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# $F$-supplemented modules

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Abstract. Let $R$ be a ring, let $M$ be a left $R$-module, and let $U, V, F$ be submodules of $M$ with $F$ proper. We call $V$ an $F$-supplement of $U$ in $M$ if $V$ is minimal in the set $F \subseteq X \subseteq M$ such that $U+X = M$, or equivalently, $F \subseteq V$, $U+V = M$ and $U \cap V$ is $F$-small in $V$. If every submodule of $M$ has an $F$-supplement, then we call $M$ an $F$-supplemented module. In this paper, we introduce and investigate $F$-supplement submodules and (amply) $F$-supplemented modules. We give some properties of these modules, and characterize finitely generated (amply) $F$-supplemented modules in terms of their certain submodules.

## Introduction

All rings considered in this paper will be associative with an identity element. Unless otherwise stated, $R$ denotes an arbitrary ring and all modules will be *left* unitary $R$-modules. Let $M$ be a module. By $X \subseteq M$, we mean $X$ is a submodule of $M$, and $X \subsetneq M$ means $X$ is a proper submodule of $M$. As usual, $\operatorname{Rad}(M)$ denotes the radical of $M$. Throughout the paper, unless otherwise stated, $F$ will be a proper submodule, and we follow the terminology and notation as in [2].

A submodule $K \subseteq M$ is called *small* in $M$, denoted by $K \ll M$, if, for every submodule $L \subseteq M$, the equality $K + L = M$ implies $L = M$. The notion of a supplement submodule was introduced in [3] in order to characterize semiperfect modules, that is projective modules whose factor modules have projective covers. For submodules $U$ and $V$ of a module $M$,

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$V$ is said to be a *supplement* of $U$ in $M$ or $U$ is said to *have a supplement* $V$ in $M$ if $U + V = M$ and $U \cap V \ll V$, and $M$ is called *supplemented* if every submodule of $M$ has a supplement in $M$. See [4, §41] and [2] for results and definitions related to supplements, supplemented modules and small submodules.

Recently, several authors have studied different generalizations of small submodules (see, for example, [6], [5], [1]). In [1], $F$-small submodules are defined and studied as a generalization of small submodules. Let $F$ be a proper submodule of a module $M$. A submodule $K \subseteq M$ is called *$F$-small in $M$*, denoted $K \ll_F M$, if, for every submodule $L \subseteq M$ containing $F$, the equality $K + L = M$ implies $L = M$. Motivated by the relation between supplement submodules and small submodules, we introduce the notion of an $F$-supplement submodule. We call a submodule $V \subseteq M$ an *$F$-supplement* of $U \subseteq M$ in $M$ if $V$ is minimal in the set $\{L \subseteq M \mid U + L = M \text{ and } F \subseteq L\}$, or equivalently, $F \subseteq V$, $U + V = M$ and $U \cap V \ll_F V$ (Proposition 1). We say $U \subseteq M$ has *ample $F$-supplements* if, for every $V \subseteq M$ with $U + V = M$, there is an $F$-supplement $V'$ of $U$ with $V' \subseteq V$. If every submodule of $M$ has an (ample) $F$-supplement, then $M$ is called an (*amply*) *$F$-supplemented* module. Like small submodules, of course, for $F = 0$, supplement submodules and $F$-supplement submodules coincide in a module. Also, any supplement submodule containing $F$ is always $F$-supplement, and the converse is true when $F \ll M$ (Remark 1). So, for instance, for a finitely generated module $M$, if we take $F = \mathrm{Rad}(M)$, then any submodule of $M$ containing $F$ is $F$-supplement if and only if it is supplement in $M$.

In Section 1, we investigate $F$-small submodules and $F$-supplement submodules, and we give some properties of $F$-supplement submodules which are adapted from supplement submodules. For instance, for submodules $K \subseteq N \subseteq M$, we show that if $N$ is an $F$-supplement in $M$, then $K$ is an $F$-supplement in $M$ if and only if $K$ is an $F$-supplement in $N$ (Theorem 1). Also, we prove that if $N$ is an $F$-supplement in $M$, then $N/K$ is an $(F + K)/K$-supplement in $M/K$, and the converse is true if, in addition, $K$ is an $F$-supplement in $M$ (Propositions 4 and 5). Moreover, we show that if $V$ is an $F$-supplement in $M$, then $\mathrm{Rad}_F(V) = V \cap \mathrm{Rad}_F(M)$, where $\mathrm{Rad}_F(M)$ is the intersection of all maximal submodules of $M$ containing $F$ (Proposition 2).

