© Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **31** (2021). Number 2, pp. 219–226
DOI:10.12958/adm1209

# On (co)pure Baer injective modules

**M. F. Hamid**

Communicated by R. Wisbauer

Abstract. For a given class of $R$-modules $\mathcal{Q}$, a module $M$ is called $\mathcal{Q}$-*copure Baer injective* if any map from a $\mathcal{Q}$-copure left ideal of $R$ into $M$ can be extended to a map from $R$ into $M$. Depending on the class $\mathcal{Q}$, this concept is both a dualization and a generalization of pure Baer injectivity. We show that every module can be embedded as $\mathcal{Q}$-copure submodule of a $\mathcal{Q}$-copure Baer injective module. Certain types of rings are characterized using properties of $\mathcal{Q}$-copure Baer injective modules. For example a ring $R$ is $\mathcal{Q}$-coregular if and only if every $\mathcal{Q}$-copure Baer injective $R$-module is injective.

## Introduction

Let $\mathcal{Q}$ be a non-empty class of left $R$-modules. An exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{1}$$

of left $R$-modules is called $\mathcal{Q}$-*copure* if every module in $\mathcal{Q}$ is injective with respect to the sequence. In this case, $f$ is called a $\mathcal{Q}$-copure monomorphism and $g$ a $\mathcal{Q}$-copure epimorphism [4, p.322]. If we denote by $\mathcal{PI}$ the class of pure injective modules then the $\mathcal{PI}$-copure sequences are exactly the pure exact ones, see [4, p.290]. So not only does this concept dualize purity but generalizes it as well. We will need the following lemma later.

**2020 MSC:** 16D50.
**Key words and phrases:** $\mathcal{Q}$-copure submodule, $\mathcal{Q}$-copure Baer injective module, pure Baer injective module..

**Lemma 1.** *[4, p.323] For a given class of modules $\mathcal{Q}$, the following hold.*

1) *Any pushout of a $\mathcal{Q}$-copure monomorphism is a $\mathcal{Q}$-copure monomorphism.*
2) *If $g \circ f$ is a $\mathcal{Q}$-copure monomorhphism, where $f : A \to B$ and $g : B \to C$, then so is $f$.*

For details about $\mathcal{Q}$-copure submodules the reader is referred to section 38 of [4].

Thani [3] introduced pure Baer injective modules as those modules which are injective with respect to all pure exact sequences with the ring $R$ as a middle term. Here we study $\mathcal{Q}$-copure Baer injective modules for some given non-empty class of left $R$-modules $\mathcal{Q}$, i.e. modules injective with respect to all $\mathcal{Q}$-copure sequences with $R$ as a middle term. Pure Baer injective modules are, now, a special case of $\mathcal{Q}$-copure Baer injectives by choosing $\mathcal{Q} = \mathcal{PI}$.

Unless otherwise stated the ring $R$ is always associative with identity, all modules are left unital $R$-modules, and $\mathcal{Q}$ is a non-empty class of modules. If there is no confusion or if the class $\mathcal{Q}$ is known we will drop the letter $\mathcal{Q}$ and just say copure sequences and copure Baer injective modules.

## 1. Copure Baer injective modules

**Definition 1.** An $R$-module $M$ is called $\mathcal{Q}$-*copure Baer injective* if any homomorphism from a $\mathcal{Q}$-copure left ideal of $R$ into $M$ has an extension to a homomorphism from $R$ into $M$.

We will often write copure Baer injective and mean $\mathcal{Q}$-copure Baer injective for some given class $\mathcal{Q}$, just like when we say module and homomorphism (or map) and mean $R$-module and $R$-homomrphism (or $R$-map) for some given ring $R$.

**Examples.** 1) Injective modules are $\mathcal{Q}$-copure Baer injective for any class $\mathcal{Q}$.

2) All pure Baer injective (and therefore all pure injective) modules are $\mathcal{PI}$-copure Baer injective.

