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RESEARCH ARTICLE

Gram matrices and Stirling numbers of a class of diagram algebras, II

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Abstract. In the paper [6], we introduced Gram matrices for the signed partition algebras, the algebra of $\mathbb{Z}_2$-relations and the partition algebras. $(s_1, s_2, r_1, r_2, p_1, p_2)$-Stirling numbers of the second kind are also introduced and their identities are established. In this paper, we prove that the Gram matrix is similar to a matrix which is a direct sum of block submatrices. As a consequence, the semisimplicity of a signed partition algebra is established.

1. Introduction

In this paper, we establish that the Gram matrices $G^k_{2s_1+s_2}$ and $\overrightarrow{G}^k_{2s_1+s_2}$ introduced in [6] are similar to matrices $\widetilde{G}^k_{2s_1+s_2}$ and $\widetilde{\overrightarrow{G}}^k_{2s_1+s_2}$ respectively and each of which is a direct sum of block sub matrices $\widetilde{A}_{2r_1+r_2,2r_1+r_2}$ and $\widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2}$ of sizes $f^{2r_1+r_2}_{2s_1+s_2}$ and $\overrightarrow{f}^{2r_1+r_2}_{2s_1+s_2}$ respectively. The diagonal entries of the matrices $\widetilde{A}_{2r_1+r_2,2r_1+r_2}$ and $\widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2}$ are the same and the diagonal element is a product of r_1 quadratic polynomials and r_2 linear polynomials which could help in determining the roots of the determinant of the Gram matrix. Similarly, we have also established that the Gram matrix G^k_s of a partition algebra is similar to a matrix $\widetilde{G}^k_s$ which is a direct sum of block matrices $\widetilde{A}_{r,r}$ of size f^r_s. The diagonal entries of

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the matrices $\widetilde{A}_{r,r}$ are the same and the diagonal element is a product of r linear polynomials which could help in determining the roots of the determinant of the Gram matrix.

Using the cellularity structure defined in [5], we show that the algebra of $\mathbb{Z}_2$-relations and signed partition algebras are semisimple over $\mathbb{K}(x)$ where $\mathbb{K}$ is a field of characteristic zero and x is an indeterminate and it is also semisimple over a field of characteristic zero except for a finite number of algebraic elements and we also prove that the algebra of $\mathbb{Z}_2$-relations and the signed partition algebras are quasi-hereditary over a field of characteristic zero. In particular, if q is an integer $\leqslant k-2$ and q is a root of the polynomial $x^2 - x - 2r', 0 \leqslant r' \leqslant k-2$ then the algebras $A_k^{\mathbb{Z}_2}(q)$ and $\overrightarrow{A}_k^{\mathbb{Z}_2}(q)$ are not semisimple.

2. Stirling numbers of second kind of the algebra of $\mathbb{Z}_2$-relations, signed partition algebras and partition algebras

Lemma 2.1.

(a) *In the algebra of $\mathbb{Z}_2$-relations, let $d_{i,\alpha}^{p_1,p_2}, d_{j,\beta}^{r_1,r_2} \in J_{2s_1+s_2}^{2k}$ with $2p_1 + p_2 < 2r_1 + r_2$ then $d_{i,\alpha}^{p_1,p_2}$ is coarser than $d_{j,\beta}^{r_1,r_2}$ if and only if $l(d_{i,\alpha}^{p_1,p_2}.d_{j,\beta}^{r_1,r_2}) = 2p_1 + p_2$ where $J_{2s_1+s_2}^{2k}$ is as in Notation 3.6(a) in [6].*

(b) *In signed partition algebras, let $d_{i,\alpha}^{p_1,p_2}, d_{j,\beta}^{r_1,r_2} \in \overrightarrow{J}_{2s_1+s_2}^{2k}$ with $2p_1 + p_2 < 2r_1 + r_2$ then $d_{i,\alpha}^{p_1,p_2}$ is coarser than $d_{j,\beta}^{r_1,r_2}$ if and only if $l(d_{i,\alpha}^{p_1,p_2}.d_{j,\beta}^{r_1,r_2}) = 2p_1 + p_2$ where $\overrightarrow{J}_{2s_1+s_2}^{2k}$ is as in Notation 3.6 (b) in [6].*

(c) *In partition algebras, let $R^{d_{i,\alpha}^p}, R^{d_{j,\beta}^r} \in J_s^k$ with $p < r$ then $R^{d_{i,\alpha}^p}$ is coarser than $R^{d_{j,\beta}^r}$ if and only if $l(R^{d_{i,\alpha}^p}.R^{d_{j,\beta}^r}) = p$ where J_s^k is as in Notation 3.6(c) in [6].*

Proof. *Part* (a): $d_{i,\alpha}^{p_1,p_2}$ is coarser than $d_{j,\beta}^{r_1,r_2}$ if and only if every $\{e\}$-through class of $d_{j,\beta}^{r_1,r_2}$ is contained in a $\{e\}$-through class of $d_{i,\alpha}^{p_1,p_2}$, every $\mathbb{Z}_2$-through class of $d_{j,\beta}^{r_1,r_2}$ is contained in a $\mathbb{Z}_2$-through class of $d_{i,\alpha}^{p_1,p_2}$, every $\{e\}$-horizontal edge of $d_{j,\beta}^{r_1,r_2}$ is contained in either a $\{e\}$ or $\mathbb{Z}_2$-horizontal edge or $\{e\}$-through class of $d_{i,\alpha}^{p_1,p_2}$ and every $\mathbb{Z}_2$-horizontal edge of $d_{j,\beta}^{r_1,r_2}$ is contained in a $\mathbb{Z}_2$-horizontal edge or $\mathbb{Z}_2$-through class of $d_{i,\alpha}^{p_1,p_2}$.

Thus, the number of loops in the product $d_{i,\alpha}^{p_1,p_2}.d_{j,\beta}^{r_1,r_2}$ is $2p_1 + p_2$.

The proof of (b) and (c) are similar to the proof of (a). □

Lemma 2.2 ([18]). *Given any two diagrams* $d_{i,\alpha}^{r_1,r_2}$ *and* $d_{j,\beta}^{r'_1,r'_2}$ *such that* $\sharp^p(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r'_1,r'_2}) = 2s_1 + s_2$ *then there exists a unique diagram which is the smallest diagram* $d_{l,\gamma}^{r''_1,r''_2}$ *among the diagrams coarser than both* $d_{i,\alpha}^{r_1,r_2}$ *and* $d_{j,\beta}^{r'_1,r'_2}$.

Also, $l(d_{l,\gamma}^{r''_1,r''_2}.d_{l,\gamma}^{r''_1,r''_2}) = l(d_{l,\gamma}^{r''_1,r''_2}.d_{i,\alpha}^{r_1,r_2}) = l(d_{l,\gamma}^{r''_1,r''_2}.d_{j,\beta}^{r'_1,r'_2})$.

Proof. The proof follows from Definition 2.13 in [6]. and [18]. □

2.1. Column operations on the Gram matrices of the algebra of $\mathbb{Z}_2$-relations, signed partition algebras and partition algebras

We now perform the column operations inductively on the Gram matrices of the algebra of $\mathbb{Z}_2$-relations, signed partition algebras and partition algebras as follows:

Let $d_{i,\alpha}^{0,0}$ be coarser than $d_{j,\beta}^{0,1}$. Then by Lemma 2.2,

$$l(d_{i,\alpha}^{0,0}.d_{i,\alpha}^{0,0}) = l(d_{i,\alpha}^{0,0}.d_{j,\beta}^{0,1}) = 0.$$

We apply the column operation: $L_{(j,\beta,0,1)} \to L_{(j,\beta,0,1)} - L_{(i,\alpha,0,0)}$ then the $((i,\alpha,0,0),(j,\beta,0,1))$-entry becomes

$$a_{(i,\alpha,0,0),(j,\beta,0,1)} - a_{(i,\alpha,0,0),(i,\alpha,0,0)} = 1 - 1 = 0.$$

Similarly, apply the column operations $L_{(j,\beta,r'_1,r'_2)} \to L_{(j,\beta,r'_1,r'_2)} - L_{(i,\alpha,r_1,r_2)}$ whenever $d_{i,\alpha}^{r_1,r_2}$ is coarser than $d_{j,\beta}^{r'_1,r'_2}$.

Then $b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)}$ denotes the $((i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2))$-entry after all the column operations are carried out

$$\begin{aligned} b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} = a_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} & \\ - \sum_{\substack{d_{l,\gamma}^{r'''_1,r'''_2} > d_{i,\alpha}^{r_1,r_2} \\ d_{l,\gamma}^{r'''_1,r'''_2} > d_{j,\beta}^{r'_1,r'_2}}} & b_{(i,\alpha,r_1,r_2),(l,\gamma,r'''_1,r'''_2)} \\ - \sum_{\substack{d_{k',\delta'}^{r''''_1,r''''_2} > d_{j,\beta}^{r'_1,r'_2} \\ d_{k',\delta'}^{r''''_1,r''''_2} \not> d_{i,\alpha}^{r_1,r_2}}} & b_{(i,\alpha,r_1,r_2),(k',\delta',r''''_1,r''''_2)} \end{aligned} \tag{2.1}$$

Lemma 2.3. (a) *In the algebra of* $\mathbb{Z}_2$*-relations and signed partition algebras, let* $(i,\alpha,r_1,r_2) < (j,\beta,r'_1,r'_2)$.

(i) *If* $d_{i,\alpha}^{r_1,r_2}$ *is coarser than* $d_{j,\beta}^{r'_1,r'_2}$ *then*

$$b_{(j,\beta,r'_1,r'_2),(i,\alpha,r_1,r_2)} = b_{(i,\alpha,r_1,r_2),(i,\alpha,r_1,r_2)}.$$

(ii) *If* $d_{i,\alpha}^{r_1,r_2}$ *is not coarser than* $d_{j,\beta}^{r'_1,r'_2}$ *and* $l(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r'_1,r'_2}) \geqslant 0$ *then*

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} = 0 \quad \textit{and} \quad b_{(j,\beta,r'_1,r'_2),(i,\alpha,r_1,r_2)} = 0.$$

(iii) *If* $d_{i,\alpha}^{r_1,r_2}$ *is coarser than* $d_{j,\beta}^{r'_1,r'_2}$ *then*

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} = 0$$

where $b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)}$ *is the* $((i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2))$*-th entry after all the column operations are carried out.*

(b) *In partition algebras, let* $(i,\alpha,r) < (j,\beta,r')$.

(i) *If* $R^{d_{i,\alpha}^r}$ *is coarser than* $R^{d_{j,\beta}^{r'}}$ *then*

$$b_{(j,\beta,r'),(i,\alpha,r)} = b_{(i,\alpha,r),(i,\alpha,r)}.$$

(ii) *If* $R^{d_{i,\alpha}^r}$ *is not coarser than* $R^{d_{j,\beta}^{r'}}$ *and* $l(R^{d_{i,\alpha}^r}.R^{d_{j,\beta}^{r'}}) \geqslant 0$ *then*

$$b_{(i,\alpha,r),(j,\beta,r')} = 0 \quad \textit{and} \quad b_{(j,\beta,r'),(i,\alpha,r)} = 0.$$

(iii) *If* $R^{d_{i,\alpha}^r}$ *is coarser than* $R^{d_{j,\beta}^{r'}}$ *then*

$$b_{(i,\alpha,r),(j,\beta,r')} = 0$$

where $b_{(i,\alpha,r),(j,\beta,r')}$ *is the* $((i,\alpha,r),(j,\beta,r'))$*-th entry after all the column operations are carried out.*

Proof. *Part* a(i): It follows from equation (2.1), for

$$\begin{aligned}
b_{(j,\beta,r'_1,r'_2),(i,\alpha,r_1,r_2)} &= a_{(j,\beta,r'_1,r'_2),(i,\alpha,r_1,r_2)} - \sum_{d_{l,\gamma}^{r''_1,r''_2} > d_{i,\alpha}^{r_1,r_2} > d_{j,\beta}^{r'_1,r'_2}} b_{(j,\beta,r'_1,r'_2),(l,\gamma,r''_1,r''_2)} \\
&= a_{(i,\alpha,r_1,r_2),(i,\alpha,r_1,r_2)} - \sum_{d_{l,\gamma}^{r''_1,r''_2} > d_{i,\alpha}^{r_1,r_2}} b_{(l,\gamma,r''_1,r''_2),(l,\gamma,r''_1,r''_2)} \\
&\quad \text{(by Lemma 2.1 and induction)} \\
&= b_{(i,\alpha,r_1,r_2),(i,\alpha,r_1,r_2)}
\end{aligned}$$

We prove the result by induction on (i,α,r_1,r_2).

Let $d_{i,\alpha}^{0,0}$ be coarser than $d_{j,\beta}^{r'_1,r'_2}$, by lemma 2.1 we have,

$$l(d_{i,\alpha}^{0,0}.d_{i,\alpha}^{0,0}) = l(d_{i,\alpha}^{0,0}.d_{j,\beta}^{r'_1,r'_2}) = 0 \tag{2.2}$$

for any diagram $d_{l,\gamma}^{r''_1,r''_2}$ which is coarser than $d_{j,\beta}^{r'_1,r'_2}$ and $d_{l,\gamma}^{r''_1,r''_2}$ but not coarser than $d_{i,\alpha}^{0,1}$, we have

$$b_{(i,\alpha,0,1),(l,\gamma,r'_1,r'_2)} = 0.$$

Thus, by applying the column operations $L_{(j,\beta,r'_1,r'_2)} \to L_{(j,\beta,r'_1,r'_2)} - L_{((i,\alpha,0,1)}$ and equation (2.1) $((i,\alpha,0,0),(j,\beta,r'_1,r'_2))$-entry becomes

$$b_{(i,\alpha,0,0),(j,\beta,r'_1,r'_2)} = a_{(i,\alpha,0,0),(j,\beta,r'_1,r'_2)} - a_{(i,\alpha,0,0),(i,\alpha,0,0)} = 1 - 1 = 0$$

by equation (2.2).

(ii) Suppose $d_{i,\alpha}^{0,1}$ and $d_{j,\beta}^{0,1}$ such that $\sharp^p(d_{i,\alpha}^{0,1}.d_{j,\beta}^{0,1}) = 2s_1 + s_2$ then by Lemma 2.1 $l(d_{i,\alpha}^{0,1}.d_{r_1,r_2}^{0,1}) = 0$ then there exists a unique diagram $d_{k,\delta}^{0,0}$ coarser than both $d_{i,\alpha}^{0,1}$ and $d_{j,\beta}^{0,1}$ such that

$$l(d_{k,\delta}^{0,0}.d_{k,\delta}^{0,0}) = l(d_{k,\delta}^{0,0}.d_{i,\alpha}^{0,1}) = l(d_{k,\delta}^{0,0}.d_{j,\beta}^{0,1}) = 0.$$

Thus, when the column operation $L_{(j,\beta,0,1)} \to L_{(j,\beta,0,1)} - L_{(k,\delta,0,0)}$ is carried out,

$$b_{(i,\alpha,0,1),(j,\beta,0,1)} = a_{(i,\alpha,0,1),(j,\beta,0,1)} - a_{(i,\alpha,0,1),(k,\delta,0,0)} = 1 - 1 = 0. \tag{2.3}$$

Part a(ii): In general, Let $d_{i,\alpha}^{r_1,r_2}$ be not coarser than $d_{j,\beta}^{r'_1,r'_2}$ such that $l(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r'_1,r'_2}) \geqslant 0$. Then by Lemma 2.2 there is a unique diagram $d_{k,\delta}^{r''_1,r''_2}$ coarser than both $d_{i,\alpha}^{r_1,r_2}$ and $d_{j,\beta}^{r'_1,r'_2}$ such that

$$l(d_{k,\delta}^{r''_1,r''_2}.d_{k,\delta}^{r''_1,r''_2}) = l(d_{k,\delta}^{r''_1,r''_2}.d_{i,\alpha}^{r_1,r_2}) = l(d_{k,\delta}^{r''_1,r''_2}.d_{j,\beta}^{r'_1,r'_2})$$

When the column operations are carried out inductively,

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} = a_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} - \sum_{\substack{d_{l,\gamma}^{r'''_1,r'''_2} > d_{i,\alpha}^{r_1,r_2} \\ d_{l,\gamma}^{r'''_1,r'''_2} > d_{j,\beta}^{r'_1,r'_2}}} b_{(i,\alpha,r_1,r_2),(l,\gamma,r'''_1,r'''_2)}$$
$$- \sum_{\substack{d_{k',\delta'}^{r''''_1,r''''_2} > d_{j,\beta}^{r'_1,r'_2} \\ d_{k',\delta'}^{r''''_1,r''''_2} \not> d_{i,\alpha}^{r_1,r_2}}} b_{(i,\alpha,r_1,r_2),(k',\delta',r''''_1,r''''_2)}$$

By induction hypothesis, each entry in the second summation becomes zero. Thus, we have

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} = a_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} - \sum_{\substack{d_{l,\gamma}^{r_1'',r_2''} > d_{i,\alpha}^{r_1,r_2} \\ d_{l,\gamma}^{r_1'',r_2''} > d_{j,\beta}^{r_1',r_2'}}} b_{(i,\alpha,r_1,r_2),(l,\gamma,r_1'',r_2'')}.$$

Also, by induction,

$$b_{(i,\alpha,r_1,r_2),(i',\alpha',r_1'''',r_2'''')} = b_{(i',\alpha',r_1'''',r_2''''),(i',\alpha',r_1'''',r_2'''')}. \tag{2.4}$$