In Section 2, we introduce and study $F$-supplemented modules. We show that every finite (direct) sum of $F$-supplemented modules is $F$-supplemented (Corollary 4), and that a module $M$ is $F$-supplemented if and only if $M/F$ is supplemented (i.e. 0-supplemented) (Theorem 2). Also, we prove that if $M$ is $F$-supplemented, then $M/\operatorname{Rad}_F(M)$ is semisimple (Proposition 8). Finally, we characterize finitely generated $F$-supplemented modules (Corollary 7). Namely, we prove the equivalence of the following statements: (1) $M$ is $F$-supplemented, (2) every maximal submodule of $M$ containing $F$ has an $F$-supplement, (3) $M$ is a sum of $F$-hollow submodules (i.e. modules in which every proper submodule containing $F$ is $F$-small), (4) $M$ is an irredundant sum of $F$-local submodules (i.e. modules $L$ with $\operatorname{Rad}_F(L)$ is the largest submodule of $L$ containing $F$).

In Section 3, we define and investigate amply $F$-supplemented modules. We prove that $M$ is amply $F$-supplemented if and only if every submodule $U \subseteq M$ is of the form $U = X + Y$, where $X$ is $F$-supplemented and $Y \ll_F M$; and if $M$ is finitely generated, these are equivalent to the statement that every maximal submodule of $M$ containing $F$ has ample $F$-supplements in $M$ (Theorem 4).

## 1. *F*-small and *F*-supplement submodules

In this section, we give some useful properties of $F$-small submodules and some results on $F$-supplement submodules.

Clearly, small submodules and $F$-small submodules are the same for $F = 0$. Moreover, small submodules are always $F$-small, but the converse is not true in general.

**Example 1.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z}$. Taking a submodule $F = 4\mathbb{Z}$ of $M$, we see that $8\mathbb{Z}$ is $F$-small in $M$ since $8\mathbb{Z} \subseteq F$. However, $8\mathbb{Z}$ is not small in $M$ since, for example, $8\mathbb{Z} + 3\mathbb{Z} = M$. In fact, 0 is the only small submodule of $M$.

We collect some known properties of $F$-small submodules which will be useful in the sequel in the following lemma (see [1]).

**Lemma 1.** *Let $M$ be a module and let $K, L$ be submodules of $M$.*

1) *If $f : M \to N$ is a homomorphism of modules, then $K \ll_F M$ implies $f(K) \ll_{f(F)} N$. In particular, if $K \ll_F M \subseteq N$, then $K \ll_F N$.*
2) *If $K \subseteq N \subseteq M$, then $N \ll_F M$ if and only if $K \ll_F M$ and $N/K \ll_{(F+K)/K} M/K$.*
3) *$K + L \ll_F M$ if and only if $K \ll_F M$ and $L \ll_F M$.*

Let $M$ be a module and let $U, V$ be submodules of $M$. Recall that $V$ is said to be an $F$-*supplement* of $U$ in $M$ if $V$ is minimal in the set $F \subseteq L \subseteq M$ with $U + L = M$.

**Proposition 1.** *Let $M$ be a module and let $U, V$ be submodules of $M$. Then $V$ is an $F$-*supplement *of $U$ in $M$ if and only if $F \subseteq V$, $U + V = M$ and $U \cap V \ll_F V$.*

*Proof.* ($\Rightarrow$) Assume that $U \cap V + X = V$ with $F \subseteq X \subseteq M$. Then $M = U + V = U + X$, and so $X = V$ by the minimality of $V$. Thus $U \cap V \ll_F V$.

