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RESEARCH ARTICLE


# Automorphisms of the endomorphism semigroup of a free abelian diband*

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Abstract. We determine all isomorphisms between the endomorphism semigroups of free abelian dibands and prove that all automorphisms of the endomorphism semigroup of a free abelian diband are inner.

## 1. Introduction

The problem of the description of automorphisms of the endomorphism semigroup for free algebras in a certain variety was raised by B. I. Plotkin in his papers on universal algebraic geometry (see, e.g., [1], [2]). At the present time there are many papers devoted to describing automorphisms of endomorphism semigroups of free finitely generated universal algebras of different varieties: groups, (inverse) semigroups and monoids [3–5], modules and semimodules [6], Lie algebras and associative algebras [7], [8], dimonoids and g-dimonoids [9], [10] and some others [11], [12].

As it is well-known, the notion of a dimonoid, and constructions of a free associative dialgebra and a free dimonoid were defined by J.-L. Loday [13]. Later on, free dimonoids and free commutative dimonoids were investigated in detail in [14] and [15], respectively. Free abelian dimonoids (this class does not coincide with the class of commutative dimonoids) were

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described in [16]. The structure of free normal dibands, free $(lr, rr)$-dibands and free $(ln, rn)$-dibands was considered in [17], [18]. For the variety of abelian idempotent dimonoids it was shown [19] that it coincides with the variety of $(ln, rn)$-dibands. In this paper, we consider the mentioned above problem of the description of automorphisms of endomorphism semigroups for free algebras in the variety of abelian dibands.

The paper is organized as follows. In Sections 2 and 3, we give necessary definitions and prove auxiliary statements, respectively. In Section 4, we describe stable automorphisms of the endomorphism semigroup of a free abelian diband. In Section 5, we establish that all automorphisms of the endomorphism semigroup of a free abelian diband are inner and show that the automorphism group of such endomorphism semigroup is isomorphic to the symmetric group.

## 2. Preliminaries

Recall that a nonempty set $D$ with two binary associative operations $\dashv$ and $\vdash$ is called a *dimonoid* if for all $x, y, z \in D$,

$$\begin{aligned}
&(\mathrm{D}_1) \quad (x \dashv y) \dashv z = x \dashv (y \vdash z),\\
&(\mathrm{D}_2) \quad (x \vdash y) \dashv z = x \vdash (y \dashv z),\\
&(\mathrm{D}_3) \quad (x \dashv y) \vdash z = x \vdash (y \vdash z).
\end{aligned}$$

A dimonoid $(D, \dashv, \vdash)$ is called *abelian* [16] if for all $x, y \in D$,

$$x \dashv y = y \vdash x.$$

A *band* is a semigroup whose elements are idempotents. If for a dimonoid $(D, \dashv, \vdash)$ the semigroups $(D, \dashv)$ and $(D, \vdash)$ are bands, then this dimonoid is called *idempotent* (or simply a *diband*).

For example, any non-singleton left zero and right zero dimonoid $(D, \dashv, \vdash)$, that is, $(D, \dashv)$ is a left zero semigroup and $(D, \vdash)$ is a right zero semigroup, is an abelian diband. More examples of abelian dibands can be found, e.g., in [18], [19].

An idempotent semigroup $S$ is called a *left regular band* if $aba = ab$ for all $a, b \in S$. If instead of the last identity, $aba = ba$ holds, then $S$ is a *right regular band.*

A dimonoid $(D, \dashv, \vdash)$ is called a $(lr, rr)$-*diband* [18] if $(D, \dashv)$ is a left regular band and $(D, \vdash)$ is a right regular band.

Let $X$ be a nonempty set and $FS(X)$ the free semilattice of all nonempty finite subsets of $X$ with respect to the operation of the set theoretical union. Define two binary operations $\dashv$ and $\vdash$ on the set $B_{lz,rz}(X) = \{(a, A) \in X \times FS(X) | a \in A\}$ as follows:

$$(x, A) \dashv (y, B) = (x, A \cup B)$$

and

$$(x, A) \vdash (y, B) = (y, A \cup B).$$

**Proposition 1.** [18] *The algebraic system* $(B_{lz,rz}(X), \dashv, \vdash)$ *is the free* $(lr, rr)$*-diband.*

A semigroup $S$ is called *left commutative* (respectively, *right commutative*) if it satisfies the identity $xya = yxa$ (respectively, $axy = ayx$).