Thus,

$$\begin{aligned} b_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} &= (a_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} \\ &\quad - \sum_{\substack{d_{l,\gamma}^{r_1''',r_2'''} > d_{i,\alpha}^{r_1,r_2} \\ d_{l,\gamma}^{r_1''',r_2'''} > d_{j,\beta}^{r_1',r_2'} \\ d_{l,\gamma}^{r_1''',r_2'''} \neq d_{k,\delta}^{r_1'',r_2''}}} b_{(i,\alpha,r_1,r_2),(l,\gamma,r_1''',r_2''')}) - b_{(k,\delta,r_1'',r_2''),(k,\delta,r_1'',r_2'')} \\ &= b_{(k,\delta,r_1'',r_2''),(k,\delta,r_1'',r_2'')} - b_{(k,\delta,r_1'',r_2''),(k,\delta,r_1'',r_2'')} \\ &= b_{(k,\delta,r_1'',r_2''),(k,\delta,r_1'',r_2'')} - b_{(k,\delta,r_1'',r_2''),(k,\delta,r_1'',r_2'')} \\ &\quad \text{(by equation (2.4))} \\ &= 0 \end{aligned}$$

Thus, $((i,\alpha,,r_1,r_2),(j,\beta,r_1',r_2'))$-entry becomes zero after applying the column operations when $d_{i,\alpha}^{r_1,r_2}$ is not coarser than $d_{j,\beta}^{r_1',r_2'}$ such that $l(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r_1',r_2'}) \geqslant 0$. Also,

$$b_{(j,\beta,r_1',r_2'),(i,\alpha,r_1,r_2)} = a_{(j,\beta,r_1',r_2'),(i,\alpha,r_1,r_2)} - \sum_{\substack{d_{l,\gamma}^{r_1'',r_2''} > d_{i,\alpha}^{r_1,r_2} \\ d_{l,\gamma}^{r_1'',r_2''} > d_{j,\beta}^{r_1',r_2'}}} b_{(j,\beta,r_1',r_2'),(l,\gamma,r_1'',r_2'')}.$$

since $b_{(j,\beta,r_1',r_2'),(k,\delta,r_1''',r_2''')}$ becomes zero by induction for all $d_{k,\delta}^{r_1''',r_2'''}$ coarser than $d_{i,\alpha}^{r_1,r_2}$ and not coarser than $d_{j,\beta}^{r_1',r_2'}$ arguing as in the proof of (ii),

$$b_{(j,\beta,r_1',r_2'),(i,\alpha,r_1,r_2)} = 0.$$

Part a(iii): In general, let $d_{i,\alpha}^{r_1,r_2}$ be coarser than $d_{j,\beta}^{r_1',r_2'}$, by Lemma 2.1

$$l(d_{i,\alpha}^{r_1,r_2}.d_{i,\alpha}^{r_1,r_2}) = l(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r_1',r_2'}) = 2r_1 + r_2.$$

By induction hypothesis,

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} = a_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} - \sum_{\substack{d_{l,\gamma}^{r_1'',r_2''} > d_{r_1,r_2}^{i,\alpha} \\ d_{l,\gamma}^{r_1'',r_2''} > d_{j,\beta}^{r_1',r_2'}}} b_{(i,\alpha,r_1,r_2),(l,\gamma,r_1'',r_2'')} \tag{2.5}$$

and

$$b_{(i,\alpha,r_1,r_2),(i,\alpha,r_1,r_2)} = a_{(i,\alpha,r_1,r_2),(i,\alpha,r_1,r_2)} - \sum_{d_{l,\gamma}^{r_1'',r_2''} > d_{i,\alpha}^{r_1,r_2}} b_{(i,\alpha,r_1,r_2),(l,\gamma,r_1'',r_2'')} \tag{2.6}$$

Thus, when the column operation $L_{(j,\beta,r_1',r_2')} \to L_{(j,\beta,r_1',r_2')} - L_{(i,\alpha,r_1,r_2)}$ is carried out the $((i,\alpha,r_1,r_2),(j,\beta,r_1',r_2'))$-th entry of the block matrix $A_{2r_1+r_2,2r_1'+r_2'}$ becomes zero. That is, $b_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} = 0$.

The proof of (b) is similar to the proof of (a). □

Theorem 2.4. (a) *After applying the column operations the diagonal entry* $x^{2r_1+r_2}$ *in the block matrix* $A_{2r_1+r_2,2r_1+r_2}$ *for* $0 \leqslant r_1 + r_2 \leqslant k - s_1 - s_2$ *and the block matrix* $\overrightarrow{A}_{2r_1+r_2,2r_1+r_2}$ *for* $0 \leqslant r_1 + r_2 \leqslant k - s_1 - s_2 - 1$ *of the algebra of* $\mathbb{Z}_2$*-relations and signed partition algebras respectively are replaced by*

(i) $\prod_{j=0}^{r_1-1}[x^2 - x - 2(s_1+j)] \prod_{l=0}^{r_2-1}[x - (s_2+l)]$ *if* $r_1 \geqslant 1$ *and* $r_2 \geqslant 1$,

(ii) $\prod_{j=0}^{r_2-1}[x - (s_2+j)]$ *if* $r_1 = 0$ *and* $r_2 \neq 0$,

(iii) $\prod_{j=0}^{r_1-1}[x^2 - x - 2(s_1+j)]$ *if* $r_1 \neq 0$ *and* $r_2 = 0$.

Also, the diagonal elements in the block matrix $A_{2r_1+r_2,2r_1+r_2}$ *and* $\overrightarrow{A}_{2r_1+r_2,2r_1+r_2}$ *are the same.*

(b) *After applying the column operations the diagonal entry* x^r *in the block matrix* $A_{r,r}$ *for* $0 \leqslant r \leqslant k$ *is replaced by*

$$\prod_{j=0}^{r-1}[x - (s+j)] \quad \textit{if } r \geqslant 1 \textit{ and } 1 \textit{ if } r = 0.$$

Also, the diagonal elements in the block matrix $A_{r,r}$ *are the same.*

Proof. *Part* (a)(i): The proof is by induction on the number of horizontal edges.

Let $d_{i,\alpha}^{r_1,r_2}$ be any diagram corresponding to the diagonal entry $x^{2r_1+r_2}$ in block matrix $A_{2r_1+r_2,2r_1+r_2}$ having $2s_1+s_2$ number of through classes and r_1 pairs of $\{e\}$-horizontal edges and r_2 number of $\mathbb{Z}_2$-horizontal edges.

After applying column operations as mentioned earlier to eliminate the entries which lie above corresponding to the diagrams coarser than $d_{i,\alpha}^{r_1,r_2}$, then by Lemma 2.1 and induction the diagonal entry $x^{2r_1+r_2}$ is replaced as

$$x^{2r_1+r_2} - \sum_{\substack{0\leqslant j\leqslant r_1\\ -r_2\leqslant j'\leqslant r_1\\ -2j+j'<0}} B^{s_1,s_2}_{2r_1+r_2,2[r_1-j]+r_2+j'} \prod_{l=0}^{r_1-j-1}[x^2-x-2(s_1+l)] \times \prod_{f=0}^{r_2+j'-1}[x-(s_2+f)] \tag{2.7}$$

where $B^{s_1,s_2}_{2r_1+r_2,2p_1+p_2}$ gives the number of diagrams which has p_1 pairs of $\{e\}$ horizontal edges and p_2 number of $\mathbb{Z}_2$ horizontal edges which lie above and coarser than $d_{i,\alpha}^{r_1,r_2}$.

Fix s and put

$$H_{2r_1+r_2,s} = -\sum_{\substack{0\leqslant j\leqslant r_1\\ -r_2\leqslant j'\leqslant r_1\\ -2j+j'<0 \text{ and } m-2j+j'\geqslant 0}} (-1)^{2j-j'} B^{s_1,s_2}_{2r_1+r_2,2[r_1-j]+r_2+j'} C_{2[r_1-j]+r_2+j',s} \tag{2.8}$$

where $C_{2r_1'+r_2',s}$ denote the coefficient of x^s in

$$\prod_{j=0}^{r_1'-1}[x^2-x-2(s_1+j)]\prod_{l=0}^{r_2'-1}[x-(s_2-l)]$$

where $m=2r_1+r_2-s$.

We shall claim that,

$$H_{2r_1+r_2,s}=(-1)^m C_{2r_1+r_2-1,s}.$$

We shall prove this by using induction on $2r_1+r_2$.

$$H_{2r_1+r_2,s} = -\sum_{\substack{0\leqslant j\leqslant r_1\\ -r_2\leqslant j'\leqslant r_1\\ -2j+j'<0 \text{ and } m-2j+j'\geqslant 0}} (-1)^{2j-j'} B^{s_1,s_2}_{2r_1+r_2,2[r_1-j]+r_2+j'} C_{2[r_1-j]+r_2+j',s}$$

where $m=2r_1+r_2-s$.

By using Lemma 3.16 in [6] and induction hypothesis, equation (2.8) becomes,

$$\begin{aligned} H_{2r_1+r_2,s} = -\sum_{\substack{0\leqslant j\leqslant r_1 \\ -r_2\leqslant j'\leqslant r_1 \\ -2j+j'<0 \text{ and } m-2j+j'\geqslant 0}} &(-1)^{2j-j'}\big\{B^{s_1,s_2}_{2r_1+r_2-1,2[r_1-j]+r_2+j'-1} \\ &+ (s_2+r_2+j')B^{s_1,s_2}_{2r_1+r_2-1,2[r_1-j]+r_2+j'}\big\} \\ &\times \big\{C_{2[r_1-j]+r_2+j'-1,s-1} + (s_2+r_2+j'-1)C_{2[r_1-j]+r_2+j'-1,s}\big\} \end{aligned}$$

The equation (2.8) can be rewritten as follows:

$$\begin{aligned} &H_{2r_1+r_2,s} \\ &= -\sum_{\substack{0\leqslant j\leqslant r_1 \\ -r_2\leqslant j'\leqslant r_1 \\ -2j+j'<0 \text{ and } m-2j+j'\geqslant 0}} (-1)^{2j-j'}B^{s_1,s_2}_{2r_1+r_2-1,2[r_1-j]+r_2+j'-1}C_{2[r_1-j]+r_2+j'-1,s-1} \\ &\quad -\sum_{\substack{0\leqslant j\leqslant r_1 \\ -r_2\leqslant j'\leqslant r_1 \\ -2j+j'<0 \text{ and } m-2j+j'\geqslant 0}} (-1)^{2j-j'}(s_2+r_2+j'-1) \\ &\qquad\qquad \times B^{s_1,s_2}_{2r_1+r_2-1,2[r_1-j]+r_2+j'-1}C_{2[r_1-j]+r_2+j'-1,s} \\ &\quad -\sum_{\substack{0\leqslant j\leqslant r_1 \\ -r_2\leqslant j'\leqslant r_1 \\ -2j+j'<0 \text{ and } m-2j+j'\geqslant 0}} (-1)^{2j-j'}(s_2+r_2+j') \\ &\qquad\qquad \times B^{s_1,s_2}_{2r_1+r_2-1,2[r_1-j]+r_2+j'}C_{2[r_1-j]+r_2+j',s} \\ &= H_{2r_1+r_2-1,s-1} + (-1)^m(s_2+r_2-1)C_{2r_1+r_2-1,s} \\ &\qquad \text{(by canceling common terms)} \\ &= (-1)^mC_{2r_1+r_2-1,s-1} + (-1)^m(s_2+r_2-1)C_{2r_1+r_2-1,s} \text{ (by induction)} \end{aligned}$$

Thus, equation (2.8) reduces to

$$\begin{aligned} H_{2r_1+r_2,s} &= (-1)^mC_{2r_1+r_2-1,s-1} + (-1)^m(s_2+r_2-1)C_{2r_1+r_2-1,s} \\ &= (-1)^mC_{2r_1+r_2,s} \end{aligned}$$

where $C_{2r_1+r_2,s} = C_{2r_1+r_2-1,s-1} + (s_2+r_2-1)C_{2r_1+r_2-1,s}$.

The same proof works for the diagonal element in the block matrix $\overrightarrow{A}_{2r_1+r_2,2r_1+r_2}$ for $0\leqslant r_1+r_2\leqslant k-s_1-s_2-1$ in signed partition algebras.

Part (a)(iii): This part can be proved in similar fashion as that of (a)(i) by using Lemma 3.17 in [6] and

$$C_{2r_1,s} = (-1)^m C_{2(r_1-1),s-2} + (-1)^m C_{2(r_1-1),s-1} - (-1)^m 2(s_1 + r_1 - 1) C_{2(r_1-1),s}.$$

The proof of (b) is same as that of the proof of (a). □

Lemma 2.5. *Let* $d_{i,\alpha}^{r_1,r_2}, d_{j,\beta}^{r'_1,r'_2} \in J_{2s_1+s_2}^{2k}$ *and* $d_{i,\alpha}^{r_1,r_2}, d_{j,\beta}^{r'_1,r'_2} \in \overrightarrow{J}_{2s_1+s_2}^{2k}$. *The* $((i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2))$*-entry of the Gram matrices* $G_{2s_1+s_2}^{k}$ *of the algebra of* $\mathbb{Z}_2$*-relations and* $\overrightarrow{G}_{2s_1+s_2}^{k}$ *of the signed partition algebras remains zero even after applying column operations inductively if the* $\mathbb{Z}_2$*-horizontal edge of* $d_{i,\alpha}^{r_1,r_2}$ *coincides with the* $\{e\}$*-through class of* $d_{j,\beta}^{r'_1,r'_2}$ *and vice versa.*

Proof. The proof follows from Definition 3.7 in [6] and there is no diagram in common which is coarser than both $d_{i,\alpha}^{r_1,r_2}, d_{j,\beta}^{r'_1,r'_2} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2}$. □

Remark 2.6. (a) Let $d_{i,\alpha}^{r_1,r_2}, d_{j,\beta}^{r'_1,r'_2} \in J_{2s_1+s_2}^{2k}$ such that $\sharp^p(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r'_1,r'_2}) < 2s_1 + s_2$. Place $d_{i,\alpha}^{r_1,r_2}$ above $d_{j,\beta}^{r'_1,r'_2}$. Choose sub diagrams $d^{r_1-t'_1,r_2-t'_2} \in J_{2(s_1-t_1)+s_2-t_2}^{2f}$ of $d_{i,\alpha}^{r_1,r_2}$ and $d^{r'_1-t''_1,r'_2-t''_2} \in J_{2(s_1-t_1)+s_2-t_2}^{2f}$ of $d_{j,\beta}^{r'_1,r'_2}$ such that

$$l(d^{r_1-t'_1,r_2-t'_2}.d^{r'_1-t''_1,r'_2-t''_2}) \geqslant 0$$

with

$$\sharp^p((d_{i,\alpha}^{r_1,r_2} \setminus d^{r_1-t'_1,r_2-t'_2}).(d_{j,\beta}^{r'_1,r'_2} \setminus d^{r'_1-t''_1,r'_2-t''_2})) < 2t_1 + t_2.$$

For the sake of convenience, we shall write

$$d_{i,\alpha}^{r_1,r_2} = d^{r_1-t'_1,r_2-t'_2} \otimes d_{l_1-f}^{l_1-f} \quad \text{and} \quad d_{j,\beta}^{r'_1,r'_2} = d^{r_1-t''_1,r_2-t''_2} \otimes d_{l_2-f}^{l_2-f}$$

where $d_{l_1-f}^{l_1-f} = d_{i,\alpha}^{r_1,r_2} \setminus d^{r'_1-t''_1,r'_2-t''_2}$ and $d_{l_2-f}^{l_2-f} = d_{j,\beta}^{r'_1,r'_2} \setminus d^{r'_1-t''_1,r'_2-t''_2}$.

(b) Let $R^{d_{i,\alpha}^{r}}, R^{d_{j,\beta}^{r'}} \in J_s^k$ such that $\sharp^p(R^{d_{i,\alpha}^{r}}.R^{d_{j,\beta}^{r'}}) < 2$. Place $R^{d_{i,\alpha}^{r}}$ above $R^{d_{j,\beta}^{r'}}$. Choose sub diagrams $R^{d^{r-t'}} \in J_{s-t}^{f}$ of $R^{d_{i,\alpha}^{r}}$ and $R^{d^{r'-t''}} \in J_{s-t}^{f}$ of $R^{d_{j,\beta}^{r'}}$ such that

$$l(R^{d^{r-t'}}.R^{d^{r'-t''}}) \geqslant 0$$

with $\sharp^p((R^{d_{i,\alpha}^{r}} \setminus R^{d^{r-t'}}).(R^{d_{j,\beta}^{r'}} \setminus R^{d^{r'-t''}})) < t$.

For the sake of convenience, we shall write

$$R^{d^r_{i,\alpha}} = R^{d^{r-t'}} \otimes d^{l_1-f}_{l_1-f} \quad \text{and} \quad R^{d^{r'}_{j,\beta}} = R^{d^{r-t''}} \otimes d^{l_2-f}_{l_2-f}$$

where $d^{l_1-f}_{l_1-f} = R^{d^r_{i,\alpha}} \setminus R^{d^{r'-t''}}$ and $d^{l_2-f}_{l_2-f} = R^{d^{r'}_{j,\beta}} \setminus R^{d^{r'-t''}}$.

Notation 2.7. (a) Let $d^{r_1,r_2}_{i,\alpha}, d^{r_1,r_2}_{j,\beta}$ be as in Remark 2.6(a) such that $\sharp^p(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{j,\beta}) < 2s_1 + s_2$, so that the $((i,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$-entry of the block matrix $A_{2r_1+r_2,2r_1+r_2}$ in algebra of $\mathbb{Z}_2$-relations is zero and $0 \leqslant r_1 + r_2 \leqslant k - s_1 - s_2$.

If $t'_1 = t''_1 = t_1, t'_2 = t''_2 = t_2$, put

$$d^{r_1,r_2}_{i,\alpha} = d^{l_1^f}_{l_1^f} \otimes d^{l_1-f}_{l_1-f} \quad \text{and} \quad d^{r_1,r_2}_{j,\beta} = d^{l_2^f}_{l_2^f} \otimes d^{l_2-f}_{l_2-f},$$

where $d^{l_1^f}_{l_1^f}(d^{l_2^f}_{l_2^f})$ is the sub diagram of $d^{r_1,r_2}_{i,\alpha}(d^{r_1,r_2}_{j,\beta})$, $d^{l_1^f}_{l_1^f}, d^{l_2^f}_{l_2^f} \in \mathbb{J}^{2t_1+t_2}_{2t_1+t_2}$ and every $\{e\}$-through class ($\mathbb{Z}_2 -$ through class) of $d^{l_1^f}_{l_1^f}$ is replaced by a $\{e\}$-horizontal edge ($\mathbb{Z}_2 -$ horizontal edge) and vice versa.