($\Leftarrow$) Assume that $U + Y = M$ for a submodule $Y$ of $M$ with $F \subseteq Y \subseteq V$. Then we have $V = M \cap V = (U + Y) \cap V = U \cap V + Y$. Since $U \cap V \ll_F V$ and $F \subseteq Y$, it follows that $V = Y$. This minimality of $V$ shows that $V$ is an $F$-supplement of $U$ in $M$. □

As a generalization of the radical of a module, $\mathrm{Rad}_F(M)$ is defined in [1, Definition 3.1] to be the intersection of all maximal submodules of $M$ that contain $F$. If there is no such maximal submodule of $M$, then $\mathrm{Rad}_F(M) = M$. Also, it was proved that $\mathrm{Rad}_F(M)$ is equal to the sum of all $F$-small submodules of $M$ (see [1, Theorem 3.3]).

**Lemma 2.** *Let $M$ be a module and let $x \in M$. Then $x \in \mathrm{Rad}_F(M)$ if and only if $Rx \ll_F M$.*

*Proof.* ($\Rightarrow$) Let $Rx + U = M$ with $F \subseteq U \subseteq M$. Assume that $U \neq M$. Then, by Zorn's Lemma, there is a submodule $L \subseteq M$ maximal with respect to $U \subseteq L$ and $x \notin L$. Since $L + Rx = M$, it follows that $L$ is a maximal submodule of $M$. So, $\mathrm{Rad}_F(M) \subseteq L$ since $F \subseteq L$, and this implies that $x \in L$. This contradiction shows that $U = M$. Hence $Rx \ll_F M$.

($\Leftarrow$) Since $\mathrm{Rad}_F(M)$ is the sum of all $F$-small submodules of $M$ and $Rx \ll_F M$, we have $Rx \subseteq \mathrm{Rad}_F(M)$. Hence $x \in \mathrm{Rad}_F(M)$. □

**Proposition 2.** *Let $M$ be a module and let $V$ be an $F$-supplement of $U$ in $M$. If $K \ll_F M$, then $K \cap V \ll_F V$ and so $\mathrm{Rad}_F(V) = V \cap \mathrm{Rad}_F(M)$.*

*Proof.* First, assume that $(K \cap V) + X = V$ with $F \subseteq X \subseteq V$. Then $M = U + V = U + (K \cap V) + X = K + (U + X)$. Since $F \subseteq U + X$ and $K \ll_F M$, we have $U + X = M$. Thus $X = V$ by the minimality of $V$. Hence $K \cap V \ll_F V$. Next, it is clear that $\mathrm{Rad}_F(V) \subseteq V \cap \mathrm{Rad}_F(M)$. Now, let $x \in V \cap \mathrm{Rad}_F(M)$. Then $x \in \mathrm{Rad}_F(M)$ and so $Rx \ll_F M$ by Lemma 2. Since $Rx \subseteq V$, it follows by the first part that $Rx = V \cap Rx \ll_F V$. Thus $x \in \mathrm{Rad}_F(V)$. □

**Corollary 1.** *Let $M$ be a module and let $V$ be an $F$-supplement of $U$ in $M$. If $\mathrm{Rad}_F(M) \ll_F M$, then $U$ is contained in a maximal submodule of $M$ containing $F$.*

*Proof.* By Proposition 2, we have $\mathrm{Rad}_F(V) = V \cap \mathrm{Rad}_F(M) \ll_F V$. Therefore, $\mathrm{Rad}_F(V) \neq V$, and so $V$ has a maximal submodule $V'$ that contains $F$. So, $M/(U + V') = (U + V)/(U + V') \cong V/V'$, and $U + V'$ is the desired maximal submodule of $M$. □

**Corollary 2.** *Let $M$ be a module, and let $V$ be an $F$-supplement of $U$ in $M$. If $U$ is a maximal submodule of $M$ containing $F$, then $U \cap V = \mathrm{Rad}_F(V)$ is the unique maximal submodule of $V$ that contains $F$.*

*Proof.* Since $U$ is a maximal submodule of $M$ containing $F$, we have $\mathrm{Rad}_F(M) \subseteq U$. So, it follows by Proposition 2 that $\mathrm{Rad}_F(V) = V \cap \mathrm{Rad}_F(M) \subseteq U \cap V$. Conversely, since $U \cap V \ll_F V$ as $V$ is an $F$-supplement of $U$, we get $U \cap V \subseteq \mathrm{Rad}_F(V)$. □

For modules which don't have maximal submodules containing $F$ (for instance, for the $\mathbb{Z}$-module $\mathbb{Q}$), we have the following result.