An idempotent semigroup $S$ is called a *left* (respectively, *right*) *normal band* if it is right (respectively, left) commutative.

A dimonoid $(D, \dashv, \vdash)$ is called a $(ln, rn)$-*diband* [18] if $(D, \dashv)$ is a left normal band and $(D, \vdash)$ is a right normal band.

**Theorem 1.** [19] *A dimonoid* $(D, \dashv, \vdash)$ *is abelian idempotent if and only if* $(D, \dashv, \vdash)$ *is a* $(ln, rn)$*-diband.*

From Theorem 1 we immediately obtain that the variety of abelian dibands coincides with the variety of $(ln, rn)$-dibands. It is well-known also [18] that the variety of $(ln, rn)$-dibands and the variety of $(lr, rr)$-dibands coincide. So, $(B_{lz,rz}(X), \dashv, \vdash)$ is the free abelian diband.

Further, we will denote $(B_{lz,rz}(X), \dashv, \vdash)$ simply by $B_{lz,rz}(X)$. We note that the cardinality of $X$ is the *rank* of the diband $B_{lz,rz}(X)$ and this diband is uniquely determined up to an isomorphism by $|X|$.

Obviously, operations of the free abelian diband $B_{lz,rz}(X)$ coincide if and only if $|X| = 1$. In this case $B_{lz,rz}(X)$ is singleton.

For every $(u_1, U) \in B_{lz,rz}(X)$, where $U = \{u_1, u_2, \ldots, u_n\}$, we have

$$(u_1, U) = (u_1, \{u_1\}) \dashv (u_2, \{u_2\}) \dashv \cdots \dashv (u_n, \{u_n\}).$$

This representation we call a *canonical form* of elements of the diband $B_{lz,rz}(X)$. It is clear that such representation is unique for $n \in \{1, 2\}$ and it is unique up to an order of $(u_i, \{u_i\}), 2 \leqslant i \leqslant n$, for $n \geqslant 3$. Moreover,

$$\langle \{(x, \{x\}) | x \in X\} \rangle = B_{lz,rz}(X),$$

that is, $X' = \{(x, \{x\}) | x \in X\}$ is a generating set of $B_{lz,rz}(X)$.

## 3. Auxiliary statements

We begin this section with the following lemma.

**Lemma 1.** *Let* $B_{lz,rz}(X)$ *and* $B_{lz,rz}(Y)$ *be free abelian dibands defined on* $X$ *and* $Y$*, respectively. Every bijection* $\varphi : X \to Y$ *induces an isomorphism* $\varepsilon_\varphi : B_{lz,rz}(X) \to B_{lz,rz}(Y)$ *such that*

$$(u, U)\varepsilon_\varphi = (u\varphi, U\varphi)$$

*for all* $(u, U) \in B_{lz,rz}(X)$*.*

*Proof.* The proof is obvious. □

Let $B_{lz,rz}(X)$ be the free abelian diband. Every endomorphism $\xi$ of $B_{lz,rz}(X)$ is uniquely determined by a mapping $\xi' : X' \to B_{lz,rz}(X)$. Indeed, to determine $\xi$, it suffices to set

$$(u_1, U)\xi = (u_1, \{u_1\})\xi' \dashv (u_2, \{u_2\})\xi' \dashv \cdots \dashv (u_n, \{u_n\})\xi'$$

for all $(u_1, U) \in B_{lz,rz}(X)$, $U = \{u_1, u_2, \ldots, u_n\}$.

In particular, the endomorphism $\xi$ of $B_{lz,rz}(X)$ is an automorphism if and only if a restriction $\xi$ on $X'$ belongs to the symmetric group $S(X')$. Therefore, the group $\mathrm{Aut}(B_{lz,rz}(X))$ is isomorphic to $S(X)$.

Let $F(X)$ be a free algebra in a variety $V$ with a generating set $X$ and $u \in F(X)$. An endomorphism $\theta_u \in \mathrm{End}(F(X))$ is called *constant* if $x\theta_u = u$ for all $x \in X$.