3) Putting the class $\mathcal{Q} = \{\mathbb{Z}\}$, we see that none of the proper ideals of $\mathbb{Z}$ is $\{\mathbb{Z}\}$-copure. Hence all $\mathbb{Z}$-modules are $\{\mathbb{Z}\}$-copure Baer injective but of course not all of them are injective.

4) We know that all $\mathbb{Z}$-modules are pure Baer injective, however, not all of them are $\mathcal{Q}$-copure Baer injective for all classes $\mathcal{Q}$. For example, let

$M$ be any injective $\mathbb{Z}$-module, so that all ideals of $\mathbb{Z}$ are $\{M\}$-copure in $\mathbb{Z}$. Hence the $\{M\}$-copure Baer injective modules are precisely the injective ones. Therefore $\mathbb{Z}$, for instance, is not $\{M\}$-copure Baer injective.

5) Let the ring $R$ be $\mathbb{Z}_4$ and $\mathcal{Q} = \{\mathbb{Z}_4\}$. Since $\mathbb{Z}_4$ is quasi injective, the sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4$ is $\mathcal{Q}$-copure exact. It is in fact the only nontrivial one! So, both of $\mathbb{Z}_4$ and $\mathbb{Z}_3$ are $\mathcal{Q}$-copure Baer injective, while $\mathbb{Z}_2$ is not. To see this consider the following diagram:

$$\begin{array}{ccc} \mathbb{Z}_2 & \hookrightarrow & \mathbb{Z}_4 \\ \big\downarrow{\scriptstyle 1_{\mathbb{Z}_2}} & & \\ \mathbb{Z}_2 & & \end{array}$$

which cannot be completed because $\mathbb{Z}_2$ is not a direct summand of $\mathbb{Z}_4$.

6) If $\mathcal{Q}$ is the class of simple modules then the class of copure Baer injective modules equals the class $\mathscr{M}$ of modules injective with respect to all inclusions $I \to R$ with $I$ an $s$-pure ideal of a commutative ring $R$, see [1].

7) Any module $Q$ is, of course, $\{Q\}$-copure Baer injective but may not, in general, be pure Baer injective.

The following proposition is easy to verify.

**Proposition 1.** 1) *The direct product (resp., direct sum) of a (finite) family of modules is copure Baer injective if and only if each factor is copure Baer injective.*

2) *An $R$-module $M$ is copure Baer injective if and only if* $\operatorname{Ext}(R/I, M) = 0$ *for every copure left ideal $I$ of $R$.*

**Proposition 2.** *The class of copure Baer injective modules is closed under extensions.*

*Proof.* Let $0 \to A \to B \to C \to 0$ be an exact sequence with $A$ and $C$ copure Baer injective. Exactness of the sequence $0 \to \operatorname{Ext}(R/I, A) \to \operatorname{Ext}(R/I, B) \to \operatorname{Ext}(R/I, C) \to 0$ gives, by Proposition 1, that $\operatorname{Ext}(R/I, B) = 0$ for any copure left ideal $I$ of $R$. □

Thani [3] introduced left pure hereditary rings as those rings whose every pure left ideal is projective. Here, we define left copure hereditary rings.

**Definition 2.** The ring $R$ is called *left $\mathcal{Q}$-copure hereditary* if every copure left ideal of $R$ is projective.

Of course, left pure hereditary rings are $\mathcal{PI}$-copure hereditary. We will just say 'left copure hereditary' when the class $\mathcal{Q}$ is known.

**Theorem 1.** *The following statements are equivalent:*

1) *The ring $R$ is left copure hereditary.*
2) *The homomorphic image of any copure Baer injective $R$-module is copure Baer injective.*
3) *The homomorphic image of any injective $R$-module is copure Baer injective.*
4) *Any finite sum of injective $R$-modules is copure Baer injective.*