(b) Let $d^{r_1,r_2}_{i,\alpha}, d^{r_1,r_2}_{j,\beta}$ be as in Remark 2.6(b) such that $\sharp^p(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{j,\beta}) < 2s_1 + s_2$, so that the $((i,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$-entry of the block matrix $\overrightarrow{A}_{2r_1+r_2,2r_1+r_2}$ in algebra of $\mathbb{Z}_2$-relations is zero and $0 \leqslant r_1 + r_2 \leqslant k - s_1 - s_2 - 1$.

If $t'_1 = t''_1 = t_1, t'_2 = t''_2 = t_2$, put

$$d^{r_1,r_2}_{i,\alpha} = d^{l_1^f}_{l_1^f} \otimes d^{l_1-f}_{l_1-f} \quad \text{and} \quad d^{r_1,r_2}_{j,\beta} = d^{l_2^f}_{l_2^f} \otimes d^{l_2-f}_{l_2-f},$$

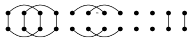
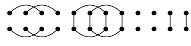
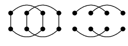
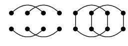
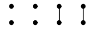
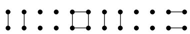
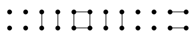
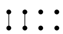
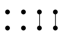
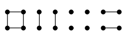
where $d^{l_1^f}_{l_1^f}(d^{l_2^f}_{l_2^f})$ is the sub diagram of $d^{r_1,r_2}_{i,\alpha}(d^{r_1,r_2}_{j,\beta}), d^{l_1^f}_{l_1^f}, d^{l_2^f}_{l_2^f} \in \overrightarrow{\mathbb{J}}^{2t_1+t_2}_{2t_1+t_2}$ and every $\{e\}$-through class ($\mathbb{Z}_2 -$ through class) of $d^{l_1^f}_{l_1^f}$ is replaced by a $\{e\}$-horizontal edge ($\mathbb{Z}_2 -$ horizontal edge) and vice versa.

(c) Let $R^{d^r_{i,\alpha}}, R^{d^r_{j,\beta}} \in \mathbb{J}^r_s$ such that $\sharp^p(R^{d^r_{i,\alpha}}.R^{d^r_{j,\beta}}) < s$, so that the $((i,\alpha,r),(j,\beta,r))$-entry of the block matrix $A_{r,r}$ in the partition algebra is zero and $0 \leqslant r \leqslant k - s$. Put

$$R^{d^r_{i,\alpha}} = d^{l_1}_{l_1} \otimes d^{l_1-f}_{l_1-f} \quad \text{and} \quad R^{d^r_{j,\beta}} = d^{l_2}_{l_2} \otimes d^{l_2-f}_{l_2-f},$$

where $d^{l_1}_{l_1}(d^{l_2}_{l_2})$ is the sub diagram of $R^{d^r_{i,\alpha}}(R^{d^r_{j,\beta}}), d^{l_1}_{l_1}, d^{l_2}_{l_2} \in \mathbb{J}^t_t$ and every through class of $d^{l_1}_{l_1}$ is replaced by a horizontal edge and vice versa.

Example 2.8. This example illustrates Notation 2.7.

s.no	$d^{2,0}_{i,(2,\Phi,2,\Phi)}$	$d^{2,0}_{j,(2,\Phi,2,\Phi)}$	$d^{l_1^f}_{l_1^f}$	$d^{l_2^f}_{l_2^f}$	$d^{l_1-f}_{l_1-f} = d^{l_2-f}_{l_2-f}$
1.					
2.					

Lemma 2.9. *Let* $(i, \alpha, r_1, r_2) < (j, \beta, r'_1, r'_2)$.

(a) *Let* $d^{r_1,r_2}_{i,\alpha}, d^{r'_1,r'_2}_{j,\beta} \in J^{2k}_{2s_1+s_2}$ *such that* $\sharp^p(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}) < 2s_1 + s_2$ *with* $d^{r_1,r_2}_{i,\alpha} = d^{r_1-t'_1,r_2-t'_2} \otimes d^{l_1-f}_{l_1-f}$ *and* $d^{r'_1,r'_2}_{j,\beta} = d^{r'_1-t''_1,r'_2-t''_2} \otimes d^{l_2-f}_{l_2-f}$ *where* $d^{r_1-t'_1,r_2-t'_2}, d^{r'_1-t''_1,r'_2-t''_2}$ *are as in Remark 2.6(a).*

(b) *Let* $d^{r_1,r_2}_{i,\alpha}, d^{r'_1,r'_2}_{j,\beta} \in \overrightarrow{J}^{2k}_{2s_1+s_2}$ *such that* $\sharp^p(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}) < 2s_1 + s_2$ *with* $d^{r_1,r_2}_{i,\alpha} = d^{r_1-t'_1,r_2-t'_2} \otimes d^{l_1-f}_{l_1-f}$ *and* $d^{r'_1,r'_2}_{j,\beta} = d^{r'_1-t''_1,r'_2-t''_2} \otimes d^{l_2-f}_{l_2-f}$ *where* $d^{r_1-t'_1,r_2-t'_2}, d^{r'_1-t''_1,r'_2-t''_2}$ *are as in Remark 2.6(a). Then*

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} = 0,$$

if any one of the following conditions hold:
(i) $2r_1 + r_2 < 2r'_1 + r'_2$ *or*
(ii) *if* $2r_1 + r_2 = 2r'_1 + r'_2$ *then* $r_1 + r_2 < r'_1 + r'_2$ *or*
(iii) $t''_1 \neq t_1$ *or* $t''_2 \neq t_2$ *or*
(iv) $2r_1 + r_2 - (2t'_1 + t'_2) < 2r'_1 + r_2 - (2t''_1 + t''_2)$

(c) *Let* $R^{d^r_{i,\alpha}}, R^{d^{r'}_{j,\beta}} \in \mathbb{J}^{r'}_s$ *such that* $\sharp^p(R^{d^r_{i,\alpha}}, R^{d^{r'}_{j,\beta}}) < s$ *with* $R^{d^r_{i,\alpha}} = d^{r-t'} \otimes R^{d^r_{i,\alpha}} \setminus d^{r-t'}$ *and* $R^{d^{r'}_{j,\beta}} = d^{r'-t''} \otimes R^{d^{r'}_{j,\beta}} \setminus d^{r'-t''}$ *where* $d^{r-t'} \in \mathbb{J}^{t'}_t, d^{r'-t''} \in \mathbb{J}^{t''}_t, R^{d^r_{i,\alpha}} \setminus d^{r-t'} \in \mathbb{J}^{r-t'}_{s-t}$ *and* $R^{d^{r'}_{j,\beta}} \setminus d^{r'-t''} \in \mathbb{J}^{r'-t''}_{s-t}$. *Then*

$$b_{(i,\alpha,r),(j,\beta,r')} = 0,$$

if any one of the following conditions hold:
(i) $r' < r$
(ii) $t'' \neq t$
(iii) $r - t' < r' - t''$

Proof. *Part* (a): The proof is by induction on the conditions
(i) $2r_1 + r_2 < 2r'_1 + r'_2$ or
(ii) if $2r_1 + r_2 = 2r'_1 + r'_2$ then $r_1 + r_2 < r'_1 + r'_2$ or
(iii) $t''_1 \neq t_1$ or $t''_2 \neq t_2$ or

(iv) $2r_1 + r_2 - (2t'_1 + t'_2) < 2r'_1 + r_2 - (2t''_1 + t''_2)$

Since $\sharp^p(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}) < 2s_1 + s_2$ which implies that $a_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} = 0$.

After applying column operations inductively we get,

$$\begin{aligned} b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} = &- \sum_{\substack{d^{r''_1,r''_2}_{l,\gamma} > d^{r_1,r_2}_{i,\alpha} \\ d^{r''_1,r''_2}_{l,\gamma} > d^{r'_1,r'_2}_{j,\beta}}} b_{(l,\gamma,r''_1,r''_2),(l,\gamma,r''_1,r''_2)} \\ &- \sum_{\substack{d^{r''_1,r''_2}_{l,\gamma} > d^{r'_1,r'_2}_{j,\beta} \\ d^{r''_1,r''_2}_{l,\gamma} \not> d^{r_1,r_2}_{i,\alpha}}} b_{(i,\alpha,r_1,r_2),(l,\gamma,r''_1,r''_2)} \end{aligned} \tag{2.9}$$

Suppose that $\sharp^p(d^{r''_1,r''_2}_{l,\gamma}.d^{r_1,r_2}_{i,\alpha}) = 2s_1 + s_2$ then by Lemma 2.2 and induction hypothesis,

$$b_{(i,\alpha,r_1,r_2),(l,\gamma,r''_1,r''_2)} = 0.$$

Suppose that $\sharp^p(d^{r''_1,r''_2}_{l,\gamma}.d^{r_1,r_2}_{i,\alpha}) < 2s_1 + s_2$ then by using induction on any one of the conditions (i), (ii), (iii) and (iv)

$$b_{(i,\alpha,r_1,r_2),(l,\gamma,r''_1,r''_2)} = 0,$$

By Lemma 2.2, there exists a unique diagram $d^{l_3-f}_{l_3-f}$ coarser than both $d^{l_2-f}_{l_2-f}$ and $d^{l_1-f}_{l_1-f}$ and $d^{l_3^f}_{l_3^f} \in \mathbb{J}^{2t_1+t_2}_{2t_1+t_2}$ which is coarser than $d^{r'_1-t''_1,r'_2-t''_2}$. Denote $d^{l_3^f}_{l_3^f} \otimes d^{l_3-f}_{l_3-f}$ by $d^{r''''_1,r''''_2}_{k,\delta}$.

It is clear that, $d^{r''''_1,r''''_2}_{k,\delta}$ is coarser than $d^{r'_1,r'_2}_{j,\beta}$. Thus, after applying the column operations $L_{(j,\beta,r'_1,r'_2)} \to L_{(j,\beta,r'_1,r'_2)} - L_{(k,\delta,r''''_1,r''''_2)}$ we get,

$$\begin{aligned} b_{(i,\alpha,r_1,r_2),(j,\beta,r'_1,r'_2)} = &- \sum_{\substack{d^{r''_1,r''_2}_{l,\gamma} > d^{r_1,r_2}_{i,\alpha} \\ d^{r''_1,r''_2}_{l,\gamma} > d^{r''''_1,r''''_2}_{k,\delta}}} b_{(l,\gamma,r'''_1,r'''_2),(l,\gamma,r'''_1,r'''_2)} \\ &- \sum_{\substack{d^{r''_1,r''_2}_{l,\gamma} > d^{r''''_1,r''''_2}_{k,\delta} \\ d^{r''_1,r''_2}_{l,\gamma} \not> d^{r_1,r_2}_{i,\alpha}}} b_{(i,\alpha,r_1,r_2),(l,\gamma,r''_1,r''_2)} - b_{(i,\alpha,r_1,r_2),(k,\delta,r''''_1,r''''_2)} \\ &= b_{(i,\alpha,r_1,r_2),(k,\delta,r''''_1,r''''_2)} - b_{(i,\alpha,r_1,r_2),(k,\delta,r''''_1,r''''_2)} = 0. \end{aligned}$$

The proof of (b) and (c) are same as that of the proof of (a). □

Notation 2.10. Put,

(i) $\phi^{s_1,s_2}_{2r_1+r_2}(x) = \prod_{j=0}^{r_1-1}[x^2 - x - 2(s_1+j)] \prod_{l=0}^{r_2-1}[x-(s_2+l)], r_1 \geqslant 1, r_2 \geqslant 1.$

(ii) $\phi^{s_1,s_2}_{2r_1+0}(x) = \prod_{j=0}^{r_1-1}[x^2 - x - 2(s_1+j)], r_2 = 0.$

(iii) $\phi^{s_1,s_2}_{2.0+r_2}(x) = \prod_{l=0}^{r_2-1}[x-(s_2+l)], r_1 = 0.$

(iv) $\phi^{s_1,s_2}_{0+0}(x) = 1$ and $\phi^{s_1,s_2}_{2r_1+r_2}(x) = 0$ if any one of $r_1, r_2 < 0$.

(v) $\phi^{s}_{r}(x) = \prod_{l=0}^{r-1}[x-(s+l)], r \geqslant 1$

(vi) $\phi^{s}_{0}(x) = 1$ and $\phi^{s}_{r} = 0$ if $r < 0$.

Now, we derive the following relation between the polynomials which are needed in the following Lemmas.

Lemma 2.11. *We have*

(i) $$\phi^{s_1+t,s_2}_{2(r_1-t)+r_2}(x) = \phi^{s_1-t,s_2}_{2(r_1-t)+r_2}(x) - \sum_{m=1}^{2t} {}_{2t}C_{m\,r_1-t}C_m 2^m m! \phi^{s_1+t,s_2}_{2(r_1-t-m)+r_2}(x).$$

(ii) $$\phi^{s_1,s_2+t}_{2r_1+r_2-t}(x) = \phi^{s_1,s_2-t}_{2r_1+r_2-t}(x) - \sum_{m=1}^{2t} {}_{2t}C_{m\,r_2-t}C_m m! \phi^{s_1,s_2+t}_{2r_1+r_2-t-m}(x).$$

(iii) *In general,*

$$\begin{aligned}
\phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1)+r_2-t_2}(x) &= \phi^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2}(x) \\
&- \sum_{k=1}^{2t_1} 2t_1C_k(r_1-t_1)C_k 2^k k! \phi^{s_1+t_1,s_2-t_2}_{2(r_1-t_1-k)+r_2-t_2}(x) \\
&- \sum_{k'=1}^{2t_2} 2t_2C_{k'}(r_2-t_2)C_{k'} k'! \phi^{s_1-t_1,s_2+t_2}_{2(r_1-t_1)+r_2-t_2-k'}(x) \\
&- \sum_{k=1}^{2t_1}\sum_{k'=1}^{2t_2} 2t_1C_k(r_1-t_1)C_k 2^k k! 2t_2C_{k'}(r_2-t_2)C_{k'} k'! \\
&\times \phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-k)+r_2-t_2-k'}(x)
\end{aligned}$$

where

$$\phi^{s_1+t,s_2}_{2(r_1-t)+r_2}(x) = \prod_{l=0}^{r_1-t-1}[x^2 - x - 2(s_1+t+l)] \prod_{l'=0}^{r_2-1}[x-(s_2+l')]$$

and

$$\phi^{s_1,s_2+t}_{2r_1+r_2-t}(x) = \prod_{l=0}^{r_1-1}[x^2-x-2(s_1+l)]\prod_{l'=0}^{r_2-t-1}[x-(s_2+t+l')].$$

Proof. Part (i): We shall prove this by using induction on $r_1 - t$ and r_2. Consider

$$\phi^{s_1-t,s_2}_{2(r_1-t)+r_2}(x) - \sum_{m=1}^{2t} {}_{2t}C_{m\,(r_1-t)}C_m 2^m m! \phi^{s_1+t,s_2}_{2[r_1-t-m]+r_2}(x) \tag{2.10}$$

$$= \phi^{s_1-t,s_2}_{2(r_1-t-1)+r_2}(x)(x^2-x-2(s_1+r_1-2t-1))$$

$$- \sum_{m=1}^{2t} {}_{2t}C_{m\,(r_1-t)}C_m 2^m m! \phi^{s_1+t,s_2}_{2(r_1-t-m)+r_2}(x)$$

$$= (\phi^{s_1-t,s_2}_{2(r_1-t-1)+r_2}(x) + \sum_{m=1}^{2t} {}_{2t}C_{m\,(r_1-t-1)}C_m 2^m m! \phi^{s_1+t,s_2}_{2(r_1-t-m-1)+r_2}(x))$$

$$(x^2-x-2(s_1+r_1-2t-1)) - \sum_{m=1}^{2t} {}_{2t}C_{m\,(r_1-t)}C_m 2^m m! \phi^{s_1+t,s_2}_{2(r_1-t-m)+r_2}(x)$$

(by induction)

$$= (\phi^{s_1-t,s_2}_{2(r_1-t-1)+r_2}(x) + \sum_{m=1}^{2t} {}_{2t}C_{m\,(r_1-t-1)}C_m 2^m m! \phi^{s_1+t,s_2}_{2(r_1-t-m-1)+r_2}(x))$$

$$\times (x^2-x-2(s_1+r_1-2t-1))$$

$$- \sum_{m=1}^{2t} {}_{2t}C_m({}_{(r_1-t-1)}C_m + {}_{r_1-t-1}C_{m-1}) 2^m m! \phi^{s_1+t,s_2}_{2(r_1-t-m-1)+r_2}(x)$$

$$\times (x^2-x-2(s_1+r_1-m-1))$$

$$= \phi^{s_1+t,s_2}_{2(r_1-t-1)+r_2}(x)(x^2-x-2(s_1+r_1-2t-1))$$

$$- \sum_{m=1}^{2t} {}_{2t}C_{m\,(r_1-t-1)}C_{m-1} 2^m m! \phi^{s_1+t,s_2}_{2(r_1-t-m)+r_2}(x)$$

$$+ \sum_{m=1}^{2t} {}_{2t}C_{m\,(r_1-t-1)}C_m 2^m m!(4t-2m) \phi^{s_1+t,s_2}_{2(r_1-t-m-1)+r_2}(x)$$

$$= \phi^{s_1+t,s_2}_{2(r_1-t-1)+r_2}(x)(x^2-x-2(s_1+r_1-2t-1)) - 4t\phi^{s_1+t,s_2}_{2(r_1-t-1)+r_2}(x)$$

(by canceling the common terms)

$$= \phi^{s_1+t,s_2}_{2(r_1-t-1)+r_2}(x)(x^2-x-2(s_1+r_1-1)) = \phi^{s_1+t,s_2}_{2(r_1-t)+r_2}(x)$$

Proof of (ii) is similar to the proof of (i) and proof of (iii) follows from (i) and (ii). □

Lemma 2.12. (a) *After performing the column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both* $d^{r_1,r_2}_{i,\alpha}$ *and* $d^{r_1,r_2}_{j,\alpha}$*, the zero in the* $((i,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$ *entry of the block matrix* $A_{2r_1+r_2,2r_1+r_2}$ *for* $0 \leqslant r_1+r_2 \leqslant k-s_1-s_2$ *in algebra of* $\mathbb{Z}_2$*-relations is replaced by*

$$-2^{t_1}t_1!t_2!x^{2(r_1-t_1)+r_2-t_2}$$

where $d^{r_1,r_2}_{i,\alpha}$ *and* $d^{r_1,r_2}_{j,\beta}$ *are as in Notation 2.7(a).*

(b) *After performing the column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both* $d^{r_1,r_2}_{i,\alpha}$ *and* $d^{r_1,r_2}_{j,\beta}$*, the zero in the* $((i,,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$ *entry of the block matrix* $\overrightarrow{A}_{2r_1+r_2,2r_1+r_2}$ *for* $0 \leqslant r_1,r_2,r_1+r_2 \leqslant k-s_1-s_2-1$ *in the signed partition algebra is replaced by*

$$-2^{t_1}t_1!t_2!x^{2(r_1-t_1)+r_2-t_2}.$$

where $d^{r_1,r_2}_{i,\alpha}$ *and* $d^{r_1,r_2}_{j,\beta}$ *are as in Notation 2.7(b).*

(c) *After performing the column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both* $R^{d^r_{i,\alpha}}$ *and* $R^{d^r_{j,\beta}}$*, the zero in the* $((i,\alpha,r),(j,\beta,r))$ *entry of the block matrix* $A_{r,r}$ *for* $0 \leqslant r \leqslant k-s$ *in partition algebra is replaced by*

$$-t!x^{r-t}.$$

where $R^{d^r_{i,\alpha}}$ *and* $R^{d^r_{j,\beta}}$ *are as in Notation 2.7(c).*

Proof. Part (a): We shall prove this by induction on t_1 and t_2.