We denote by $\mathbb{N}$ the set of all positive integers. For every nonempty subset $G$ of $B_{lz,rz}(X)$ and $i \in \mathbb{N}$ we put

$$\Theta_G = \{\theta_{(a,A)} | (a, A) \in G\}$$

and

$$U_i^X = \{(a, A) \in B_{lz,rz}(X) : |A| = i\}.$$

**Lemma 2.** *Let* $\Psi : \mathrm{End}(B_{lz,rz}(X)) \to \mathrm{End}(B_{lz,rz}(Y))$ *be an isomorphism. For every* $i \in \mathbb{N}$ *there exists* $j \in \mathbb{N}$ *such that* $\Theta_{U_i^X}\Psi = \Theta_{U_j^Y}$*.*

*Proof.* For $|X| = 1$ the statement is trivial. Let $|X| \geqslant 2$ and $\theta_{(a,A)} \in \mathrm{End}(B_{lz,rz}(X))$, where $(a, A) \in U_i^X$. It is not hard to see that all elements $\theta_{(u,U)}$, $(u, U) \in B_{lz,rz}(X)$, and only they have the property that $\xi\theta_{(u,U)} = \theta_{(u,U)}$ for all $\xi \in \mathrm{End}(B_{lz,rz}(X))$. Therefore, $\theta_{B_{lz,rz}(X)}\Psi = \theta_{B_{lz,rz}(Y)}$, so that $\theta_{(a,A)}\Psi = \theta_{(b,B)}$ for some $(b, B) \in B_{lz,rz}(Y)$.

Take arbitrary $(a', A') \in U_i^X$, $(a', A') \neq (a, A)$, and an automorphism $\varphi : B_{lz,rz}(X) \to B_{lz,rz}(X)$ such that $(a, A)\varphi = (a', A')$. It is clear that there exists a such automorphism, since $|A| = |A'|$. Then

$$\begin{aligned}\theta_{(a',A')}\Psi &= \theta_{(a,A)\varphi}\Psi = (\theta_{(a,A)}\varphi)\Psi = (\theta_{(a,A)}\Psi)(\varphi\Psi)\\ &= \theta_{(b,B)}(\varphi\Psi) = \theta_{(b,B)(\varphi\Psi)}.\end{aligned}$$

Since $\varphi\Psi$ is an automorphism of $B_{lz,rz}(Y)$, then putting $(b, B)(\varphi\Psi) = (b', B')$ we obtain $|B| = |B'|$. Hence $(b', B') \in U_j^Y$, where $j = |B|$. It means that $\Theta_{U_i^X}\Psi \subseteq \Theta_{U_j^Y}$. Since $\Psi^{-1}$ is an isomorphism and obviously $|Y| \geqslant 2$, we can analogously obtain the inclusion $\Theta_{U_j^Y}\Psi^{-1} \subseteq \Theta_{U_i^X}$, consequently $\Theta_{U_i^X}\Psi = \Theta_{U_j^Y}$. □

The following statement shows that any isomorphism of the endomorphism monoids of free abelian dibands induces a uniquely determined bijection between the generating sets of these dibands.

**Lemma 3.** *Every isomorphism* $\Psi : \mathrm{End}(B_{lz,rz}(X)) \to \mathrm{End}(B_{lz,rz}(Y))$ *implies that* $\Theta_{U_1^X}\Psi = \Theta_{U_1^Y}$.

*Proof.* Let $\Psi$ be an arbitrary isomorphism of monoids $\mathrm{End}(B_{lz,rz}(X))$ and $\mathrm{End}(B_{lz,rz}(Y))$, and $|X| \geqslant 2$ (the case $|X| = 1$ is trivial). By Lemma 2, $\Theta_{U_1^X}\Psi = \Theta_{U_\alpha^Y}$ for some $\alpha \in \mathbb{N}$.

Assume that $\alpha \neq 1$, then there exist some $(a', C)$ and $(b', C)$ of $U_\alpha^Y$ with distinct $a'$ and $b'$. Using Lemma 2, there exists $\beta \in \mathbb{N}$ such that $\Theta_{U_\beta^X}\Psi = \Theta_{U_1^Y}$. It is easy to see that $\beta \neq 1$. Let $(s, S)$ and $(t, T)$ be two distinct pairs from $U_\beta^X$. Then

$$\begin{gathered}\theta_{(a,\{a\})}\Psi = \theta_{(a',C)}, \quad \theta_{(b,\{b\})}\Psi = \theta_{(b',C)},\\ \theta_{(s,S)}\Psi = \theta_{(x,\{x\})} \quad \text{and} \quad \theta_{(t,T)}\Psi = \theta_{(y,\{y\})}\end{gathered}$$

for some $a, b \in X$ and $x, y \in Y$. Evidently, $a \neq b$ and $x \neq y$.