*Proof.* (1) $\Rightarrow$ (2) Consider the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \hookrightarrow & R \\ & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{g} & K & \longrightarrow & 0 \end{array}$$

of $R$-modules, where $I$ is a copure left ideal in $R$ and $M$ is a copure Baer injective module. Projectivity of $I$ gives the existence of a $\phi : I \to M$ such that $g\phi = f$. Copure Baer injectivity of $M$ gives a map $\phi' : R \to M$ extending $\phi$, hence $g\phi'$ extends $f$ and $K$ is copure Baer injective. (2) $\Rightarrow$ (3) is trivial. (3) $\Rightarrow$ (1) Let $I$ be a copure left ideal of $R$ and consider the following diagram for a given $R$-module $M$:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \overset{\iota}{\hookrightarrow} & R \\ & & \downarrow{\scriptstyle f} & & \\ \mathrm{E}(M) & \xrightarrow{g} & K & \longrightarrow & 0 \end{array}$$

where $\mathrm{E}(M)$ denotes the injective envelope of $M$. Since $K$ is copure Baer injective, there is a map $h : R \to K$ such that $h|_I = f$. Projectivity of $R$ gives a $\sigma : R \to E(M)$ such that $g\sigma = h$, i.e. $g\sigma\iota = h\iota = f$. This means $I$ is $\mathrm{E}(M)$-projective, i.e. $I$ is projective. (3) $\Rightarrow$ (4) is clear. (4) $\Rightarrow$ (3) Similar to the proof of (4) $\Rightarrow$ (3) in [3, Theorem 2.2]. $\square$

## 2. Embedding in copure baer injective modules

The main result of this section is the following:

**Theorem 2.** *Let $\mathcal{Q}$ be a non-empty class of $R$-modules. Every module can be embedded as a $\mathcal{Q}$-copure submodule in some $\mathcal{Q}$-copure Baer injective module.*

We break the proof into the following three lemmas.

**Lemma 2.** *Every module can be embedded in a copure Baer injective module.*

*Proof.* Given a module $A$, we want to show the existence of a copure Baer injective module that contains $A$ as a submodule. Consider the copure left ideals $I$ of $R$ and the set $\mathscr{F}$ of all maps $f : I \to A$. Thus, for any $f \in \mathscr{F}$ there is a pushout $B$ and a map $g : R \to B$ with $g|_I = f$. The module $B$ may not be copure Baer injective, so put $A_0 = A$, $A_1 = B$ and repeat the above process with $A$ replaced by $A_1$ to give $A_2$ and $A_0 \subseteq A_1 \subseteq A_2$. Continuing in this manner, we get a sequence $A_0 \subseteq \cdots \subseteq A_n \subseteq A_{n+1} \subseteq \cdots$, for all $n \in \mathbb{N}$. Put $A_\omega = \bigcup A_n$. Now, for each nonlimit ordinal repeat the above process. If we get to a limit ordinal, say $l$, define $A_l = \bigcup\{A_s, s < l\}$. Let $t$ be the smallest ordinal with cardinality bigger than that of the ring $R$, i.e. $|t| = |R|^+$ (the successor cardinal of $|R|$). For each $s < t$, we have $|s| < |t|$, $t$ is an initial ordinal and $A_t = \bigcup\{A_s, s < t\}$. Now, $A_t$ is our copure Baer injective module. To see this, let $I$ be a copure left ideal of $R$ and $f : I \to A_t$ any map. For each $r \in I$, let $s(r)$ be the smallest ordinal such that $f(r) \in A_{s(r)}$. Then $s(r) < t$ and $|s(r)| < |t| = |R|^+$. Hence $|s(r)| \leqslant |R|$. Put $p = \sup\{s(r), r \in R\}$. As each $|s(r)| \leqslant |R|$, we must have $|p| \leqslant |R| < |t|$. Hence, $p < t$. Since $t$ is a limit ordinal, we have $p + 1 < t$. Therefore, for each $r \in R$, $r \in A_{s(r)} \subseteq A_p \subseteq A_{p+1} \subseteq A_t$. So, $f(I) \subseteq A_p$. Moreover, the map $f : I \to A_p$ can be extended to a map $g : R \to A_{p+1}$ with $g|_I = f$. View $g$ now as a map $R \to A_t$. (The proof is adapted from [2, p. 295].) □

Of course, we know that every module can be embedded in an injective (hence copure Baer injective) module. But this, unlike the next lemmas, does not guarantee that the embedding is copure.