Case (i): Let $t_1 = 1$ and $t_2 = 1$. We know that the diagrams coarser than both $d^{r_1,r_2}_{i,\alpha}$ and $d^{r_1,r_2}_{j,\beta}$ are obtained if and only if the pair of $\{e\}$-through classes and the pair of $\{e\}$-horizontal edges of $d^{l_1^f}_{l_1^f}$ or $d^{l_2^f}_{l_2^f}$ is connected by an $\{e\}$-horizontal edge which can be done in two ways and $\mathbb{Z}_2$ horizontal edge and $\mathbb{Z}_2$-through class of $d^{l_1^f}_{l_1^f}$ or $d^{l_2^f}_{l_2^f}$ is connected by a $\mathbb{Z}_2$-edge which can be done in one way. Also $d^{l_1-f}_{l_1-f}$ and $d^{l_2-f}_{l_2-f}$ have $2(r_1-1)+r_2-1$ horizontal edges then after performing the column operations the zero in the $((i,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$-entry of the block matrix $A_{2r_1+r_2,2r_1+r_2}$ is replaced by

$$-2\sum_{l=0}^{r_1-1}\sum_{l'=0}^{r_2-1+l} B^{s_1,s_2}_{2(r_1-1)+r_2-1,2(r_1-1-l)+r_2-1+l'}\phi^{s_1,s_2}_{2[r_1-1-l]+r_2-1+l'}(x)$$

which is equal to

$$- 2\phi^{s_1,s_2}_{2[r_1-1]+r_2-1}(x) - 2\sum_{l=1}^{r_1-1}\sum_{l'=1}^{r_2-1+l} B^{s_1,s_2}_{2(r_1-1)+r_2,2(r_1-1-l)+r_2-1+l'}\phi^{s_1,s_2}_{2[r_1-1-l]+r_2-1+l'}(x)$$

By Theorem 2.4 we know that,

$$\phi^{s_1,s_2}_{2[r_1-1]+r_2-1}(x) = x^{2(r_1-1)+r_2-1} - \sum_{l=1}^{r_1-1}\sum_{l'=1}^{r_2-1+l} B^{s_1,s_2}_{2(r_1-1)+r_2-1,2(r_1-1-l)+r_2-1+l'}\phi^{s_1,s_2}_{2[r_1-1-l]+r_2-1+l'}(x). \tag{2.11}$$

Substituting equation (2.11) in the above expression and canceling the common terms we get,

$$-2x^{2(r_1-1)+r_2-1}.$$

In general, the diagrams coarser than both $d^{r_1,r_2}_{i,\alpha}$ and $d^{r_1,r_2}_{j,\beta}$ are obtained if and only if t_1 pairs of $\{e\}$-through classes (t_2 number of $(\mathbb{Z}_2)$-through classes) and t_1 pairs of $\{e\}$-horizontal edges (t_2 number of $(\mathbb{Z}_2)$-horizontal edges) of $d^{l_1^f}_{l_1^f}$ or $d^{l_2^f}_{l_2^f}$ is connected by an $\{e\}$-horizontal edges($(\mathbb{Z}_2)$-horizontal edges) which can be done in $2^{t_1}t_1!t_2!$ ways. Also $d^{l_1-f}_{l_1-f}$ and $d^{l_2-f}_{l_2-f}$ have $2(r_1-t_1)+r_2-t_2$ horizontal edges then after performing the column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both $d^{r_1,r_2}_{i,\alpha}$ and $d^{r_1,r_2}_{j,\alpha}$ the zero in the $((i,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$-entry of the block matrix $A_{2r_1+r_2,2r_1+r_2}$ is replaced by

$$- 2^{t_1}t_1!t_2!\sum_{l=0}^{r_1-t_1}\sum_{l'=0}^{r_2-t_2+l} B^{s_1,s_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2+l'} \times \phi^{s_1,s_2}_{2[r_1-t_1-l]+r_2-t_2+l'}(x)$$

which is equal to

$$- 2^{t_1}t_1!t_2!(\phi^{s_1,s_2}_{2[r_1-t_1]+r_2-t_2}(x) - \sum_{l=1}^{r_1-t_1}\sum_{l'=1}^{r_2-t_2+l} B^{s_1,s_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2+l'}\phi^{s_1,s_2}_{2[r_1-t_1-l]+r_2-t_2+l'}(x)).$$

Substituting equation (2.11) in the above expression and canceling the common terms we get,

$$-2^{t_1}t_1!t_2!x^{2(r_1-t_1)+r_2-t_2}.$$

The proof of (b) and (c) are similar to the proof of (a). $\square$

Proposition 2.13. (a) *For* $0 \leqslant r_1 + r_2 \leqslant k - s_1 - s_2$*, after performing the column operations to eliminate the non-zero entries which lie above corresponding to the diagrams coarser than* $d_{j,\beta}^{r_1,r_2}$*, then the* $((i,\alpha,r_1,r_2), (j,\beta,r_1,r_2))$*-entry of the block matrix* $A_{2r_1+r_2,2r_1+r_2}$ *and*

(b) *For* $0 \leqslant r_1 + r_2 \leqslant k - s_1 - s_2 - 1$*, after performing the column operations to eliminate the non-zero entries which lie above corresponding to the diagrams coarser than* $d_{j,\beta}^{r_1,r_2}$*, then the* $((i,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$*-entry of the block matrix* $\overrightarrow{A}_{2r_1+r_2,2r_1+r_2}$ *for* $0 \leqslant r_1 + r_2, r_1, r_2 \leqslant k - s_1 - s_2 - 1$ *are replaced as*

(i) $(-1)^{t_1+t_2}(t_1)!(t_2)!2^{t_1}\prod_{j=t_1}^{r_1-1}[x^2 - x - 2(s_1+j)]\prod_{l=t_2}^{r_2-1}[x-(s_2+l)]$ *if* $r_1 \geqslant 1$ *and* $r_2 \geqslant 1$*,*

(ii) $(-1)^{t_2}(t_2)!\prod_{l=t_2}^{r_2-1}[x-(s_2+l)]$ *if* $r_1 = 0$ *and* $r_2 \neq 0$*,*

(iii) $(-1)^{t_1}(t_1)!2^{t_1}\prod_{j=t_1}^{r_1-1}[x^2 - x - 2(s_1+j)]$ *if* $r_1 \neq 0$ *and* $r_2 = 0$*,*

where $d_{j,\beta}^{r_1,r_2}$ *is as in Notation 2.7.*

(c) *After performing the column operations to eliminate the non-zero entries which lie above corresponding to the diagrams coarser than* $R^{d_{j,\beta}^{r'}}$*, then the* $((i,\alpha,r),(j,\beta,r))$*-entry is replaced by*

$$(-1)^t t!\prod_{j=t}^{r-1}[x-(s+l)].$$

where $R^{d_{j,\beta}^{r}}$ *is as in Notation 2.7.*

Proof. Part (a): We shall prove this by using induction on t_1, t_2, the number of horizontal edges and the index of the diagram (j,β,r_1,r_2).

By Lemma 2.9 the $((i,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$ entry $b_{(i,\alpha,r_1,r_2),(j,\beta,r_1,r_2)}$ is given by

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r_1,r_2)} = -\sum_{\substack{d_{l,\gamma}^{r_1'',r_2''} > d_{i,\alpha}^{r_1,r_2} \\ d_{l,\gamma}^{r_1'',r_2''} > d_{j,\beta}^{r_1',r_2'}}} b_{(l,\gamma,r_1'',r_2''),(l,\gamma,r_1'',r_2'')} - \sum_{\substack{d_{l,\gamma}^{r_1'',r_2''} > d_{j,\beta}^{r_1,r_2} \\ d_{l,\gamma}^{r_1'',r_2''} \not> d_{i,\alpha}^{r_1,r_2}}} b_{(i,\alpha,r_1,r_2),(l,\gamma,r_1'',r_2'')}. \tag{2.12}$$

Case (i): Let $t_1 = 0, t_2 = 1, r_1 = 0, r_2 = 1$ and $d_{l_1-f}^{l_1-f}$ and $d_{l_2-f}^{l_2-f}$ have $2s_1 + s_2 - 1$ through classes and no horizontal edge. After applying column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both $d_{l_1-f}^{l_1-f}$ and $d_{l_2-f}^{l_2-f}$ then by Lemma 2.12 and equation (2.12) the $((i,\alpha,0,1),(j,\beta,0,1))$-entry $b_{(i,\alpha,0,1),(j,\beta,0,1)}$ of the block matrix $A_{2\times 0+1,2\times 0+1}$ is given by

$$b_{(i,\alpha,0,1),(j,\beta,0,1)} = (-1)1!\,.$$

Since there is no diagram coarser than $d_{j,\beta}^{0,1}$ alone.

Case (ii): Let $t_1 = 1, t_2 = 0, r_1 = 1, r_2 = 0$ and $d_{l_1-f}^{l_1-f}$ and $d_{l_2-f}^{l_2-f}$ have $2(s_1 - 1) + s_2$ through classes and no horizontal edge. After applying column operations to eliminate the non-zero entries corresponding to the diagrams coarser than both $d_{i,\alpha}^{1,0}$ and $d_{j,\beta}^{1,0}$ then by Lemma 2.12 and equation (2.12) the $((i,\alpha,1,0),(j,\beta,1,0))$-entry $b_{(i,\alpha,1,0),(j,\beta,1,0)}$ of the block matrix $A_{2\times 1+0,2\times 1+0}$ is given by

$$b_{(i,\alpha,1,0),(j,\beta,1,0)} = (-1)21!\,.$$

Since there is no diagram coarser than $d_{j,\beta}^{1,0}$ alone.

In general, suppose that the diagrams $d_{l_1-f}^{l_1-f}$ and $d_{l_2-f}^{l_2-f}$ have $2(s_1 - t_1) + s_2 - t_2$ through classes and have $r_1 - t_1$ pair of $\{e\}$-horizontal edges and $r_2 - t_2$ number of $\mathbb{Z}_2$-horizontal edges then after performing column operations to eliminate the coarser elements of $d_{i,\alpha}^{r_1,r_2}$ and $d_{j,\beta}^{r_1,r_2}$ having t' pair of $\{e\}$-through classes ($\{e\}$-horizontal edges) with $t' < t$, the 0 in the $((i,\alpha,r_1,r_2),(j,\beta,r_1,r_2))$-entry $b_{(i,\alpha,r_1,r_2),(j,\beta,r_1,r_2)}$ of the block matrix $A_{2r_1+r_2,2r_1+r_2}$ is replaced by $-(t_1)!(t_2)!2^{t_1}x^{2(r_1-t_1)+r_2-t_2}$ inductively.

For, $0 \leqslant f' \leqslant t_1$ and $0 \leqslant f'' \leqslant t_2$, the number of diagrams obtained by joining f' pairs of $\{e\}$ through classes (f'' numbers of $\mathbb{Z}_2$ through classes) with f' pairs of $\{e\}$-horizontal edges(f'' numbers $\mathbb{Z}_2$ horizontal edges) in $d^{l_2^f}_{l_2^f}$ is given by $(t_1 C_{f'})^2 (t_2 C_{f''})^2 2^{f'} f'! f''!$. The number of diagrams which are coarser than $d^{r_1,r_2}_{j,\beta}$ but not coarser than $d^{r_1,r_2}_{i,\alpha}$ having $(r_1 - t_1 - l)$-pairs of $\{e\}$-horizontal edges and $r_2 - t_2 - l'$ number of $\mathbb{Z}_2$-horizontal edges is given by

$$\sum_{m=0}^{2t_1-2f'} \sum_{m'=0}^{2t_2-2f''} (r_1 - t_1 - l + m) C_m (2t_1 - 2f') \\ \times C_m 2^m m! (r_2 - t_2 - l' + m') C_{m'} (2t_2 - 2f'') C_{m'} \\ \times (m')! (t_1 C_{f'})^2 f'! 2^{f'} (t_2 C_{f''})^2 f''! \\ \times B^{s_1-t_1, s_2-t_2}_{2(r_1-t_1)+r_2-t_2, 2(r_1-t_1-l+m)+r_2-t_2-l'+m'}. \tag{2.13}$$

Here (2.13) is obtained by choosing m pairs of $\{e\}$-horizontal edges (m' number of $\mathbb{Z}_2$-horizontal edges) for every diagram coarser than $d^{l_2-f}_{l_2-f}$ having $r_1 - t_1 - (l - m)$ pairs of $\{e\}$-horizontal edges ($r_2 - t_2 - (l' - m')$ number of $\mathbb{Z}_2$-horizontal edges) and choose m pairs of $\{e\}$-connected components(m' number of $\mathbb{Z}_2$-connected components) from $d^{l_2^f}_{l_2^f}$. Connecting the chosen m pairs of $\{e\}$-horizontal edges from $d^{l_2-f}_{l_2-f}$ to the m pairs of $\{e\}$-connected components of $d^{l_2^f}_{l_2^f}$ by $\{e\}$-horizontal edge will give $2^m m!(m')!$ number of diagrams having $r_1 - t_1 - l$ pairs of $\{e\}$-horizontal edges. m and m' cannot exceed $2t_1 - 2f'$ and $2t_2 - 2f''$ respectively, since $d^{l_2^f}_{l_2^f}$ has $2t_1 - 2f'$-pairs of $\{e\}$-components and $2t_2 - f''$ number of $\mathbb{Z}_2$-components