Now we take an arbitrary mapping $\eta' : U_1^X \to U_\beta^X$ such that

$$(a, \{a\})\eta' = (s, S) \quad \text{and} \quad (b, \{b\})\eta' = (t, T).$$

This mapping is uniquely extended (see reasoning after Lemma 1) to an endomorphism of $B_{lz,rz}(X)$ that we will denote by $\eta$. Thus,

$$\begin{aligned}\theta_{(x,\{x\})} &= \theta_{(s,S)}\Psi = (\theta_{(a,\{a\})}\eta)\Psi = (\theta_{(a,\{a\})}\Psi)(\eta\Psi)\\ &= \theta_{(a',C)}(\eta\Psi) = \theta_{(a',C)(\eta\Psi)}.\end{aligned}$$

Analogously, we can obtain $\theta_{(y,\{y\})} = \theta_{(b',C)(\eta\Psi)}$. From here it follows that $(a',C)(\eta\Psi) = (x,\{x\})$ and $(b',C)(\eta\Psi) = (y,\{y\})$.

Further, we have

$$((a',C) \dashv (b',C))(\eta\Psi) = (a',C)(\eta\Psi) = (x,\{x\})$$

and

$$(a',C)(\eta\Psi) \dashv (b',C)(\eta\Psi) = (x,\{x\}) \dashv (y,\{y\}) = (x,\{x,y\}),$$

that contradicts the fact that $\eta\Psi$ is an endomorphism of $B_{lz,rz}(Y)$. Hence $\alpha = 1$, that is, $\Theta_{U_1^X}\Psi = \Theta_{U_1^Y}$. □

Let $\Psi : \mathrm{End}(B_{lz,rz}(X)) \to \mathrm{End}(B_{lz,rz}(Y))$ be an arbitrary isomorphism, $X'$ and $Y'$ generating sets of $B_{lz,rz}(X)$ and of $B_{lz,rz}(Y)$, respectively. By Lemma 3, for every $\mathfrak{x} = (x,\{x\}) \in X'$ there exists $\mathfrak{y} = (y,\{y\}) \in Y'$ such that $\theta_{\mathfrak{x}}\Psi = \theta_{\mathfrak{y}}$. Define a bijection $\psi : X \to Y$ putting $x\psi = y$ if $\theta_{\mathfrak{x}}\Psi = \theta_{\mathfrak{y}}$. In this case we say that $\psi$ is induced by the isomorphism $\Psi$.

## 4. Stable automorphisms of End($B_{lz,rz}(X)$)

Let $F(X)$ be a free algebra in a variety $V$ over a set $X$. An automorphism $\Psi$ of the endomorphism monoid $\mathrm{End}(F(X))$ is called *stable* if $\Psi$ induces the identity permutation of $X$, that is, $\theta_x\Psi = \theta_x$ for all $x \in X$.

An endomorphism $\theta$ of the free algebra $F(X)$ is called *linear* if $x\theta \in X$ for all $x \in X$.

Throughout this section, $X$ denotes an arbitrary set with $|X| \geqslant 2$.

**Lemma 4.** *Let $\Psi$ be a stable automorphism of* $\mathrm{End}(B_{lz,rz}(X))$, $(a,A)$, $(b,B) \in B_{lz,rz}(X)$ *and* $\xi \in \mathrm{End}(B_{lz,rz}(X))$. *Then*

(i) $\xi\Psi = \xi$, *if* $\xi$ *is linear;*

(ii) $A = B$, *if* $\theta_{(a,A)}\Psi = \theta_{(b,B)}$.

*Proof.* (i) Using the linearity of $\xi$ and the stability of $\Psi$, we obtain

$$\begin{aligned}\theta_{(x,\{x\})(\xi\Psi)} &= \theta_{(x,\{x\})}(\xi\Psi) = (\theta_{(x,\{x\})}\Psi)(\xi\Psi)\\ &= (\theta_{(x,\{x\})}\xi)\Psi = \theta_{(x,\{x\})\xi}\Psi = \theta_{(x,\{x\})\xi}, \quad x \in X.\end{aligned}$$

From here $(x,\{x\})(\xi\Psi) = (x,\{x\})\xi$ for all $x \in X$.