**Proposition 3.** *Let $M$ be a module. Then $\mathrm{Rad}_F(M) = M$ if and only if every finitely generated submodule of $M$ is $F$-small in $M$.*

*Proof.* (⇒) Let $N$ be any finitely generated submodule of $M$. Then $N = Rm_1 + Rm_2 + \cdots + Rm_k$ for some $m_i \in M$. Since, by assumption, $m_i \in \mathrm{Rad}_F(M)$, it follows that $Rm_i \ll_F M$ for all $i = 1, 2, \ldots k$ by Lemma 2. Therefore $N \ll_F M$ by Lemma 1-(3).

(⇐) Clearly, $\mathrm{Rad}_F(M) \subseteq M$. To show that $M \subseteq \mathrm{Rad}_F(M)$, let $x \in M$. Then $Rx \ll_F M$ by assumption, and so $Rx \subseteq \mathrm{Rad}_F(M)$. Thus $x \in \mathrm{Rad}_F(M)$ as desired. □

**Corollary 3.** *Let $M$ be a module. If $\mathrm{Rad}_F(M) = M$, then every finitely generated submodule of $M$ has an $F$-supplement in $M$.*

*Proof.* Since every finitely generated submodule $U$ of $M$ is $F$-small in $M$ by Proposition 3, then $M$ is an $F$-supplement of $U$ in $M$. □

The following result shows $F$-supplements in submodules.

**Theorem 1.** *Let $K \subseteq N \subseteq M$ be submodules.*

1) *If $K$ is an $F$-supplement in $M$, then $K$ is an $F$-supplement in $N$.*
2) *If $N$ is an $F$-supplement in $M$, then*
   (a) *$K$ is an $F$-supplement in $M$ if and only if $K$ is an $F$-supplement in $N$;*
   (b) *$K \ll_F M$ if and only if $K \ll_F N$.*

*Proof.* 1) Since $K$ is an $F$-supplement in $M$, there is a submodule $P \subseteq M$ such that $K + P = M$ and $K \cap P \ll_F K$. By modular law, we have $N = K + N \cap P$. Moreover, $K \cap N \cap P = K \cap P \ll_F K$. Thus $K$ is an $F$-supplement of $N \cap P$ in $N$.

2) Suppose that $N$ is $F$-supplement in $M$. Then there is a submodule $L \subseteq M$ such that $N + L = M$ and $N \cap L \ll_F N$.

(a) ($\Rightarrow$) It follows by (1).
($\Leftarrow$) Since $K$ is an $F$-supplement in $N$, there is a submodule $T \subseteq N$ such that $K+T = N$ and $K \cap T \ll_F K$. So, $K+(T+L) = N+L = M$. Assume that $K' + (T + L) = M$ for a submodule $F \subseteq K' \subseteq K$. Since $F \subseteq K' + T$ and $N$ is $F$-supplement of $L$ in $M$, it follows that $K' + T = N$ by the minimality of $N$. Now by the minimality of $K$, we conclude that $K' = K$. This means that $K$ is $F$-supplement of $T + L$ in $M$.

(b) ($\Rightarrow$) Assume that $K + X = N$ for a submodule $F \subseteq X \subseteq N$. Then $K + X + L = N + L = M$, and so $X + L = M$ since $F \subseteq X + L$ and $K \ll_F M$ by assumption. Thus by modular law, we have $N = X + N \cap L$. Since $N \cap L \ll_F N$, it follows that $X = N$. Hence $K \ll_F N$.
($\Leftarrow$) It is always true by Lemma 1-(1). □

Supplement submodules and $F$-supplement submodules coincide in a module under some extra conditions over $F$.

**Remark 1.** Clearly, supplement submodules and 0-supplement submodules coincide in a module, and any supplement submodule containing $F$ is always $F$-supplement. However an $F$-supplement submodule need not be supplement in general (since $F$-smallness need not imply smallness, see Example 1). But, for example, if $F \ll M$ and $V$ is an $F$-supplement of $U$ in $M$, then we have $F \ll V$ (by Theorem 1-(2b)), and so $U \cap V \ll_F V$ implies that $U \cap V \ll V$ (by [1, Proposition 2.3]). This means that $V$ is a supplement of $U$ in $M$.