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r_1,r_2)} = -2^{t_1} t_1! t_2! x^{2(r_1-t_1)+r_2-t_2} \\ - (-1)^{t_1+t_2} (t_1)! (t_2)! 2^{t_1} \Bigg\{ \sum_{\substack{l=0 \\ (l,l')\neq(0,0)}}^{r_1-t_1} \sum_{l'=-l}^{r_2-t_2} \sum_{m=0}^{2t_1} \sum_{m'=0}^{2t_2} 2t_1 C_m \\ \times (r_1 - t_1 - l + m) C_m 2^m m! 2t_2 C_{m'} (r_2 - t_2 - l' + m') C_{m'} (m')! \\ \times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2, 2(r_1-t_1-l+m)+r_2-t_2-l'+m'} \phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \Bigg\} \\ - \sum_{\substack{f'=0 \\ (f',f'')\neq(0,0) \\ (f',f'')\neq(t_1,t_2)}}^{t_1} \sum_{f''=0}^{t_2} \sum_{l=0}^{r_1-t_1} \sum_{l'=-l}^{r_2-t_2} \sum_{m=0}^{2t_1-2f'} \sum_{m'=0}^{2t_2-f''} ({}_{t_1}C_{f'})^2 2^{f'} f'! ({}_{t_2}C_{f''})^2 f''! \\ \times (-1)^{t_1-f'} 2^{t_1-f'} (t_1 - f')! (-1)^{t_2-f''} (t_2 - f'')! 2(t_1 - f')$$

$$
\begin{aligned}
&\times C_m(r_1-t_1-l+m)C_m 2^m m! 2(t_2-f'')C_{m'}(r_2-t_2-l'+m')\\
&\times C_{m'}(m')! B^{s_1-t_1+f',s_2-t_2+f''}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}\\
&\times \phi^{s_1+t_1-f',s_2+t_2-f''}_{2(r_1-t_1-l)+r_2-t_2-l'}(x)\\
=&-2^{t_1}t_1!t_2!x^{2(r_1-t_1)+r_2-t_2}-(-1)^{t_1+t_2}(t_1)!(t_2)!2^{t_1}\\
&\times\Big\{\sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{m=0}^{2t_1}\sum_{m'=0}^{2t_2}2t_1C_m(r_1-t_1-l+m)C_m2^m m!2t_2\\
&\times C_{m'}(r_2-t_2-l'+m')C_{m'}(m')!\\
&\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}\phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'}(x)\Big\}\\
&-\sum_{\substack{f'=0\\(f',f'')\neq(0,0)\\(f',f'')\neq(t_1,t_2)}}^{t_1}\sum_{f''=0}^{t_2}({}_{t_1}C_{f'})^2 2^{f'}f'!({}_{t_2}C_{f''})^2f''!(-1)^{t_1-f'}2^{t_1-f'}(t_1-f')!\\
&\times(-1)^{t_2-f''}(t_2-f'')!\Big\{\sum_{l=0}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{m=0}^{2t_1-2f'}\sum_{m'=0}^{2t_2-f''}2(t_1-f')\\
&\times C_m(r_1-t_1-l+m)C_m 2^m m! 2(t_2-f'')C_{m'}(r_2-t_2-l'+m')\\
&\times C_{m'}(m')! B^{s_1-t_1+f',s_2-t_2+f''}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}\\
&\times \phi^{s_1+t_1-f',s_2+t_2-f''}_{2(r_1-t_1-l)+r_2-t_2-l'}(x)-\phi^{s_1+t_1-f',s_2+t_2-f''}_{2(r_1-t_1)+r_2-t_2}(x)\Big\}\\
=&-2^{t_1}t_1!t_2!x^{2(r_1-t_1)+r_2-t_2}-(-1)^{t_1+t_2}(t_1)!(t_2)!2^{t_1}\\
&\times\Big\{\sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{m=0}^{2t_1}\sum_{m'=0}^{2t_2}2t_1C_m(r_1-t_1-l+m)C_m2^m m!2t_2\\
&\times C_{m'}(r_2-t_2-l'+m')C_{m'}(m')!\\
&\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}\phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'}(x)\Big\}\\
&-\sum_{f'=0}^{t_1}\sum_{f''=0}^{t_2}(t_1C_{f'})^2(-1)^{t_1-f'}2^{f'}f'!(-1)^{t_2-f''}(t_2C_{f''})^2f''!2^{t_1-f'}\\
&\times(t_1-f')!(t_2-f'')!x^{2(r_1-t_1)+r_2-t_2}\\
&+(-1)^{t_1+t_2}2^{t_1}t_1!t_2!x^{2(r_1-t_1)+r_2-t_2}+2^{t_1}t_1!t_2!x^{2(r_1-t_1)+r_2-t_2}
\end{aligned}
$$

$$= (-1)^{t_1+t_2}(t_1)!(t_2)!2^{t_1}\Bigg\{x^{2(r_1-t_1)+r_2-t_2}$$

$$- \sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{m=0}^{2t_1}\sum_{m'=0}^{2t_2} 2t_1C_m(r_1-t_1-l+m)C_m 2^m m!$$

$$\times 2t_2C_{m'}(r_2-t_2-l'+m')C_{m'}(m')!$$

$$\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}\phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'}(x)\Bigg\}$$

expanding and using Lemma 2.12 we get,

$$= (-1)^{t_1+t_2}(t_1)!(t_2)!2^{t_1}\Bigg\{x^{2(r_1-t_1)+r_2-t_2}$$

$$- \sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2} B^{s_1-t_1,s_2-t_2}_{2(r_1-_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'}\phi^{s_1-t_1,s_2-t_2}_{2(r_1-t_1-l)+r_2-t_2-l'}(x)$$

(putting $m=0, m'=0$)

$$+ \sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{k'=1}^{2t_2} 2t_2C_{k'}(r_2-t_2-l')C_{k'}k'!$$

$$\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'}\phi^{s_1-t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'-k'}(x) \tag{2.14}$$

$$- \sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{m'=1}^{2t_2} 2t_2C_{m'}(r_2-t_2-l'+m')C_{m'}m'!$$

$$\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'+m'}\phi^{s_1-t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \tag{2.15}$$

$$+ \sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{k=1}^{2t_1} 2t_1C_k(r_1-t_1-l)C_k 2^k k!$$

$$\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'}\phi^{s_1+t_1,s_2-t_2}_{2(r_1-t_1-l-k)+r_2-t_2-l'}(x) \tag{2.16}$$

$$- \sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{m=1}^{2t_1} 2t_1C_m(r_1-t_1-l+m)C_m 2^m m!$$

$$\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'}\phi^{s_1+t_1,s_2-t_2}_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \tag{2.17}$$

$$-\sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{k=1}^{2t_1}\sum_{k'=1}^{2t_2} 2t_1C_k(r_1-t_1-l)C_k2^kk!2t_2C_{k'}(r_2-t_2-l')C_{k'}k'!$$

$$\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'}\phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l-k)+r_2-t_2-l'-k'}(x) \tag{2.18}$$

$$-\sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{m=1}^{2t_1}\sum_{k'=1}^{2t_2} 2t_1C_m(r_1-t_1-l+m)C_m2^mm!2t_2$$

$$\times C_{k'}(r_2-t_2-l')C_{k'}k'!B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'}$$

$$\times \phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'-k'}(x) \tag{2.19}$$

$$+\sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{k=1}^{2t_1}\sum_{m'=1}^{2t_2} 2t_1C_k(r_1-t_1-l)C_k2^kk!2t_2C_{m'}(r_2-t_2-l'+m')$$

$$\times C_{m'}m'!B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l)+r_2-t_2-l'+m'}$$

$$\times \phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l-k)+r_2-t_2-l'}(x) \tag{2.20}$$

$$+\sum_{\substack{l=0\\(l,l')\neq(0,0)}}^{r_1-t_1}\sum_{l'=-l}^{r_2-t_2}\sum_{m=1}^{2t_1}\sum_{m'=1}^{2t_2} 2t_1C_m(r_1-t_1-l+m)C_m2^mm!2t_2$$

$$\times C_{m'}(r_2-t_2-l'+m')C_{m'}m'!$$

$$\times B^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2,2(r_1-t_1-l+m)+r_2-t_2-l'+m'}\phi^{s_1+t_1,s_2+t_2}_{2(r_1-t_1-l)+r_2-t_2-l'}(x) \tag{2.21}$$

Putting $(l=0, l'=m')$ in equation (2.15), $(l=m, l'=0)$ in equation (2.17), $(l=m, l'=m')$ in equation (2.21) and canceling the common terms , we get

$$b_{(i,\alpha,r_1,r_2),(j,\beta,r_1,r_2)}$$

$$= (-1)^{t_1+t_2}(t_1)!(t_2)!2^{t_1}\Bigg\{\phi^{s_1-t_1,s_2-t_2}_{2(r_1-t_1)+r_2-t_2}(x)$$

$$-\sum_{m=1}^{2t_1}2t_1C_m(r_1-t_1)C_m2^mm!\phi^{s_1+t_1,s_2-t_2}_{2(r_1-t_1-m)+r_2-t_2}(x)$$

$$- \sum_{m'=1}^{2t_2} 2t_2 C_{m'}(r_2 - t_2) C_{m'} m'! \phi^{s_1-t_1, s_2+t_2}_{2(r_1-t_1)+r_2-t_2-m'}(x)$$
$$- \sum_{m=1}^{2t_1} \sum_{m'=1}^{2t_2} 2t_1 C_m(r_1 - t_1) C_m 2^m m! 2t_2 C_{m'}(r_2 - t_2) C_{m'} m'!$$
$$\times \phi^{s_1+t_1, s_2+t_2}_{2(r_1-t_1-m)+r_2-t_2-m'}(x) \Big\}$$
$$= (-1)^{t_1+t_2}(t_1)!(t_2)! 2^{t_1} \phi^{s_1+t_1, s_2+t_2}_{2(r_1-t_1)+r_2-t_2}(x)$$

Therefore the $((i, \alpha, r_1, r_2), (j, \beta, r_1', r_2'))$-entry in the block matrix $A_{2r_1+r_2, 2r_1+r_2}$ is replaced as

$$(-1)^{t_1+t_2}(t_1)!(t_2)! 2^{t_1} \prod_{l=t_1}^{r_1-1} [x^2 - x - 2(s_1 + l)] \prod_{m=t_2}^{r_2-1} [x - (s_2 + m)].$$

Proof (b) and (c) are similar to the proof of (a). □

Now, we have the main theorem of this section.

2.2. Main theorem

Theorem 2.14. (a) *Let $\widetilde{G}^k_{2s_1+s_2}$ be the matrix similar to the Gram matrix $G^k_{2s_1+s_2}$ of the algebra of $\mathbb{Z}_2$-relations which is obtained after the column operations and the corresponding row operations on $G^k_{2s_1+s_2}$. Then*

$$\widetilde{G}^k_{2s_1+s_2} = \left(\bigoplus_{0 \leqslant r_1+r_2 \leqslant k-s_1-s_2} \widetilde{A}_{2r_1+r_2, 2r_1+r_2} \right)$$

(b) *Let $\widetilde{\overrightarrow{G}}^k_{2s_1+s_2}$ be the matrix similar to the Gram matrix $\overrightarrow{G}^k_{2s_1+s_2}$ of signed partition algebras which is obtained after the column operations and the corresponding row operations on $\overrightarrow{G}^k_{2s_1+s_2}$. Then*

$$\widetilde{\overrightarrow{G}}^k_{2s_1+s_2} = \left(\bigoplus_{\substack{0 \leqslant r_1 \leqslant k-s_1-s_2-1 \\ 0 \leqslant r_2 < k-s_1-s_2-1 \\ 2r_1+r_2 \leqslant 2k-2s_1-2s_2-1}} \widetilde{\overrightarrow{A}}_{2r_1+r_2, 2r_1+r_2} \right) \bigoplus \widetilde{\overrightarrow{A}}_{\rho}$$

where

(i) *the diagonal element of* $\widetilde{A}_{2r_1+r_2,2r_1+r_2}$ *and* $\widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2}$ *are given by*

1. $\prod_{j=0}^{r_1-1}[x^2-x-2(s_1+j)]\prod_{l=0}^{r_2-1}[x-(s_2+l)] \quad if \ \ r_1 \geqslant 1, \ \ r_2 \geqslant 1;$

2. $\prod_{j=0}^{r_1-1}[x^2-x-2(s_1+j)] \quad if \ \ r_2 = 0;$

3. $\prod_{l=0}^{r_2-1}[x-(s_2+l)] \quad if \ \ r_1 = 0.$

(ii) *The entry* $b_{(i,\alpha,r_1,r_2),(j,\beta,r_1,r_2)}$ *of the block matrix* $\widetilde{A}_{2r_1+r_2,2r_1+r_2}$ *and* $\widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2}$ *are replaced by*

1. $$(-1)^{t_1+t_2}2^{t_1}(t_1)!(t_2)!\prod_{j=0}^{r_1-t_1-1}[x^2-x-2(s_1+t_1+j)]$$
$$\times\prod_{l=0}^{r_2-t_2-1}[x-(s_2+t_2+l)] \quad if \ \ r_1 \geqslant 1, r_2 \geqslant 1;$$

2. $(-1)^{t_1}2^{t_1}(t_1)!\prod_{j=0}^{r_1-t_1-1}[x^2-x-2(s_1+t_1+j)] \quad if \ \ r_2 = 0;$

3. $(-1)^{t_2}(t_2)!\prod_{l=0}^{r_2-t_2-1}[x-(s_2+t_2+l)] \quad if \ \ r_1 = 0.$

whenever $d^{r_1,r_2}_{i,\alpha}$ *and* $d^{r_1,r_2}_{j,\beta}$ *is as in Remark 2.6(a), Notation 2.7 and Proposition 2.13.*

(iii) *All other entries in the block matrix* $\widetilde{A}_{2r_1+r_2,2r_1+r_2}$ *and* $\widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2}$ *are zero.*

The underlying partitions of the diagrams corresponding to the entries of the block matrix $\widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2}$ *are* $\alpha = [\alpha_1]^1[\alpha_2]^2[\alpha_3]^3[\alpha_4]^4$ *with* $\alpha_1 = (\alpha_{11},\ldots,\alpha_{1s_1}), \alpha_2 = (\alpha_{21},\ldots,\alpha_{2s_2}), \alpha_3 = (\alpha_{31},\ldots,\alpha_{3r_1}), \alpha_4 = (\alpha_{41},\ldots,\alpha_{4r_2})$ *such that atleast one of* $\alpha_{1i},\alpha_{2j},\alpha_{3l},\alpha_{4m}$ *is greater than* 1 *for* $1 \leqslant i \leqslant s_1, 1 \leqslant j \leqslant s_2, 1 \leqslant l \leqslant r_1$ *and* $1 \leqslant m \leqslant r_2$.

Since the diagrams corresponding to the partition $\rho = [\rho_1]^1[\rho_2]^2[\rho_3]^3[\rho_4]^4$ *with* $|\rho_{1i}| = 1 \ \forall 1 \leqslant i \leqslant s_1$, $|\rho_{2j}| = 1 \ \forall 1 \leqslant j \leqslant s_2$, $|\rho_{3m}| = 0 \ \forall 1 \leqslant m \leqslant r_1$ *and* $|\rho_{4l}| = 1 \ \forall 1 \leqslant l \leqslant r_2$ *does not belong to the signed partition algebra.*

Thus the block corresponding to the diagrams whose underlying partition is ρ is studied separately.

(b)′ *Let $\overrightarrow{\widetilde{A}}_\rho$ where the partition ρ is such that each $\rho_{1i}, \rho_{2j}, \rho_{3l}, \rho_{4m}$ is equal to 1 for $1 \leqslant i \leqslant s_1$, $1 \leqslant j \leqslant s_2$, $1 \leqslant l \leqslant r_1$ and $1 \leqslant m \leqslant r_2$ and $\overrightarrow{\widetilde{A}}_\rho$ is the block sub matrix corresponding to the diagrams whose underlying partition is ρ.*

(i) *The $((i, \rho, r'_1, r'_2), (i, \rho, r'_1, r'_2))$-entry $x^{2r'_1+r'_2}$ in the matrix $\overrightarrow{\widetilde{A}}_\rho$ is replaced by*

$$\prod_{j=0}^{r'_1-1}[x^2 - x - 2(s_1 + j)] \prod_{l=0}^{l=r'_2-1} [x - (s_2 + l)] + \prod_{l=0}^{k-s_1-s_2-1} [x - (s_2 + l)]$$

where $1 \leqslant r'_1 \leqslant k - s_1 - s_2$ and $r'_2 = k - s_1 - s_2 - r'_1$.

(ii) *The zero in the $((i, \rho, r'_1, r'_2), (j, \rho, r'_1, r'_2))$-entry in the block matrix $\overrightarrow{\widetilde{A}}_\rho$ is replaced by*

$$(-1)^{t_1+t_2}2^{t_1}(t_1)!(t_2)! \prod_{j=0}^{r'_1-t_1-1} [x^2 - x - 2(s_1 + t_1 + j)]$$
$$\times \prod_{l=0}^{r'_2-t_2-1} [x - (s_2 + l + t_2)] + \prod_{l=0}^{k-s_1-s_2-1} [x - (s_2 + l)]$$

where $d^{r'_1,r'_2}_{i,\rho}$ and $d^{r'_1,r'_2}_{j,\rho}$ are as in Remark 2.6(a), Notation 2.7 and Proposition 2.13 where $1 \leqslant i, j \leqslant 2k - 2s_1 - 2s_2$ and $i \neq j$.

(iii) *If $\sharp^p(d^{r'_1,k-s_1-s_2-r'_1}_{i,\rho} . d^{r_1,k-s_1-s_2-r_1}_{j,\rho}) = 2s_1 + s_2$ then the $((i, \rho, r'_1, k - s_1 - s_2 - r'_1), (j, \rho, r_1, k - s_1 - s_2 - r_1))$-entry in the block matrix $\overrightarrow{\widetilde{A}}_\rho$ is replaced by*

$$(-1)^{r_1+r'_1} \prod_{l=0}^{k-s_1-s_2-1} [x - (s_2 + l)]$$

where $1 \leqslant i, j \leqslant 2k - 2s_1 - 2s_2$ and $i \neq j$.

(iv) *All other entries in the block matrix $\overrightarrow{\widetilde{A}}_\rho$ are zero.*

(c) *Let $\widetilde{G}^k_s$ be the matrix similar to the Gram matrix G^k_s which is obtained after the column operations and the row operations on G^k_s. Then*

$$\widetilde{G}^k_s = \left(\bigoplus_{0\leqslant r\leqslant k-s} \widetilde{A}_{r,r} \right)$$

where

(i) *The diagonal element of* $\widetilde{A}_{r,r}$ *is given by*

$$\prod_{l=0}^{r-1}[x-(s+l)]$$

(ii) *The entry* $b_{(i,\alpha,r),(j,\beta,r)}$ *of the block matrix* $\widetilde{A}_{r,r}$ *is replaced by*

$$(-1)^t(t)!\prod_{j=t}^{r-1}[x-(s+l)]$$

whenever $R^{d^r_{i,\alpha}}$ *and* $R^{d^r_{j,\beta}}$ *are as in Remark 2.6(b), Notation 2.7 and Proposition 2.13.*

(iii) *All other entries in the block matrix* $\overrightarrow{A}_{r,r}$ *are zero.*

Proof. Part (a): Every entry $x^{2r_1+r_2}$ in the sub block matrix $\widetilde{A}_{2r_1+r_2,2r'_1+r'_2}$ is also replaced by

$$\prod_{j=0}^{r_1-1}[x^2-x-2(s_1+j)]\prod_{l=0}^{r_2-1}[x-(s_2+l)]$$

We continue to do the column operations for all the diagrams whose underlying partition is α where $\alpha=[\alpha_1]^1[\alpha_2]^2[\alpha_3][\alpha_4]^4$ with $\alpha_1=(\alpha_{11},\ldots,\alpha_{1s_1})$, $\alpha_2=(\alpha_{21},\ldots,\alpha_{2s_2})$, $\alpha_3=(\alpha_{31},\ldots,\alpha_{3r_1})$, $\alpha_4=(\alpha_{41},\ldots,\alpha_{4r_2})$ such that at least one of α_{1i}, α_{2j}, α_{3l}, α_{4m} is greater than 1 and hence the above entry gets eliminated.