**Lemma 3.** *Suppose that $A_0 \subseteq A_1 \subseteq \cdots$ is an ascending chain of modules such that $A_i$ is a copure submodule of $A_{i+1}$ for all $i$. Then, $A_0$ is copure in $\bigcup A_i$.*

*Proof.* Let $M$ be a member of the class $\mathcal{Q}$ and $f_0 : A_0 \to M$ a map which extends, by assumption, to a map $f_1 : A_1 \to M$, which in turn extends to $f_2 : A_2 \to M$, and so on. View the maps $f_i$ as sets of ordered pairs $(a_i, f(a_i))$ with $a_i \in A_i$ for all $i$. Hence, it is clear that $f_i \subseteq f_{i+1}$ for all $i$ and if $(x, y_1)$, $(x, y_2) \in f_i$ for some $i$ then $y_1 = y_2$. Now, claim that $f = \cup f_i$ is a (well-defined) homomorphism. To see this, let $x \in \cup A_i$, i.e. $x \in A_i$ for some $i$ and $(x, f_i(x)) \in f_i \subseteq f$. If $(x, y_1), (x, y_2) \in f$, then

$(x, y_1) \in f_i$ and $(x, y_2) \in f_j$ for some $i$ and $j$. Without loss of generality, assume $i \leqslant j$, so that $(x, y_1)$ and $(x, y_2)$ are both in $f_j$ and therefore $(x, y_1) = (x, y_2)$ and $f$ is well-defined. To finish the proof, let $x, y \in \cup A_i$ so that $x \in A_i$ and $y \in \cup A_j$ for some $i$ and $j$. Again assume $i \leqslant j$, so $f_j(x) = f_i(x)$. Now, for any $r, s \in R$, $f_j(rx + sy) = rf_j(x) + sf_j(y)$. So $f(rx + sy) = rf(x) + sf(y)$. □

**Lemma 4.** *The embedding in Lemma 2 is copure.*

*Proof.* The construction of $A_i$ in Lemma 2 shows, by (1) of Lemma 1, that $A_i$ is copure in $A_{i+1}$ for all $i$, and by Lemma 3, $A$ is copure in $\bigcup A_n = A_\omega$. Again by Lemma 3, $A_\omega$ is copure in $A_{\omega+1}$ and $A_{\omega+1}$ is copure in $A_{\omega+2}$ and so on. In other words, for every ordinal $s < |R|^+$, we have either $A$ is copure in $A_s$ if $s$ is not a limit ordinal, or $A_s = \bigcup_{u<s} A_u$ if $s$ is a limit ordinal. In either case, $A$ is copure in $A_t$, as desired. □

The embedding Theorem can be used in characterizing some copure exact sequences.

**Theorem 3.** *The sequence* $0 \to I \overset{\iota}{\hookrightarrow} R \to R/I \to 0$ *is copure exact if and only if every copure Baer injective $R$-module is injective with respect to it.*

*Proof.* Necessity is clear. To prove sufficiency, let $j : I \to C$ be a copure embedding in a copure Baer injective module $C$ (Theorem 2). Therefore, there exists a map $f : R \to C$ such that $f\iota = j$. But $j$ is a copure monomorphism, so by (2) of Lemma 1 $\iota$ is a copure monomorphism. □

## 3. Characterization of rings using copure Baer injectivity

Thani [3] proved that for a left self injective ring $R$, the condition that $R/I$ is pure Baer injective for every essential left ideal $I$ of $R$ is enough to make $R/I$ pure Baer injective for all left ideals $I$ of $R$. Using the same line of argument, we generalize this to copure Baer injectivity.