(ii) Suppose that $A \setminus B \neq \varnothing$ and take $z \in A \setminus B,\ x \in X, x \neq z$ and $\xi \in \mathrm{End}(B_{lz,rz}(X))$ such that $(z,\{z\})\xi = (x,\{x\})$ and $(y,\{y\})\xi = (y,\{y\})$

for all $y \in X, y \neq z$. So $\xi$ is linear, in addition, $(b, B)\xi = (b, B)$. By (i) of this lemma, $\xi\Psi = \xi$ and

$$\begin{aligned}\theta_{(a,A)}\Psi = \theta_{(b,B)} = \theta_{(b,B)\xi} = \theta_{(b,B)}\xi = (\theta_{(a,A)}\Psi)(\xi\Psi)\\ = (\theta_{(a,A)}\xi)\Psi = \theta_{(a,A)\xi}\Psi.\end{aligned}$$

Using the injectivity of $\Psi$, we have $\theta_{(a,A)} = \theta_{(a,A)\xi}$. From here $(a, A) = (a, A)\xi$ which contradicts to the definition of $\xi$, so $A \setminus B = \varnothing$. In the similar way we can prove that $B \setminus A = \varnothing$. Therefore, $A = B$. □

**Corollary 1.** *Let $\Psi$ be a stable automorphism of* $\mathrm{End}(B_{lz,rz}(X))$ *and $a, b \in X$, $a \neq b$. Then*

$$\theta_{(a,\{a,b\})}\Psi = \theta_{(a,\{a,b\})} \quad or \quad \theta_{(a,\{a,b\})}\Psi = \theta_{(b,\{a,b\})}.$$

*Proof.* From Lemma 2 it follows that $\theta_{(a,\{a,b\})}\Psi = \theta_{(u,U)}$ for some $(u, U)$ of $B_{lz,rz}(X)$. By (ii) of Lemma 4, $U = \{a, b\}$. Thus, $(u, U) = (a, \{a, b\})$ or $(u, U) = (b, \{a, b\})$. □

**Corollary 2.** *Let $\Psi$ be a stable automorphism of* $\mathrm{End}(B_{lz,rz}(X))$. *Then for all $x \in X$, $\theta_{(x,\{x\})\xi}\Psi = \theta_{(x,\{x\})(\xi\Psi)}$.*

*Proof.* It follows from the proof of the condition (i) of Lemma 4. □

We denote by $\Phi_0$ the identity automorphism of $\mathrm{End}(B_{lz,rz}(X))$.

**Lemma 5.** *Let $\Psi$ be a stable automorphism of* $\mathrm{End}(B_{lz,rz}(X))$ *and $a, b \in X$, $a \neq b$. If $\theta_{(a,\{a,b\})}\Psi = \theta_{(a,\{a,b\})}$, then $\Psi = \Phi_0$.*

*Proof.* Show firstly that $\theta_{(u,U)}\Psi = \theta_{(u,U)}$ for all $(u, U) \in B_{lz,rz}(X)$ by induction on $|U|$. Since $\Psi$ is stable, $\theta_{(u,\{u\})}\Psi = \theta_{(u,\{u\})}$ for all $u \in X$. Assume that $\theta_{(v,V)}\Psi = \theta_{(v,V)}$ for all $(v, V) \in B_{lz,rz}(X)$ with $|V| < n$, and let $(u, U) \in B_{lz,rz}(X)$, where $U = \{u_1, u_2, \ldots, u_n\}, u = u_1$ and $n \geqslant 2$. We put $v_1 = (u, U \setminus \{u_n\})$, $v_2 = (u_n, \{u_n\})$ and take the endomorphism $f$ of $B_{lz,rz}(X)$ such that $(a, \{a\})f = v_1$, $(b, \{b\})f = v_2$ and $(y, \{y\})f = (y, \{y\})$ for all $y \in X \setminus \{a, b\}$. Then for all $x \in X$,

$$\begin{aligned}(x, \{x\})(\theta_{(a,\{a,b\})}f) = ((a, \{a\}) \dashv (b, \{b\}))f = (a, \{a\})f \dashv (b, \{b\})f\\ = (u, U) = (x, \{x\})\theta_{(u,U)}.\end{aligned}$$