Thus, from Lemmas 2.1 and 2.9 it follows that the rectangular sub matrix $\widetilde{A}_{2r_1+r_2,2r'_1+r'_2}$ with $2r_1+r_2\neq 2r'_1+r'_2$ becomes zero after all the column operations are carried out.

After applying the row operations corresponding to the column operations performed in Lemmas 2.5, 2.9, Proposition 2.13, and Theorem 2.4, the Gram matrix $G^k_{2s_1+s_2}$ which is similar to a matrix $\widetilde{G}^k_{2s_1+s_2}$ decomposes as a direct sum of block matrices, i.e.

$$\widetilde{G}^k_{2s_1+s_2}=\left(\bigoplus_{0\leqslant r_1+r_2\leqslant k-s_1-s_2}\widetilde{A}_{2r_1+r_2,2r_1+r_2}\right)$$

where the diagonal element of $\widetilde{A}_{2r_1+r_2,2r_1+r_2}$ is given by

$$\prod_{j=0}^{r_1-1}[x^2-x-2(s_1+j)]\prod_{l=0}^{r_2-1}[x-(s_2+l)].$$

Result (i) follows from Theorem 2.4(a), result (ii) follows from Proposition 2.13(a) and result (iii) follow from Lemmas 2.3, 2.5, and 2.9(a) respectively.

Part (b)′: The column operations corresponding to the diagrams whose underlying partition is ρ where

$\rho = [\rho_1]^1[\rho_2]^2[\rho_3]^3[\rho_4]^4$ where $|\rho_{1i}| = 1, \forall 1 \leqslant i \leqslant s_1, |\rho_{2j}| = 1 \forall 1 \leqslant j \leqslant s_2, |\rho_{3m}| = 0, \forall 1 \leqslant m \leqslant r_1$ and $|\rho_{4l}| = 1 \forall 1 \leqslant l \leqslant r_2$ such that $s_1 + s_2 + r_2 = k$ with $s_1 \lneqq k$ cannot be carried out for the block matrix $\widetilde{\overrightarrow{A}}_\rho$, since those diagrams do not belong to the signed partition algebra.

Part (i): We prove the result by induction.

Case (i): Let $d^{1,k-s_1-s_2-1}_{i,\rho}$ be a diagram in $\overrightarrow{\mathbb{J}}^{2.1+k-s_1-s_2-1}_{2s_1+s_2}$, after the column operations the $((i,\rho,1,k-s_1-s_2-1),(i,\rho,1,k-s_1-s_2-1))$-entry corresponding to the diagram $d^{1,k-s_1-s_2-1}_{i,\rho}$ will be replaced by

$$\phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)$$

since the signed partition algebra does not contain diagrams with $k-s_1-s_2$ number of $\mathbb{Z}_2$-horizontal edges.

Case (ii): Let $d^{2,k-s_1-s_2-2}_{i,\rho}$ be a diagram in $\overrightarrow{\mathbb{J}}^{2.2+k-s_1-s_2-2}_{2s_1+s_2}$.

After applying the column operations

$$L_{(i,\rho,2,k-s_1-s_2-2)} \to L_{(i,\rho,2,k-s_1-s_2-2)} - L_{(k,\alpha,r_1,r_2)}$$

for all $d^{r_1,r_2}_{k,\alpha}$ where $d^{r_1,r_2}_{k,\alpha} \in \overrightarrow{\mathbb{J}}^{2r_1+r_2}_{2s_1+s_2}$ with $r_1 + r_2 + s_1 + s_2 \leqslant k-1$, the $((i,\rho,2,k-s_1-s_2-2),(i,\rho,2,k-s_1-s_2-2))$-entry will be replaced by

$$\phi^{s_1,s_2}_{2.2+k-s_1-s_2-2}(x) + 2\phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)$$

Again applying the column operations inside the block matrix $\widetilde{\overrightarrow{A}}_\rho$, the $((i,\rho,2,k-s_1-s_2-2),(i,\rho,2,k-s_1-s_2-2))$-entry becomes

$$\begin{aligned}
&\phi^{s_1,s_2}_{2.2+k-s_1-s_2-2}(x) + 2\phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)\\
&\quad - 2\left[\phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)\right]\\
&= \phi^{s_1,s_2}_{2.2+k-s_1-s_2-2}(x) - \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x).
\end{aligned}$$

After applying the row operations corresponding to the column operations which is performed to obtain the above $((i,\rho,2,k-s_1-s_2-2),(i,\rho,2,k-s_1-s_2-2))$-entry, the $((i,\rho,2,k-s_1-s_2-2),(i,\rho,2,k-s_1-s_2-2))$-entry further becomes

$$\begin{aligned}
&\phi^{s_1,s_2}_{2.2+k-s_1-s_2-2}(x) - \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) + 2\phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)\\
&= \phi^{s_1,s_2}_{2.2+k-s_1-s_2-2}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x).
\end{aligned}$$

In general, let $d_{i,\rho}^{j,k-s_1-s_2-j}$ be a diagram in $\overrightarrow{\mathbb{J}}_{2s_1+s_2}^{2.j+k-s_1-s_2-j}$. After applying the column operations, by induction the $((i,\rho,j,k-s_1-s_2-j),(i,\rho,j,k-s_1-s_2-j))$-entry of the matrix $\widetilde{\overrightarrow{A}}_\rho$ becomes

$$\begin{aligned}
&\phi_{2j+k-s_1-s_2-j}^{s_1,s_2}(x)+\sum_{m=1}^{j-1}{}_jC_m\phi_{2(j-m)+k-s_1-s_2-j+m}^{s_1,s_2}(x)+\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)\\
&\quad-\sum_{m=1}^{j-1}{}_jC_m(\phi_{2(j-m)+k-s_1-s_2-j+m}^{s_1,s_2}(x)+\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x))\\
&\quad=\phi_{2j+k-s_1-s_2-j}^{s_1,s_2}(x)-\sum_{m=1}^{j-1}{}_jC_m\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)+\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)
\end{aligned}$$

After applying the row operations the $((i,\rho,j,k-s_1-s_2-j),(i,\rho,j,k-s_1-s_2-j))$-entry further becomes

$$\begin{aligned}
&\phi_{2j+k-s_1-s_2-j}^{s_1,s_2}(x)-\sum_{m=1}^{j-1}{}_jC_m\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)+\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)\\
&\quad+\sum_{m=1}^{j-1}{}_jC_m\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)\\
&\quad=\phi_{2j+k-s_1-s_2-j}^{s_1,s_2}(x)+\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)
\end{aligned}$$

Thus, for a diagram $d_{i,\rho}^{r'_1,k-s_1-s_2-r'_1}\in\overrightarrow{\mathbb{J}}_{2s_1+s_2}^{2r'_1+k-s_1-s_2-r'_1}$ the $((i,\rho,r'_1,k-s_1-s_2-r'_1),(i,\rho,r'_1,k-s_1-s_2-r'_1))$-entry is replaced as
$\prod_{j=0}^{r'_1-1}[x^2-x-2(s_1+j)]\prod_{l=0}^{k-s_1-s_2-r'_1-1}[x-(s_2+l)]+\prod_{l=0}^{k-s_1-s_2-1}[x-(s_2+l)].$

Part (ii): The proof follows from Proposition 2.13(b) and it is similar to the Proof of (1), whenever $d_{i,\rho}^{r'_1,r'_2}$ and $d_{j,\rho}^{r'_1,r'_2}$ are as in Notation 2.7.

Part (iii). *Case* (i): Let $d_{i,\rho}^{1,k-s_1-s_2-1}\in\overrightarrow{\mathbb{J}}_{2s_1+s_2}^{2.1+k-s_1-s_2-1}$ and $d_{j,\rho}^{2,k-s_1-s_2-2}\in\overrightarrow{\mathbb{J}}_{2s_1+s_2}^{2.2+k-s_1-s_2-2}$ such that number of $\{e\}$-horizontal edges in $d_{i,\rho}^{1,k-s_1-s_2-1}$ is greater than the number of $\{e\}$-horizontal edges in $d_{j,\rho}^{2,k-s_1-s_2-2}$ then $l(d_{i,\rho}^{1,k-s_1-s_2-1}.d_{j,\rho}^{2,k-s_1-s_2-2})\leqslant 2.1+k-s_1-s_2-1$.

There will be two diagrams say $d_{i',\rho}^{1,k-s_1-s_2-1}$ and $d_{i'',\rho}^{1,k-s_1-s_2-1}$ coarser than $d_{j,\rho}^{2,k-s_1-s_2-2}$.

Subcase (i): Suppose $l(d_{i,\rho}^{1,k-s_1-s_2-1}.d_{j,\rho}^{2,k-s_1-s_2-2}) = 2.1+k-s_1-s_2-1$ then

$$a_{(i,\rho,1,k-s_1-s_2-1),(j,\rho,2,k-s_1-s_2-2)} = \phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x).$$

Also,

$$a_{(i,\rho,1,k-s_1-s_2-1),(i',\rho,1,k-s_1-s_2-1)} = \phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)$$

and

$$a_{(i,\rho,1,k-s_1-s_2-1),(i'',\rho,1,k-s_1-s_2-1)} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x),$$

or

$$a_{(i,\rho,1,k-s_1-s_2-1),(i',\rho,1,k-s_1-s_2-1)} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)$$

and

$$a_{(i,\rho,1,k-s_1-s_2-1),(i'',\rho,1,k-s_1-s_2-1)} = \phi^{s_1,s_2}_{2.1+k-s_1-s_2-1}(x) + \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x).$$

After applying the column operations the $((i,\rho,1,k-s_1-s_2-1),$ $(j,\rho,2,k-s_1-s_2-2))$-entry in $\overrightarrow{\widetilde{A}}_\rho$ becomes

$$\begin{aligned} b_{(i,\rho,1,k-s_1-s_2-1),(j,\rho,2,k-s_1-s_2-2)} &= a_{(i,\rho,1,k-s_1-s_2-1),(j,\rho,2,k-s_1-s_2-2)} \\ &\quad - a_{(i,\rho,1,k-s_1-s_2-1),(i',\rho,1,k-s_1-s_2-1)} \\ &\quad - a_{(i,\rho,1,k-s_1-s_2-1),(i'',\rho,1,k-s_1-s_2-1)} \\ &= -\phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x). \end{aligned}$$

Subcase (ii): Suppose $l(d_{i,\rho}^{1,k-s_1-s_2-1}.d_{j,\rho}^{2,k-s_1-s_2-2}) < 2.1+k-s_1-s_2-1$ then

$$a_{(i,\rho,1,k-s_1-s_2-1),(j,\rho,2,k-s_1-s_2-2)} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x).$$

Also,

$$a_{(i,\rho,1,k-s_1-s_2-1),(i',\rho,1,k-s_1-s_2-1)} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)$$

and

$$a_{(i,\rho,1,k-s_1-s_2-1),(i'',\rho,1,k-s_1-s_2-1)} = \phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x)$$

After applying the column operations the $(i,\rho,1,k-s_1-s_2-1),(j,\rho,2,k-s_1-s_2-2)$-entry in $\overrightarrow{\widetilde{A}}_\rho$ becomes

$$\begin{aligned} b_{(i,\rho,1,k-s_1-s_2-1),(j,\rho,2,k-s_1-s_2-2)} &= a_{(i,\rho,1,k-s_1-s_2-1),(j,\rho,2,k-s_1-s_2-2)} \\ &\quad - a_{(i,\rho,1,k-s_1-s_2-1),(i',\rho,1,k-s_1-s_2-1)} \\ &\quad - a_{(i,\rho,1,k-s_1-s_2-1),(i'',\rho,1,k-s_1-s_2-1)} \\ &= -\phi^{s_1,s_2}_{2.0+k-s_1-s_2}(x) \end{aligned}$$

In general, let $d_{i,\rho}^{r_1',k-s_1-s_2-r_1'} \in \overrightarrow{\mathbb{J}}_{2s_1+s_2}^{2r_1'+k-s_1-s_2-r_1'}$ and $d_{j,\rho}^{r_1,k-s_1-s_2-r_1} \in \overrightarrow{\mathbb{J}}_{2s_1+s_2}^{2r_1+k-s_1-s_2-r_1}$ such that the number of $\{e\}$-horizontal edges in $d_{j,\rho}^{r_1,k-s_1-s_2-r_1}$ is strictly greater than the number of $\{e\}$-horizontal edges in $d_{i,\rho}^{r_1',k-s_1-s_2-r_1'}$ then $l(d_{i,\rho}^{r_1',k-s_1-s_2-r_1'}.d_{j,\rho}^{r_1,k-s_1-s_2-r_1}) \leqslant 2r_1'+k-s_1-s_2-r_1'$.

After applying the column operations the $((i,\rho,r_1',k-s_1-s_2-r_1'),(j,\rho,r_1,k-s_1-s_2-r_1))$-entry becomes

$$\begin{aligned}&b_{(i,\rho,r_1',k-s_1-s_2-r_1'),(j,\rho,r_1,k-s_1-s_2-r_1)}\\ &\qquad\times\left(\sum_{m=1}^{r_1-1}(-1)^{m-1}{}_{r_1}C_m-1\right)\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)\\ &\quad=(-1)^{r_1+1}\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)\end{aligned}$$

After applying row operations the $((i,\rho,r_1',k-s_1-s_2-r_1'),(j,\rho,r_1,k-s_1-s_2-r_1))$-entry further becomes

$$\begin{aligned}&b_{(i,\rho,r_1',k-s_1-s_2-r_1'),(j,\rho,r_1,k-s_1-s_2-r_1)}\\ &\quad=(\sum_{m=1}^{r_1'-1}(-1)^{m-1}{}_{r_1'}C_m-1)(-1)^{r_1+1}\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)\\ &\quad=(-1)^{r_1'+r_1}\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x)\end{aligned}$$

Thus, the $((i,\rho,r_1',k-s_1-s_2-r_1'),(j,\rho,r_1,k-s_1-s_2-r_1))$-entry of the block matrix $\overrightarrow{\widetilde{A}}_\rho$ is given by

$$(-1)^{r_1+r_1'}\phi_{2.0+k-s_1-s_2}^{s_1,s_2}(x).$$

The proof of (b) and (c) is similar to the proof of (a). □

Remark 2.15. (a) $\widetilde{G}_{0+0}^k = \bigoplus_{0\leqslant r_1+r_2\leqslant k}\widetilde{A}_{2r_1+r_2,2r_1+r_2}$.

(b) $\overrightarrow{\widetilde{G}}_{0+0}^k = \bigoplus_{\substack{0\leqslant r_1\leqslant k-1\\ 0\leqslant r_2\leqslant k-1\\ 2r_1+r_2\leqslant 2k-1}}\overrightarrow{\widetilde{A}}_{2r_1+r_2,2r_1+r_2}^k \oplus \overrightarrow{\widetilde{A}}_\rho$, where $\widetilde{A}_{2r_1+r_2,2r_1+r_2}$ and $\overrightarrow{\widetilde{A}}_{2r_1+r_2,2r_1+r_2}$ are the diagonal block matrices whose diagonal entry is given by

(i) $\prod_{j=0}^{r_1-1}[x^2-x-2j]\prod_{l=0}^{r_2-1}[x-l], \quad r_1\geqslant 1,\ r_2\geqslant 1,$

(ii) $\prod_{l=0}^{r_2-1}[x-l], \quad r_1=0,$

(iii) $\prod_{j=0}^{r_1-1}[x^2-x-2j], \quad r_2=0.$

(b)′ Let $\overset{\rightarrow}{\widetilde{A}}_\rho$ where the partition ρ is such that $\rho_{1i}=\Phi$, $\rho_{2j}=\Phi$, $\rho_{3l}=1$, $\rho_{4m}=1$ for $1\leqslant i\leqslant s_1$, $1\leqslant j\leqslant s_2$, $1\leqslant l\leqslant r_1$ and $1\leqslant m\leqslant r_2$ and $\overset{\rightarrow}{\widetilde{A}}_\rho$ is the block sub matrix corresponding to the diagrams whose underlying partition is ρ.

The $((i,\rho,r_1',r_2'),(i,\rho,r_1',r_2'))$-entry $x^{2r_1'+r_2'}$ of the matrix $\overset{\rightarrow}{\widetilde{A}}_\rho$ is replaced by

$$\prod_{j=0}^{r_1'-1}[x^2-x-2j]\prod_{l=0}^{r_2'-1}[x-l]+\prod_{l=0}^{r_1'+r_2'-1}[x-l].$$

(c) $G_0^k=\underset{0\leqslant r\leqslant k}{\oplus}\widetilde{A}_{r,r}$

$$\prod_{l=0}^{r-1}[x-l].$$

3. Semisimplicity of signed partition algebras

Semisimplicity of the algebra of $\mathbb{Z}_2$-relations and partition algebras are already discussed in [15] and [2] respectively. In this paper, we give an alternate approach to show that the partition algebras and the algebra of $\mathbb{Z}_2$-relations are semisimple. We also study about the semisimplicity of signed partition algebras.

Definition 3.1. [5] Let $s=2s_1+s_2$. For $0\leqslant s\leqslant 2k$ and $((s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu))\in\Lambda'$ $(((s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu))\in\overrightarrow{\Lambda}')$, put $\lambda=(\lambda_1,\lambda_2)$.