**Proposition 3.** *Let $R$ be a left self injective ring. If $R/J$ is copure Baer injective for any essential left ideal $J$ of $R$, then $R/I$ is copure Baer injective for any left ideal $I$ of $R$.*

*Proof.* Since $R$ is injective, the injective envelope $\mathrm{E}(I)$ of $I$ must be a direct summand of $R$, for any left ideal $I$ of $R$. Therefore, $\mathrm{E}(I) = Re$ for some idempotent $e \in \mathrm{E}(I)$. Now for the map $f : R \to Re$, defined by

$1 \mapsto e$, since $I$ is essential in $Re$, $f^{-1}(I)$ must as well be essential in $R$ and, therefore by assumption, $R/f^{-1}(I)$ is copure Baer injective. Define $\bar{f} : Re/I \to R/f^{-1}(I)$ by $re + I \mapsto r + f^{-1}(I)$ and proceed as in the proof of [3, Proposition 2.3]. □

By a *left $\mathcal{Q}$-copure split* ring we mean a ring every $\mathcal{Q}$-copure left ideal of which is a direct summand (hence a principal ideal). Clearly, every left pure split ring is left $\mathcal{PI}$-copure split and if a ring $R$ is left copure-split then it is left copure hereditary. The $\mathcal{Q}$-copure split rings are characterized in the following Theorem.

**Theorem 4.** *The following statements are equivalent:*

1) *The ring $R$ is left copure split.*
2) *Every $R$-module is copure Baer injective.*
3) *Any copure left ideal of $R$ is copure Baer injective.*
4) (a) *$R$ is left copure hereditary, and*
   (b) *Every free left $R$-module is copure Baer injective.*

*Proof.* (1) ⇒ (2) Let $M$ be an $R$-module. Since every left ideal $I$ of $R$ is a direct summand, every map $I \to M$ into any $R$-module can easily be extended to a map $R \to M$. (2) ⇒ (3) is obvious. (3) ⇒ (1) Let $I$ be a copure left ideal of $R$. Copure Baer injectivity of $I$ gives a homomorphism $R \to I$ that extends the identity map of $I$, which means $I$ is a direct summand of $R$. (1) ⇒ (4)(a) and (2) ⇒ (4)(b) are immediate. (4) ⇒ (3) Let $I$ be a copure left ideal of $R$, hence projective by (a) and therefore a direct summand of some free $R$-module $F$. From (b) it follows that $F$ is copure Baer injective and by Proposition 1, so is $I$. □

Recall that a ring $R$ is called *left coregular* if every left ideal of $R$ is copure in $R$ [4, p.324].

**Theorem 5.** *For a ring $R$ the following statements are equivalent:*

1) *The ring $R$ is left coregular.*
2) *Every copure Baer injective $R$-module is injective.*
3) *Every copure Baer injective $R$-module is quasi injective.*

*Proof.* (1) ⇒ (2) ⇒ (3) are obvious. (2) ⇒ (1) By assumption, every copure Baer injective $R$-module is injective with respect to any sequence $0 \to I \to R \to R/I \to 0$, which must, therefore, be copure exact by Theorem 3. (3) ⇒ (2) Let $M$ be a copure Baer injective $R$-module. Hence, by Proposition 1, so is $M \oplus \mathrm{E}(R)$ which must be quasi injective by assumption. Therefore, $M$ is injective with respect to $\mathrm{E}(R)$. In particular, $M$ is $R$-injective or injective by Baer condition. □

## References

[1] Iuliu Crivei, *s-Pure submodules*, Int. J. Math. Math. Sci. 2005, 2005, pp.491–497.

[2] John Dauns, *Modules and Rings*, Cambridge University Press, 1994.

[3] Nada M. Al Thani, *Pure Baer injective modules*, Int. J. Math. Math. Sci. 20, 1997, pp.529–538.

[4] Robert Wisbauer, *Foundations of Module and Ring Theory.* Gordon and Breach, 1991.

### Contact information

**Mohanad Farhan Hamid**

Department of Production Engineering and Metallurgy, University of Technology-Iraq, Baghdad, Iraq

*E-Mail(s)*: 70261@uotechnology.edu.iq, mohanadfhamid@yahoo.com

Received by the editors: 30.06.2018.