Therefore, $\theta_{(a,\{a,b\})}f = \theta_{(u,U)}$ for all $(u, U) \in B_{lz,rz}(X)$ with $|U| \geqslant 2$. Using the induction hypothesis and Corollary 2, we have

$$\theta_{(a,\{a\})(f\Psi)} = \theta_{(a,\{a\})f}\Psi = \theta_{v_1}\Psi = \theta_{v_1} = \theta_{(a,\{a\})f},$$
$$\theta_{(b,\{b\})(f\Psi)} = \theta_{(b,\{b\})f}\Psi = \theta_{v_2}\Psi = \theta_{v_2} = \theta_{(b,\{b\})f}$$

and

$$\begin{aligned}\theta_{(y,\{y\})(f\Psi)} &= \theta_{(y,\{y\})f}\Psi = \theta_{(y,\{y\})}\Psi \\ &= \theta_{(y,\{y\})} = \theta_{(y,\{y\})f}, \quad y \in X \setminus \{a,b\}.\end{aligned}$$

Thus, $f\Psi = f$ and for all $(u,U) \in B_{lz,rz}(X)$ with $|U| \geqslant 2$,

$$\theta_{(u,U)}\Psi = (\theta_{(a,\{a,b\})}f)\Psi = (\theta_{(a,\{a,b\})}\Psi)(f\Psi) = \theta_{(a,\{a,b\})}f = \theta_{(u,U)}.$$

So, $\theta_{(u,U)}\Psi = \theta_{(u,U)}$ for all $(u,U) \in B_{lz,rz}(X)$. Moreover, for all $x \in X$ and $\varphi \in \mathrm{End}(B_{lz,rz}(X))$ we obtain

$$\theta_{(x,\{x\})(\varphi\Psi)} = \theta_{(x,\{x\})\varphi}\Psi = \theta_{(x,\{x\})\varphi}.$$

This means that $\varphi\Psi = \varphi$ for all $\varphi \in \mathrm{End}(B_{lz,rz}(X))$ and so $\Psi = \Phi_0$. □

**Lemma 6.** *Let $a, b \in X$ be distinct. There is no a stable automorphism $\Psi$ of* $\mathrm{End}(B_{lz,rz}(X))$ *such that* $\theta_{(a,\{a,b\})}\Psi = \theta_{(b,\{a,b\})}$.

*Proof.* Assume that there exists a stable automorphism $\Psi$ of the monoid $\mathrm{End}(B_{lz,rz}(X))$ such that $\theta_{(a,\{a,b\})}\Psi = \theta_{(b,\{a,b\})}$. Using the condition (ii) of Lemma 4, $\theta_{(b,\{a,b\})}\Psi = \theta_{(a,\{a,b\})}$.

Let $g \in \mathrm{End}(B_{lz,rz}(X))$ such that $(a,\{a\})g = (a,\{a\})$, $(b,\{b\})g = (a,\{a,b\})$ and $(x,\{x\})g = (x,\{x\})$ for all $x \in X \setminus \{a,b\}$. It is easy to see that $\theta_{(a,\{a\})}g = \theta_{(a,\{a\})}$ and $\theta_{(b,\{b\})}g = \theta_{(a,\{a,b\})}$. Then

$$\theta_{(a,\{a\})} = \theta_{(a,\{a\})}\Psi = (\theta_{(a,\{a\})}g)\Psi = \theta_{(a,\{a\})}(g\Psi) = \theta_{(a,\{a\})(g\Psi)}$$

and

$$\theta_{(b,\{a,b\})} = \theta_{(a,\{a,b\})}\Psi = (\theta_{(b,\{b\})}g)\Psi = \theta_{(b,\{b\})}(g\Psi) = \theta_{(b,\{b\})(g\Psi)}.$$

From here $(a,\{a\})(g\Psi) = (a,\{a\})$ and $(b,\{b\})(g\Psi) = (b,\{a,b\})$.

Using the equality $\theta_{(b,\{a,b\})}g = \theta_{(a,\{a,b\})}$, on the one hand we obtain

$$\theta_{(b,\{a,b\})} = \theta_{(a,\{a,b\})}\Psi = (\theta_{(b,\{a,b\})}g)\Psi = \theta_{(a,\{a,b\})}(g\Psi) = \theta_{(a,\{a,b\})(g\Psi)},$$

and so $(a,\{a,b\})(g\Psi) = (b,\{a,b\})$. On the other hand,

$$\begin{aligned}(a,\{a,b\})(g\Psi) &= ((a,\{a\}) \dashv (b,\{b\}))(g\Psi) \\ &= (a,\{a\}) \dashv (b,\{a,b\}) = (a,\{a,b\})\end{aligned}$$

that contradicts the previous expression for $(a,\{a,b\})(g\Psi)$. □

## 5. The automorphism group of $\mathrm{End}(B_{lz,rz}(X))$

Firstly, we describe all isomorphisms between the endomorphism semigroups of free abelian dibands.