The left cell module

$$W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)] \qquad (\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)])$$

for the cellular algebra $\mathscr{A}\left[A_k^{\mathbb{Z}_2}\right]$ $(\mathscr{A}\left[\overrightarrow{A}_k^{\mathbb{Z}_2}\right])$ is defined as follows:

(i) $W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ is a free $\mathscr{A}$-module with basis

$$\left\{C_S^{m_{s_\lambda}^\lambda m_{s_\mu}^\mu}\,\middle|\,S=(d,P)\in M^k[(s,(s_1,s_2))]\right\}$$

and $A_k^{\mathbb{Z}_2}$-action is defined on the basis element by a

$$aC_S^{m^{\lambda}_{s_\lambda} m^{\mu}_{s_\mu}} \equiv \sum_{(S',s')\in M'^k\left[\left(s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} C_{S'}^{r_a(S',S) m^{\lambda}_{s'_\lambda} m^{\mu}_{s'_\mu}} \mod A_k^{\mathbb{Z}_2}\Big(< \big(s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\Big),$$

where $(S,w) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)), (S',s') = ((d',P'),((s'_{\lambda_1},s'_{\lambda_2}),s'_\mu))$, $r_a(S',S)$ is as in 3(a)(i) and (b)(i) of Theorem 5.4.

(ii) $\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ is a free $\mathscr{A}$-module with basis

$$\left\{\overrightarrow{C}_{\overrightarrow{S}}^{m^{\lambda}_{s_\lambda} m^{\mu}_{s_\mu}} \,\middle|\, \overrightarrow{S} = (d,P) \in \overrightarrow{M}^k\left[(s,(s_1,s_2))\right]\right\}$$

and $\overrightarrow{A}_k^{\mathbb{Z}_2}$-action is defined on the basis element by $\overrightarrow{a}$

$$\overrightarrow{a}\,\overrightarrow{C}_S^{m^{\lambda}_{s_\lambda} m^{\mu}_{s_\mu}} \equiv \sum_{(S',s')\in \overrightarrow{M}'^k\left[\left(s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} \overrightarrow{C}_{S'}^{r_{\overrightarrow{a}}(S',S) m^{\lambda}_{s'_\lambda} m^{\mu}_{s'_\mu}} \mod \overrightarrow{A}_k^{\mathbb{Z}_2}\Big(< \big(s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big)\Big),$$

where $(S,w) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)), (S',s') = ((d',P'),((s'_{\lambda_1},s'_{\lambda_2}),s'_\mu))$, $r_a(S',S)$ is as in 3(a)(ii) and (b)(ii) of Theorem 5.4.

Lemma 3.2 ([5]).

$$\text{(i) } C_{S,S}^{m^{\lambda}_{s_\lambda},s_\lambda m^{\mu}_{s_\mu},s_\mu} C_{T,T}^{m^{\lambda}_{t_\lambda},t_\lambda m^{\mu}_{t_\mu},t_\mu} \equiv \Phi_1((S,s),(T,t)) C_{S,T}^{m^{\lambda}_{s_\lambda},t_\lambda m^{\mu}_{s_\mu},t_\mu} \mod \left[A_k^{\mathbb{Z}_2} < (s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]$$

where

$$\Phi_1((S,s),(T,t)) = \begin{cases} x^{l(P\vee P')}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\phi^{\mu}_{\delta_2}(s_\mu,t_\mu) & \textit{when conditions (a) and (b) of Definition 4.6 in [5] hold,} \\ 0 & \textit{otherwise.} \end{cases}$$

$$\text{(ii) } \overrightarrow{C}_{S,S}^{m^{\lambda}_{s_\lambda},s_\lambda m^{\mu}_{s_\mu},s_\mu} \overrightarrow{C}_{T,T}^{m^{\lambda}_{t_\lambda},t_\lambda m^{\mu}_{t_\mu},t_\mu} \equiv \overrightarrow{\Phi}_1((S,s),(T,t)) C_{S,T}^{m^{\lambda}_{s_\lambda},t_\lambda m^{\mu}_{s_\mu},t_\mu} \mod \left[\overrightarrow{A}_k^{\mathbb{Z}_2} < (s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]$$

where

$$\overrightarrow{\Phi}_1((S,s),(T,t)) = \begin{cases} x^{l(P\vee P')}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\phi^{\mu}_{\delta_2}(s_\mu,t_\mu) & \textit{when conditions (a) and (b)} \\ & \textit{of Definition 4.6 in [5] hold,} \\ 0 & \textit{otherwise.} \end{cases}$$

Here $(S,s) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu))$, $(T,t) = ((d',P'),((t_{\lambda_1},t_{\lambda_2}),t_\mu))$, *and* $l(P\vee P')(l(P\vee P'))$ *denotes the number of connected components in* $d'.d''$ *excluding the union of all the connected components of* P *and* P',

$$m^{\lambda}_{s_\lambda,s_\lambda}\delta_1 m^{\lambda}_{t_\lambda,t_\lambda} \equiv \phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)m^{\lambda}_{s'_\lambda,t_\lambda} \mod \mathscr{H}(<(\lambda_1,\lambda_2)), m^{\mu}_{s_\mu,s_\mu}\delta_2 m^{\mu}_{t_\mu,t_\mu} \equiv \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)m^{\mu}_{s'_\mu,t_\mu} \mod \mathscr{H}'(<\mu)$$

as in Lemma 1.7 in [1].

Definition 3.3 ([5])**.** For

$$(s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)) \in \Lambda' \quad (s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)) \in \overrightarrow{\Lambda}'$$

the bilinear map $\phi^{\lambda,\mu}_{s_1,s_2}(\overrightarrow{\phi}^{\lambda,\mu}_{s_1,s_2})$ is defined as

(i) $\phi^{\lambda,\mu}_{s_1,s_2}\big(C^{m^{\lambda}_{s_\lambda,s_\lambda}m^{\mu}_{s_\mu,s_\mu}}_{(d,P)}, C^{m^{\lambda}_{t_\lambda,t_\lambda}m^{\mu}_{t_\mu,t_\mu}}_{(d',P')}\big) = \Phi_1((S,s),(T,t))$,
$(S,s),(T,t) \in M'^k\big[s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big]$;

(ii) $\overrightarrow{\phi}^{\lambda,\mu}_{s_1,s_2}\big(\overrightarrow{C}^{m^{\lambda}_{s_\lambda,s_\lambda}m^{\mu}_{s_\mu,s_\mu}}_{(d,P)}, \overrightarrow{C}^{m^{\lambda}_{t_\lambda,t_\lambda}m^{\mu}_{t_\mu,t_\mu}}_{(d',P')}\big) = \Phi_1((S,s),(T,t))$,
$(S,s),(T,t) \in \overrightarrow{M}'^k\big[s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big]$,

where $\Phi_1((S,s),(T,t))(\overrightarrow{\Phi}_1((S,s),(T,t)))$ is as in Lemma 3.2.

Put

(i) $G^{\lambda,\mu}_{2s_1+s_2} = (\Phi_1((S,s),(T,t)))_{(S,s),(T,t)\in M'^k\left[s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]}$,

where

$$\Phi_1((S,s),(T,t)) = \begin{cases} x^{l(P_i\vee P_j)}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\phi^{\mu}_{\delta_2}(s_\mu,t_\mu) & \text{when conditions (a) and (b)} \\ & \text{of Definition 4.6 in [5] hold,} \\ 0 & \text{otherwise,} \end{cases}$$

where $(S,s) = ((d_i,P_i),((s_{\lambda_1},s_{\lambda_2}),s_\mu))$, $(T,t) = ((d_j,P_j),((t_{\lambda_1},t_{\lambda_2}),t_\mu))$.

(ii) $\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2} = (\overrightarrow{\Phi}_1((S,s),(T,t)))_{(S,s),(T,t)\in\overrightarrow{M}'^k\left[s,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]}$,
where

$$\overrightarrow{\Phi}_1((S,s),(T,t)) = \begin{cases} x^{l(P_i\vee P_j)}\phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)\phi^{\mu}_{\delta_2}(s_\mu,t_\mu) & \text{when conditions (a) and (b)} \\ & \text{of Definition 4.6 in [5] hold,} \\ 0 & \text{otherwise,} \end{cases}$$

where $(S,s) = ((d_i,P_i),((s_{\lambda_1},s_{\lambda_2}),s_\mu))$, $(T,t) = ((d_j,P_j),((t_{\lambda_1},t_{\lambda_2}),t_\mu))$, and $l(P_i \vee P_j)$ denotes the number of connected components in $d'.d''$ excluding the union of all the connected components of P_i and P_j,

$$m^{\lambda}_{s_\lambda,s_\lambda}\delta_1 m^{\lambda}_{t_\lambda,t_\lambda} \equiv \phi^{\lambda}_{\delta_1}(s_\lambda,t_\lambda)m^{\lambda}_{s'_\lambda,t_\lambda} \mod \mathscr{H}(< (\lambda_1,\lambda_2))$$

and

$$m^{\mu}_{s_\mu,s_\mu}\delta_2 m^{\mu}_{t_\mu,t_\mu} \equiv \phi^{\mu}_{\delta_2}(s_\mu,t_\mu)m^{\mu}_{s'_\mu,t_\mu} \mod \mathscr{H}'(< \mu)$$

as in Lemma 1.7 in [1].

$G^{\lambda,\mu}_{2s_1+s_2}(\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2})$ is called the *Gram matrix of the cell module*

$$W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)] \qquad (\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)).$$

Definition 3.4. Let

$$\left\{C^{m^{\lambda}_{s_\lambda}m^{\mu}_{s_\mu}}_{S^{r_1,r_2}_{i,\alpha}}\right\}_{(S^{r_1,r_2}_{i,\alpha},t_l)\in M'^k[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]}$$
$$\left(\left\{\overrightarrow{C}^{m^{\lambda}_{s_\lambda}m^{\mu}_{s_\mu}}_{S^{r_1,r_2}_{i,\alpha}}\right\}_{(S^{r_1,r_2}_{i,\alpha},t_l)\in\overrightarrow{M}'k[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]}\right)$$

be the basis of the cell module

$$W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)] \qquad (\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]),$$

where $S^{r_1,r_2}_{i,\alpha} = (d_i,P_i)$, $t_l = ((t^l_{\lambda_1},t^l_{\lambda_2}),t^l_\mu)$.

Now, we shall introduce the ordering on the basis of the cell module $W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ as follows:

$$(S^{r_1,r_2}_{i,\alpha},t_l) < (S^{r'_1,r'_2}_{j,\beta},t_k)$$

(i) if $(i,\alpha,r_1,r_2) < (j,\beta,r'_1,r'_2)$ as in Definition 3.7 in PMK and

(ii) if $(i,\alpha,r_1,r_2) = (j,\beta,r'_1,r'_2)$ then $(S^{r_1,r_2}_{i,\alpha},t_l),(S^{r'_1,r'_2}_{j,\beta},t_k)$ can be indexed arbitrarily.

The above ordering can be used for the basis of the cell module $\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$

Arrange the basis of the cell module $W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ and $\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ according to the order defined above and obtain the Gram matrix $G^{\lambda,\mu}_{2s_1+s_2}$ and $\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2}$ corresponding to the cell modules $W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ and $\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ respectively.

Theorem 3.5. (i) *The algebra of $\mathbb{Z}_2$-relations $A^{\mathbb{Z}_2}_k(x)$, signed partition algebras $\overrightarrow{A}^{\mathbb{Z}_2}_k(x)$ and partition algebras $A_k(x)$ are semisimple over $\mathbb{K}(x)$ where $\mathbb{K}$ is a field of characteristic zero where x is an indeterminate.*

(ii) *Suppose that the characteristic of the field $\mathbb{K}$ is 0, then*

(a) *the algebra of $\mathbb{Z}_2$-relations $A^{\mathbb{Z}_2}_k(q)$ is semisimple if and only if q is not a root of the polynomial $f(x)$ where $f(x) = \prod\limits_{\lambda,\mu} \prod\limits_{2s_1+s_2=0}^{2k} \det G^{\lambda,\mu}_{2s_1+s_2}$ where $x = q$ and $q \in \mathbb{C}$.*

(b) *the signed partition algebra $\overrightarrow{A}^{\mathbb{Z}_2}_k(q)$ is semisimple if and only if q is not a root of the polynomial $f(x)$ where $f(x) = \prod\limits_{\lambda,\mu} \prod\limits_{2s_1+s_2=0}^{2k} \det \overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2}$.*

(c) *the partition algebra $A_k(x)$ is semisimple if and only if q is not a root of the polynomial $f(x)$ where $f(x) = \prod\limits_{\lambda} \prod\limits_{s=0}^{k} \det G^{\lambda}_s$.*

(iii) *In particular,*

(a) *$G^{\lambda,\mu}_{2s_1+s_2}$ coincides with $G^{k}_{2s_1+s_2}$ if*

1. *$\lambda = ([s_1],\Phi)$ and $\mu = [s_2]$ when $s_1, s_2 \neq 0$,*
2. *$\lambda = (\Phi,\Phi)$ and $\mu = [s_2]$ when $s_1 = 0, s_2 \neq 0$,*
3. *$\lambda = ([s_1],\Phi)$ and $\mu = \Phi$ when $s_1 \neq 0, s_2 = 0$*
4. *$\lambda = (\Phi,\Phi)$ and $\mu = \Phi$ when $s_1, s_2 = 0$,*

for $0 \leqslant s_1 \leqslant k, 0 \leqslant s_2 \leqslant k, 0 \leqslant s_1 + s_2 \leqslant k$.

(b) *$\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2}$ coincides with $\overrightarrow{G}^{k}_{2s_1+s_2}$ if*

1. *$\lambda = ([s_1],\Phi)$ and $\mu = [s_2]$ when $s_1, s_2 \neq 0$,*
2. *$\lambda = (\Phi,\Phi)$ and $\mu = [s_2]$ when $s_1 = 0, s_2 \neq 0$,*
3. *$\lambda = ([s_1],\Phi)$ and $\mu = \Phi$ when $s_1 \neq 0, s_2 = 0$*
4. *$\lambda = (\Phi,\Phi)$ and $\mu = \Phi$ when $s_1, s_2 = 0$,*

for $0 \leqslant s_1 \leqslant k-1, 0 \leqslant s_2 \leqslant k-1, 0 \leqslant s_1 + s_2 \leqslant k-1$.

(c) *G^{λ}_s coincides with G^{k}_s if*

1. *$\lambda = s$ when $s \neq 0$,*
2. *$\lambda = \Phi$ when $s = 0$*

for $0 \leqslant s \leqslant k$.

(iii)$'$ (a) *If q is a root of the polynomial*

$$f(x) = \prod_{2s_1+s_2=0}^{2k} \det G^k_{2s_1+s_2}$$

where $\det G^k_{2s_1+s_2} = \prod_{\substack{0\leqslant r_1\leqslant k-s_1-s_2\\ 0\leqslant r_2\leqslant k-s_1-s_2\\ 2r_1+r_2\leqslant 2k-2s_1-2s_2}} \det \widetilde{A}_{2r_1+r_2,2r_1+r_2}$, $\widetilde{A}_{2r_1+r_2,2r_1+r_2}$ *is as in Theorem 2.14 then the algebra* $A^{\mathbb{Z}_2}_k(q)$ *is not semisimple.*

In particular, q is an integer such that $0 \leqslant q \leqslant k-2$ and q is a root of the polynomial $x^2 - x - 2r', 0 \leqslant r' \leqslant k-2$ then $A^{\mathbb{Z}_2}_k(q)$ is not semisimple.

(b) *If q is a root of the polynomial*

$$f(x) = \prod_{2s_1+s_2=0}^{2k} \det \overrightarrow{G}^k_{2s_1+s_2}$$

where $\det \overrightarrow{G}^k_{2s_1+s_2} = \prod_{\substack{0\leqslant r_1\leqslant k-s_1-s_2-1\\ 0\leqslant r_2\leqslant k-s_1-s_2-1\\ 2r_1+r_2\leqslant 2k-2s_1-2s_2-1}} \det \widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2} \prod \det \widetilde{\overrightarrow{A}}_\rho,$

$\widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2}$ *and* $\widetilde{\overrightarrow{A}}_\rho$ *are as in Theorem 2.14 then the algebra* $\overrightarrow{A}^{\mathbb{Z}_2}_k(q)$ *is not semisimple.*

In particular, q is an integer such that $0 \leqslant q \leqslant k-2$ and q is a root of the polynomial $x^2 - x - 2r', 0 \leqslant r' \leqslant k-2$ then $\overrightarrow{A}^{\mathbb{Z}_2}_k(q)$ is not semisimple.

(c) *If q is a root of the polynomial*

$$f(x) = \prod_{s=0}^{k} \det G^k_s$$

where $\det G^k_s = \prod_{0\leqslant r\leqslant k-s} \det \widetilde{A}_{r,r}$, $\widetilde{A}_{r,r}$ *is as in Theorem 2.14 then the algebra $A_k(q)$ is not semisimple.*

(iv) *The algebra of $\mathbb{Z}_2$-relations ($A^{\mathbb{Z}_2}_k(q)$), signed partition algebra ($\overrightarrow{A}^{\mathbb{Z}_2}_k(q)$) and the partition algebra ($A_k(q)$) over a field of characteristics 0 are quasi-hereditary for $q \neq 0$.*

Proof. Part (i): The matrix of the bilinear form associated to the cell module $\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ as defined in Definition 4.3(ii) with respect to the ordering of the basis as in Definition 3.4 is rewritten as follows:

$$\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2} = \left(g_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')}\right)_{1\leqslant (i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')\leqslant \overrightarrow{f}_{2s_1+s_2}}$$

where $g_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} = a_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} B^{\lambda,\mu}_{\delta_1,\delta_2}$,

$$a_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} = \begin{cases} x^{l(P_i \vee P_j)} & \text{if conditions (a) and (b)} \\ & \text{of Definition 4.6 in [5] are satisfied,} \\ 0 & \text{otherwise,} \end{cases}$$

$$B^{\lambda,\mu}_{\delta_1,\delta_2} = B^{\lambda}_{\delta_1} \otimes B^{\mu}_{\delta_2}$$

with $B^{\lambda}_{\delta_1} = (\phi^{\lambda}_{\delta_1}(s_\lambda, t_\lambda))$ and $B^{\mu}_{\delta_2} = (\phi^{\mu}_{\delta_2}(s_\mu, t_\mu))$, $B^{\lambda}_{\delta_1}$ and $B^{\mu}_{\delta_2}$ are the matrices of the non-degenerate bilinear forms associated to the cell module W^λ and W^μ of the cellular algebras of $K[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}]$ and $K[\mathfrak{S}_{s_2}]$ respectively as in Theorem 3.8 in [1] and δ_1 and δ_2 depends on the product of the diagrams $d^{r_1,r_2}_{i,\alpha}$ and $d^{r_1',r_2'}_{j,\beta}$.

$$\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2} = (a_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} B^{\lambda,\mu}_{\delta_1,\delta_2})_{1 \leqslant (i,\alpha,r_1,r_2),(j,\beta,r_1',r_2') \leqslant \overrightarrow{f}_{2s_1+s_2}}$$

The $g_{(i,\alpha,r_1,r_2),(i,\alpha,r_1,r_2)} = a_{(i,\alpha,r_1,r_2),(i,\alpha,r_1,r_2)} A$ where $A = B^{\lambda,\mu}_{1,1} = B^{\lambda}_1 \otimes B^{\mu}_1$. Thus, the leading coefficient of the Gram matrix is

$$(\det A)^{\overrightarrow{f}_{2s_1+s_2} \times \dim \overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]}$$

which is non-zero over a characteristic zero. Therefore, the algebra $\overrightarrow{A}^{\mathbb{Z}_2}_k(x)$ is semisimple. The proof for the algebra of $\mathbb{Z}_2$-relations and the partition algebras are similar to the above proof.