**Proposition 4.** *Let $M$ be a module. If $N$ is an $F$-supplement of $U$ in $M$, then for $K \subseteq U$, $(N + K)/K$ is an $(F + K)/K$-supplement of $U/K$ in $M/K$.*

*Proof.* Since $N$ is an $F$-supplement of $U$ in $M$, we have $U + N = M$ and $U \cap N \ll_F N$. So, we have $(U/K)+(N+K)/K = (U+N+K)/K = M/K$, and by modular law, $(U/K) \cap (N + K)/K = (U \cap N + K)/K$. Since $U \cap N \ll_F N$, it follows by Lemma 1-(1) that $(U \cap N + K)/K \ll_{(F+K)/K}$

$(N+K)/K$ (by considering the epimorphism $f: N \to (N+K)/K$). Thus $(N+K)/K$ is an $(F+K)/K$-supplement of $U/K$ in $M/K$. □

The converse of the previous statement is true under a special condition.

**Proposition 5.** *Let $K \subseteq N \subseteq M$ be submodules. If $K$ is an $F$-supplement in $M$, and $N/K$ is an $(F+K)/K$-supplement in $M/K$, then $N$ is an $F$-supplement in $M$.*

*Proof.* First of all, since $F \subseteq K$, we see by assumption that $N/K$ is a supplement in $M/K$. Now, let $K$ be an $F$-supplement of a submodule $K'$ in $M$. Then $K + K' = M$ and $K \cap K' \ll_F K$. Moreover, let $N/K$ be a supplement of $N'/K$ in $M/K$ where $K \subseteq N'$. Then we have $N/K + N'/K = M/K$ and $N/K \cap N'/K = (N \cap N')/K \ll N/K$. Since $M = N \cap N' + K'$ as $K \subseteq N \cap N'$, and $N + N' = M$ it follows that $M = N + K' \cap N'$ by [2, 1.24]. Since $N = K + (N \cap K')$ by modular law, we get $N \cap (K' \cap N')/(K \cap K') \ll N/(K \cap K')$ by [2, 2.3-(1)]. So, clearly, we have $N \cap (K' \cap N')/(K \cap K') \ll_{F+(K\cap K')/(K\cap K')} N/(K \cap K')$. Thus, the fact that $K \cap K' \ll_F N$, which follows from $K \cap K' \ll_F K$, implies that $N \cap (K' \cap N') \ll_F N$ by Lemma 1-(2). This means that $N$ is $F$-supplement of $K' \cap N'$ in $M$. □

## 2. *F*-supplemented modules

In this section, we define the concept of $F$-supplemented modules and we give a characterization for finitely generated $F$-supplemented modules.

A module $M$ is called $F$-*supplemented* if every submodule of $M$ has an $F$-supplement in $M$.

**Proposition 6.** *Let $M$ be a module. Assume that $U$ and $M_1$ are submodules of $M$, where $M_1$ is $F$-supplemented. If $M_1 + U$ has an $F$-supplement in $M$, then $U$ also has an $F$-supplement in $M$.*

*Proof.* Let $X$ be an $F$-supplement of $M_1 + U$ in $M$. Since $M_1$ is $F$-supplemented, the submodule $(X+U) \cap M_1$ has an $F$-supplement in $M_1$, say $Y$. We claim that $X + Y$ is an $F$-supplement of $U$ in $M$. First, we have $M = X + M_1 + U = X + (Y + (X+U) \cap M_1) + U = (X+Y) + U$. Next, since $Y \subseteq M_1$, we have $Y \cap (X+U) = (Y \cap M_1) \cap (X+U) = Y \cap [(X+U) \cap M_1] \ll_F Y$, and since $Y + U \subseteq M_1 + U$, it follows that $X \cap (Y+U) \ll_F X$. Therefore, the inclusion $(X+Y) \cap U \subseteq X \cap (Y+U) + Y \cap (X+U)$ implies that $(X+Y) \cap U \ll_F X + Y$ by Lemma 1, as claimed. □

**Remark 2.** Since the zero submodule 0 is small in any module $M$, we observe that $M$ is always a supplement of 0 in $M$, and vice versa. Likewise, since any submodule of $M$ contained in $F$ is $F$-small in $M$, we see that $F$ is $F$-small in $M$. So, $M$ is an $F$-supplement of $F$ in $M$ by Proposition 1, because $F + M = M$ and $F \cap M = F \ll_F M$. In fact, $F$ is also an $F$-supplement of $M$ in $M$ since $F \ll_F F$.