Recall that $\varepsilon_\varphi$ denotes the isomorphism $B_{lz,rz}(X) \to B_{lz,rz}(Y)$ which is induced by the bijection $\varphi : X \to Y$ (see Lemma 1).

**Theorem 2.** *Every isomorphism* $\Phi : \mathrm{End}(B_{lz,rz}(X)) \to \mathrm{End}(B_{lz,rz}(Y))$ *is induced by the isomorphism* $\varepsilon_f$ *of* $B_{lz,rz}(X)$ *to* $B_{lz,rz}(Y)$ *for a uniquely determined bijection* $f : X \to Y$.

*Proof.* The case $|X| = 1$ is trivial, so that we suppose further $|X| > 1$. Let $\Phi : \mathrm{End}(B_{lz,rz}(X)) \to \mathrm{End}(B_{lz,rz}(Y))$ be an arbitrary isomorphism. By Lemma 3, $\Phi$ induces a uniquely determined bijection $f : X \to Y$ such that $\theta_{(x,\{x\})}\Phi = \theta_{(xf,\{xf\})}$ for every $x \in X$. By Lemma 1, $f$ induces the isomorphism $\varepsilon_f : B_{lz,rz}(X) \to B_{lz,rz}(Y)$. It is not hard to check that the mapping

$$E_f : \mathrm{End}(B_{lz,rz}(X)) \to \mathrm{End}(B_{lz,rz}(Y)) : \eta \mapsto \varepsilon_f^{-1}\eta\varepsilon_f$$

is an isomorphism. From here it follows that $\Omega = \Phi E_f^{-1}$ is an automorphism of $\mathrm{End}(B_{lz,rz}(X))$. Since for all $x \in X$ we have

$$\theta_{(x,\{x\})}\Omega = (\theta_{(x,\{x\})}\Phi)E_f^{-1} = \theta_{(xf,\{xf\})}E_f^{-1} = \theta_{(xff^{-1},\{xff^{-1}\})} = \theta_{(x,\{x\})},$$

then $\Omega$ is stable.

By Corollary 1, Lemma 5 and Lemma 6, $\Omega$ is an identity automorphism $\Phi_0$. From $\Phi E_f^{-1} = \Phi_0$ we obtain $\Phi = E_f$, i.e., $\Phi$ is an isomorphism induced by $\varepsilon_f$. □

Let $F(X)$ be a free algebra in a variety $V$ over a set $X$. An automorphism $\Phi$ of $\mathrm{End}(F(X))$ is called *inner* if there exists an automorphism $\alpha$ of $F(X)$ such that $\beta\Phi = \alpha^{-1}\beta\alpha$ for all $\beta \in \mathrm{End}(F(X))$.

Finally, we characterize the automorphism group of the endomorphism monoid of a free abelian diband.

**Theorem 3.** *All automorphisms of* $\mathrm{End}(B_{lz,rz}(X))$ *are inner. In addition, the automorphism group* $\mathrm{Aut}(\mathrm{End}(B_{lz,rz}(X)))$ *is isomorphic to the symmetric group* $S(X)$.

*Proof.* Let $X = Y$ in Theorem 2, then it will be the first part of the given theorem. By Theorem 2, every automorphism $\Phi$ of $\mathrm{End}(B_{lz,rz}(X))$ has a form $\Phi = E_f$, where $\eta\Phi = \varepsilon_f^{-1}\eta\varepsilon_f$ for all $\eta \in \mathrm{End}(B_{lz,rz}(X))$ and a suitable bijection $f : X \to X$. As follows from Lemma 1 (see Section 3), $\varepsilon_f \in \mathrm{Aut}(B_{lz,rz}(X))$ for all $f \in S(X)$, therefore all automorphisms of $\mathrm{End}(B_{lz,rz}(X))$ are inner.

Define a mapping $\zeta : \mathrm{Aut}(\mathrm{End}(B_{lz,rz}(X))) \to S(X)$ as follows: $E_f\zeta = f$ for all $E_f \in \mathrm{Aut}(\mathrm{End}(B_{lz,rz}(X)))$. An immediate check shows that $\zeta$ is an isomorphism. □

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