Part (ii): By Theorem 3.8 in [1], $\overrightarrow{A}^{\mathbb{Z}_2}_k$ is semisimple if and only if $\det G^{\lambda,\mu}_{2s_1+s_2} \neq 0$ for all s_1, s_2 and for all λ, μ, since

$$\det G^{\lambda,\mu}_{2s_1+s_2} \neq 0 \quad \text{if and only if} \quad \Phi \text{ is non-degenerate.}$$

Part (iii)(b): Now, $\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2} = \overrightarrow{G}^{k}_{2s_1+s_2}$ if

1) $\lambda = ([s_1], \Phi)$ and $\mu = [s_2]$ when $s_1, s_2 \neq 0$,
2) $\lambda = (\Phi, \Phi)$ and $\mu = [s_2]$ when $s_1 = 0, s_2 \neq 0$,
3) $\lambda = ([s_1], \Phi)$ and $\mu = \Phi$ when $s_1 \neq 0, s_2 = 0$

for $0 \leqslant s_1 \leqslant k-1$, $0 \leqslant s_2 \leqslant k-1$, $0 \leqslant s_1 + s_2 \leqslant k-1$, since A is the 1×1 identity matrix,

If $\lambda = (\Phi, \Phi)$ and $\mu = \Phi$ when $s_1, s_2 = 0$, then $\overrightarrow{G}^{\Phi,\Phi}_{2s_1+s_2}$ coincides with $\overrightarrow{G}^{k}_{0+0}$.

Part (iii)$'$(b): If q is a root of

$$f(x) = \prod_{\substack{0\leqslant r_1\leqslant k-s_1-s_2-1\\ 0\leqslant r_2\leqslant k-s_1-s_2-1\\ 2r_1+r_2\leqslant 2k-2s_1-2s_2-1}} \det \widetilde{\overrightarrow{A}}_{2r_1+r_2,2r_1+r_2} \prod \det \widetilde{\overrightarrow{A}}_{\rho},$$

then $\det \overrightarrow{G}^k_{2s_1+s_2} = 0 = \det \overrightarrow{G}^{(([s_1],\Phi),[s_2])}_{2s_1+s_2}$. Thus, the algebra $\overrightarrow{A}^{\mathbb{Z}_2}_k$ is not semisimple.

In particular, by Remark 2.15 if q is an integer such that $0 \leqslant q \leqslant k-2$ and q is a root of polynomial $x^2 - x - 2r', 0 \leqslant r' \leqslant k-2$ then the algebra $\overrightarrow{A}^{\mathbb{Z}_2}_k$ is not semisimple.

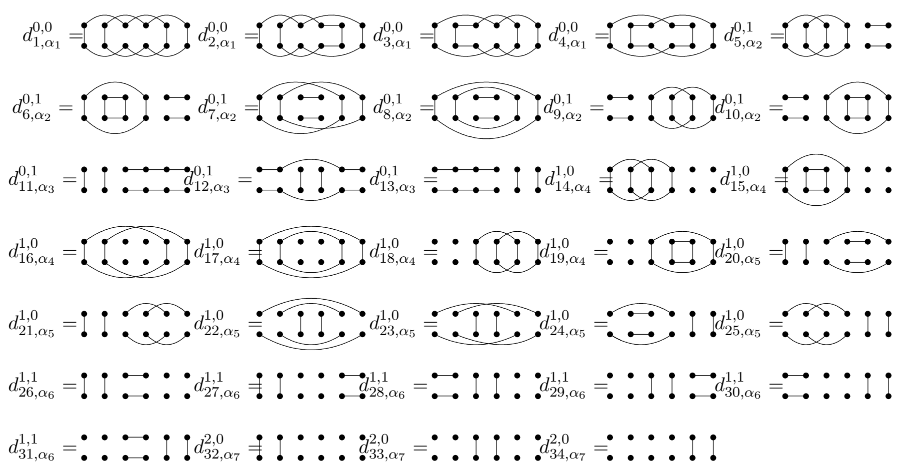
The proof of (a) and (c) is similar to the proof of (b).

Part (iv): It follows from Remark 3.10 in [1] and Theorem 5.4 in [5]. $\square$

Appendix

The following is an example of Gram matrix in $\overrightarrow{A}^{\mathbb{Z}_2}_3(x)$.

Let $s_1 = 1$ and $s_2 = 0$. The following are the diagrams in $J^6_{2\times1+0}$.



where $\alpha_1 = (3,\Phi,\Phi,\Phi)$, $\alpha_2 = (2,\Phi,\Phi,1)$, $\alpha_3 = (1,\Phi,\Phi,2)$, $\alpha_4 = (2,\Phi,1,\Phi)$, $\alpha_5 = (1,\Phi,0,2)$, $\alpha_6 = (1,\Phi,1,1)$, $\alpha_7 = (1,0,1^2,0)$ and $d^{r_1,r_2}_{i,\alpha}$ is a diagram having r_1 number of pairs of $\{e\}$-horizontal edges, r_2 number of $\mathbb{Z}_2$-horizontal edges and α is the underlying partition of $d^{r_1,r_2}_{i,\alpha}$.

	$d^{0,0}_{1,\alpha_1}$	$d^{0,0}_{2,\alpha_1}$	$d^{0,0}_{3,\alpha_1}$	$d^{0,0}_{4,\alpha_1}$	$d^{0,1}_{5,\alpha_2}$	$d^{0,1}_{6,\alpha_2}$	$d^{0,1}_{7,\alpha_2}$	$d^{0,1}_{8,\alpha_2}$	$d^{0,1}_{9,\alpha_2}$	$d^{0,1}_{10,\alpha_2}$	$d^{0,1}_{11,\alpha_3}$	$d^{0,1}_{12,\alpha_3}$	$d^{0,1}_{13,\alpha_3}$	$d^{1,0}_{14,\alpha_4}$	$d^{1,0}_{15,\alpha_4}$	$d^{1,0}_{16,\alpha_4}$	$d^{1,0}_{17,\alpha_4}$	$d^{1,0}_{18,\alpha_4}$	$d^{1,0}_{19,\alpha_4}$	$d^{1,0}_{20,\alpha_5}$	$d^{1,0}_{21,\alpha_5}$	$d^{1,0}_{22,\alpha_5}$	$d^{1,0}_{23,\alpha_5}$	$d^{1,0}_{24,\alpha_5}$	$d^{1,0}_{25,\alpha_5}$	$d^{1,1}_{26,\alpha_6}$	$d^{1,1}_{27,\alpha_6}$	$d^{1,1}_{28,\alpha_6}$	$d^{1,1}_{29,\alpha_6}$	$d^{1,1}_{30,\alpha_6}$	$d^{1,1}_{31,\alpha_6}$	$d^{2,0}_{32,\alpha_7}$	$d^{2,0}_{33,\alpha_7}$	$d^{2,0}_{34,\alpha_7}$
$d^{0,0}_{1,\alpha_1}$	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	1	1	1
$d^{0,0}_{2,\alpha_1}$	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	1	0	1	0	1	1	0	0	0	0	0	0	0	1	1	1
$d^{0,0}_{3,\alpha_1}$	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1	0	1	0	0	1	0	1	0	0	0	0	0	0	1	1	1
$d^{0,0}_{4,\alpha_1}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0	1	1	0	0	1	0	0	0	0	0	0	1	1	1
$d^{0,1}_{5,\alpha_2}$	0	0	0	0	x	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	0	0	0	0	x	0	x	0	0	x	x	0
$d^{0,1}_{6,\alpha_2}$	0	0	0	0	0	x	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	0	0	0	x	0	x	0	0	x	x	0
$d^{0,1}_{7,\alpha_2}$	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	0	x	0	0	0	0	x	x	0	x
$d^{0,1}_{8,\alpha_2}$	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	x	0	0	0	0	x	x	0	x
$d^{0,1}_{9,\alpha_2}$	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	0	x	0	x	0	0	x	x
$d^{0,1}_{10,\alpha_2}$	0	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	x	0	x	0	0	x	x
$d^{0,1}_{11,\alpha_3}$	0	0	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	x	x	0	0	0	0	x	x	0	0	0	0	x	0	0
$d^{0,1}_{12,\alpha_3}$	0	0	0	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	0	x	x	0	0	0	0	x	x	0	0	0	x	0
$d^{0,1}_{13,\alpha_3}$	0	0	0	0	0	0	0	0	0	0	0	0	x	0	0	0	0	0	0	0	0	0	0	x	x	0	0	0	0	x	x	0	0	x
$d^{1,0}_{14,\alpha_4}$	1	1	0	0	x	0	0	0	0	0	0	0	0	x^2	0	1	1	1	1	1	1	1	1	0	0	0	x	0	x	x	0	x^2	x^2	0
$d^{1,0}_{15,\alpha_4}$	0	0	1	1	0	x	0	0	0	0	0	0	0	0	x^2	1	1	1	1	1	1	1	1	0	0	0	x	0	x	x	0	x^2	x^2	0
$d^{1,0}_{16,\alpha_4}$	1	0	0	1	0	0	x	0	0	0	0	0	0	1	1	x^2	0	1	1	1	1	0	0	1	1	x	0	0	0	0	x	x^2	0	x^2
$d^{1,0}_{17,\alpha_4}$	0	1	1	0	0	0	0	x	0	0	0	0	0	1	1	0	x^2	1	1	1	1	0	0	1	1	x	0	0	0	0	x	x^2	0	x^2
$d^{1,0}_{18,\alpha_4}$	1	0	1	0	0	0	0	0	x	0	0	0	0	1	1	1	1	x^2	0	0	0	1	1	1	1	0	0	x	0	x	0	0	x^2	x^2
$d^{1,0}_{19,\alpha_4}$	0	1	0	1	0	0	0	0	0	x	0	0	0	1	1	1	1	0	x^2	0	0	1	1	1	1	0	0	x	0	x	0	0	x^2	x^2
$d^{1,0}_{20,\alpha_5}$	1	0	1	0	0	0	0	0	0	0	x	0	0	1	1	1	1	0	0	x^2	x	1	1	1	1	x	x	0	0	0	0	x^2	0	0
$d^{1,0}_{21,\alpha_5}$	0	1	0	1	0	0	0	0	0	0	x	0	0	1	1	1	1	0	0	x	x^2	1	1	1	1	x	x	0	0	0	0	x^2	0	0
$d^{1,0}_{22,\alpha_5}$	1	0	0	1	0	0	0	0	0	0	0	x	0	1	1	0	0	1	1	1	1	x^2	x	1	1	0	0	x	x	0	0	0	x^2	0
$d^{1,0}_{23,\alpha_5}$	0	1	1	0	0	0	0	0	0	0	0	x	0	1	1	0	0	1	1	1	1	x	x^2	1	1	0	0	x	x	0	0	0	x^2	0
$d^{1,0}_{24,\alpha_5}$	1	1	0	0	0	0	0	0	0	0	0	0	x	0	0	1	1	1	1	1	1	1	1	x^2	x	0	0	0	0	x	x	0	0	x^2
$d^{1,0}_{25,\alpha_5}$	0	0	1	1	0	0	0	0	0	0	0	0	x	0	0	1	1	1	1	1	1	1	1	x	x^2	0	0	0	0	x	x	0	0	x^2
$d^{1,1}_{26,\alpha_6}$	0	0	0	0	0	0	x	x	0	0	x	0	0	0	0	x	x	0	0	x	x	0	0	0	0	x^3	x^2	0	0	0	0	x^3	0	0
$d^{1,1}_{27,\alpha_6}$	0	0	0	0	x	x	0	0	0	0	x	0	0	x	x	0	0	0	0	x	x	0	0	0	0	x^2	x^3	0	0	0	0	x^3	0	0
$d^{1,1}_{28,\alpha_6}$	0	0	0	0	0	0	0	0	x	x	0	x	0	x	x	0	0	x	x	0	0	x	x	0	0	0	0	x^3	x^2	0	0	0	x^3	0
$d^{1,1}_{29,\alpha_6}$	0	0	0	0	x	x	0	0	0	0	0	x	0	x	x	0	0	0	0	0	0	x	x	0	0	0	0	x^2	x^3	0	0	0	x^3	0
$d^{1,1}_{30,\alpha_6}$	0	0	0	0	0	0	0	0	x	x	0	0	x	0	0	0	0	x	x	0	0	0	0	x	x	0	0	0	0	x^3	x^2	0	0	x^3
$d^{1,1}_{31,\alpha_6}$	0	0	0	0	0	0	x	x	0	0	0	0	x	0	0	x	x	0	0	0	0	0	0	x	x	0	0	0	0	x^2	x^3	0	0	x^3
$d^{2,0}_{32,\alpha_7}$	1	1	1	1	x	x	x	x	0	0	x	0	0	x^2	x^2	x^2	x^2	0	0	x^2	x^2	0	0	0	0	x^3	x^3	0	0	0	0	x^4	0	0
$d^{2,0}_{33,\alpha_7}$	1	1	1	1	x	x	0	0	x	x	0	x	0	x^2	x^2	0	0	x^2	x^2	0	0	x^2	x^2	0	0	0	0	x^3	x^3	0	0	0	x^4	0
$d^{2,0}_{34,\alpha_7}$	1	1	1	1	0	0	x	x	x	x	0	0	x	0	0	x^2	x^2	x^2	x^2	0	0	0	0	x^2	x^2	0	0	0	0	x^3	x^3	0	0	x^4

After applying the column operations and by Theorem 2.14 the matrix $\overrightarrow{G}^3_{2.1+0}$ reduces as follows:

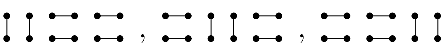
$$\overrightarrow{A}_{0,0} \sim \widetilde{\overrightarrow{A}}_{0,0} = I_4, \quad \overrightarrow{A}_{1,1} \sim \widetilde{\overrightarrow{A}}_{1,1} = xI_9,$$
$$\overrightarrow{A}_{2,2} \sim \widetilde{\overrightarrow{A}}_{2,2} = (x^2 - x - 2)I_{12} + (-2)I'_{12}.$$

where I_n denotes $n \times n$ identity matrix and I'_n denotes $n \times n$ off-diagonal matrix.

After applying the row and column operations, the matrix $\overrightarrow{A}_\rho$ is reduced as follows: $\overrightarrow{A}_\rho \sim$

	$d^{1,1}_{26,\alpha_6}$	$d^{1,1}_{27,\alpha_6}$	$d^{1,1}_{28,\alpha_6}$	$d^{1,1}_{29,\alpha_6}$	$d^{1,1}_{30,\alpha_6}$	$d^{1,1}_{31,\alpha_6}$	$d^{2,0}_{32,\alpha_7}$	$d^{2,0}_{33,\alpha_7}$	$d^{2,0}_{34,\alpha_7}$
$d^{1,1}_{26,\alpha_6}$	x^3-3x	x^2-x	0	0	0	$-2x$	$-x^2+x$	0	0
$d^{1,1}_{27,\alpha_6}$	x^2-x	x^3-3x	0	$-2x$	0	0	$-x^2+x$	0	0
$d^{1,1}_{28,\alpha_6}$	0	0	x^3-3x	x^2-x	$-2x$	0	0	$-x^2+x$	0
$d^{1,1}_{29,\alpha_6}$	0	$-2x$	x^2-x	x^3-3x	0	0	0	$-x^2+x$	0
$d^{1,1}_{30,\alpha_6}$	0	0	$-2x$	0	x^3-3x	x^2-x	0	0	$-x^2+x$
$d^{1,1}_{31,\alpha_6}$	$-2x$	0	0	0	x^2-x	x^3-3x	0	0	$-x^2+x$
$d^{2,0}_{32,\alpha_7}$	$-x^2+x$	$-x^2+x$	0	0	0	0	$x^4-2x^3-4x^2+5x+8$	$-2x^2+2x+8$	$-2x^2+2x+8$
$d^{2,0}_{33,\alpha_7}$	0	0	$-x^2+x$	$-x^2+x$	0	0	$-2x^2+2x+8$	$x^4-2x^3-4x^2+5x+8$	$-2x^2+2x+8$
$d^{2,0}_{34,\alpha_7}$	0	0	0	0	$-x^2+x$	$-x^2+x$	$-2x^2+2x+8$	$-2x^2+2x+8$	$x^4-2x^3-4x^2+5x+8$

The entry $x^2 - x$ in the above matrix cannot be eliminated while applying column operations since the following diagrams do not belong to $\overrightarrow{A}^{\mathbb{Z}_2}_3(x)$.



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