0, 1]$ and finite Łukasiewicz chains. □

Remember that a *block* of an orthomodular L is a maximal Boolean subalgebra of L.

Theorem 3.10. *Let an orthomodular lattice L satisfies* (H_1) *and* (H_2)*. If every block of L is complete subalgebra and* q'*-compact, then L is a horizontal sum of strongly algebraically closed algebras.*

Proof. Suppose that L is an orthomodular lattice and L satisfies (H_1). By Proposition 3.1, L is a lattice effect algebra and every block of L is subalgebra which is an MV-algebra. Moreover, L satisfies (H_1) and (H_2), then L is a horizontal sum of complete MV-algebras, in particular, L is a horizontal sum of complete Boolean algebra ([5], Theorem 2). Finally, we have that every block of L is q'-compact, then L is a horizontal sum of strongly algebraically closed algebras. □

We recall from [28] that orthoalgebras play an important role in the empirical logic approach to the foundations of quantum mechanics initiated by D. J. Foulis and C. H. Randall. Such an algebra appears in that theory as the (empirical) logic associated with a manual of operations. Any orthomodular lattice or orthomodular poset is also an orthoalgebra.

Corollary 3.11. *Let an orthomodular lattice L satisfies* (H_1) *and* (H_2)*. If every subalgebra of L is orthoalgebra and* q'*-compact and compact Hausdorff MV-algebra, then L is a horizontal sum of strongly algebraically closed algebras.*

Proof. We know that every MV-algebra, which is an orthoalgebra, is a Boolean algebra. Since L satisfies (H_1) and (H_2) and every subalgebra of L admits a topology making it a compact and Hausdorff topological MV-algebra, then L is a horizontal sum of complete Boolean algebras. By assumption we have that every subalgebra of L is q'-compact, then L is a horizontal sum of strongly algebraically closed algebras. □

Definition 3.12. [51] A quantale is a triple $(Q, \vee, \otimes)$ such that

(1) $(Q, \vee)$ is a $\vee$-semilattice,

(2) $(Q, \otimes)$ is a semigroup,

(3) $q \otimes (\vee S) = \vee_{s \in S}(q \otimes s)$ and $(\vee S) \otimes q = \vee_{s \in S}(s \otimes q)$ for every $q \in Q$ and every $S \subseteq Q$.

Also, we can say that a *quantale* $(Q, \vee, \cdot, \perp)$ is a sup-lattice equipped with a monoid operation " $\cdot$ ", which distributes over arbitrary joins. For example, let $(A, \cdot)$ be a semigroup. The powerset $\mathcal{P}(A)$ is a quantale, where $\vee$ is union and $S \otimes T = \{s \cdot t \mid s \in S, t \in T\}$.

Corollary 3.13. *Let an orthomodular lattice L satisfies (H_1) and (H_2). If every subalgebra of L is compact Hausdorff MV-algebra, then L is isomorphic with a horizontal sum of quantales.*

Proof. Since L is an orthomodular lattice such that every subalgebra of L admits a topology making it a compact and Hausdorff topological MV-algebra and L satisfies (H_1) and (H_2), L is a horizontal sum of complete MV-algebras. On the other hand, it is proved that every complete MV-algebra is isomorphic with quantales [44]. As a result, L is isomorphic with a horizontal sum of quantales. □

In this final section we consider Sheffer stroke basic algebras. In 1913, Sheffer [47] presented the following 3-basis for Boolean algebra in terms of the Sheffer stroke. We recall from [42] the following definition:

Definition 3.14. An algebra $(A, |)$ of type (2) is called a Sheffer stroke basic algebra if the following identities hold:

(1) $(x|(x|x))|(x|x) = x$,

(2) $(x|(y|y))|(y|y) = (y|(x|x))|(x|x)$,

(3) $(((x|(y|y))|(y|y))|(z|z))|((x|)z|z))|(x|(z|z))) = x|(x|x)$,

for all $x, y, z \in A$.

Theorem 3.15. *Let a basic algebra $(A, \oplus, \neg, 0)$ satisfies (H_1) and (H_2). We define $x|y = \neg x \oplus \neg y$. If every subalgebra of A is q'-compact and complete, then A is a horizontal sum of strongly algebraically closed algebras.*

Proof. Using [42], the first we claim that with definition $x|y = \neg x \oplus \neg y$, the structure $(A, |)$ is a Sheffer stroke basic algebra. From definition of Sheffer stroke basic algebra have the following.

(1) $(x|(x|x)))|(x|x) = \neg(\neg x \oplus \neg(\neg x \oplus \neg x)) \oplus \neg(\neg x \oplus \neg x) = \neg(\neg x \oplus \neg x) \oplus x = 0 \oplus x = 1$.

(2) $(x|(y|y)))|(y|y) = \neg(\neg x \oplus \neg(\neg y \oplus \neg y)) \oplus \neg(\neg x \oplus \neg y) = \neg(\neg y \oplus \neg y) \oplus x = (y|(x|x)))|(x|x)$.

Also, for (3) of Sheffer stroke basic algebra satisfies have that:

$$
\begin{aligned}
&(((x|(y|y))|(y|y))|(z|z))|((x|(z|z))|(x|(z|))) \\
&\quad = \neg(\neg(\neg(\neg x\neg(\neg y))\neg(\neg z)))\neg(\neg(\neg x\neg(\neg z))\neg(\neg x\neg(\neg z))) \\
&\quad = 1 = x|(x|x).
\end{aligned}
$$

Therefore, $(A, |)$ is a Sheffer Stroke basic algebra. On the other hand, by ([42], Theorem 3.18), every subalgebra $(B, |)$ Sheffer stroke basic algebra of A with the least element 0 and the greatest element 1, and its induced lattice $(B, \vee, \wedge, 0, 1)$ with an antitone involution $x \longrightarrow x^0$ is a Boolean algebra. By ([5], Corollary 2), since induced lattice of every subalgebra is Boolean algebra and A satisfies (H_1) and (H_2), then A is a horizontal sum of Boolean algebras and here the Boolean algebras are complete. By the assumption we have that every subalgebra of A is q'-compact. This implies that A is a horizontal sum of strongly algebraically closed algebras ([39], Theorem 3.5). □

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