**Proposition 7.** *If $M_1$ and $M_2$ are $F$-supplemented modules, then $M = M_1 + M_2$ is also an $F$-supplemented module.*

*Proof.* Let $U \subseteq M$. Since $M_1 + (M_2 + U)$ has trivially an $F$-supplement $F$ in $M$, it follows, by Proposition 6, that there is an $F$-supplement for $M_2 + U$, and so there is an $F$-supplement for $U$ in $M$. □

**Corollary 4.** *Every finite (direct) sum of $F$-supplemented modules is $F$-supplemented.*

Now, we investigate factor modules of $F$-supplemented modules.

**Proposition 8.** *If $M$ is an $F$-supplemented module, then $M/\operatorname{Rad}_F(M)$ is semisimple (and so supplemented).*

*Proof.* Let $X/\operatorname{Rad}_F(M) \subseteq M/\operatorname{Rad}_F(M)$. Since $M$ is $F$-supplemented, there is an $F$-supplement $Y$ of $X$ in $M$, that is, $X + Y = M$ and $X \cap Y \ll_F Y$. So, we have $(X/\operatorname{Rad}_F(M)) + (Y + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) = M/\operatorname{Rad}_F(M)$, and $(X/\operatorname{Rad}_F(M)) \cap (Y + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) = X \cap (Y + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) = (X \cap Y + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) = 0$, because $X \cap Y \ll_F Y$ implies that $X \cap Y \subseteq \operatorname{Rad}_F(M)$. Thus $X/\operatorname{Rad}_F(M)$ is a direct summand of $M/\operatorname{Rad}_F(M)$. Hence $M/\operatorname{Rad}_F(M)$ is semisimple. □

**Proposition 9.** *Let $M$ be a module and let $K \subseteq M$ be a submodule. If $M$ is $F$-supplemented, then $M/K$ is $(F + K)/K$-supplemented.*

*Proof.* Take any submodule $N/K$ of $M/K$ where $K \subseteq N \subseteq M$. Since $M$ is $F$-supplemented, $N$ has an $F$-supplement in $M$, say $V$. Thus $(V+K)/K$ is an $(F + K)/K$-supplement of $N/K$ in $M/K$ by Proposition 4. Hence $M/K$ is $(F + K)/K$-supplemented. □

**Theorem 2.** *Let $M$ be a module. Then $M$ is $F$-supplemented if and only if $M/F$ is supplemented.*

*Proof.* ($\Rightarrow$) Suppose that $M$ is $F$-supplemented. Then, by Proposition 9, we see that $M/F$ is 0-supplemented, that is, supplemented.

($\Leftarrow$) Take any submodule $U \subseteq M$. Since $M/F$ is supplemented, $(U+F)/F$ has a supplement in $M/F$, say $V/F$. Then $[(U+F)/F]+V/F = M/F$ which implies that $U+V=M$, and $[U\cap V)+F]/F = [(U+F)/F]\cap V/F \ll V/F$ from which it follows that $U\cap V \ll_F V$ by [1, Proposition 2.9]. Thus $V$ is $F$-supplement of $U$ in $M$. Hence $M$ is $F$-supplemented. □

Taking $F=0$, we obtain the following corollary.

**Corollary 5.** *A module $M$ is 0-supplemented if and only if it is supplemented.*

A nonzero module $M$ is called *hollow* if every proper submodule is small in $M$; and *local* if it has a largest submodule (namely $\mathrm{Rad}(M)$). It is clear that local modules are hollow, and that any finitely generated module is hollow if and only if it is local (see, for example, [2, p. 15]).

Let $M$ be a nonzero module and let $F$ be a proper submodule of $M$. We call $M$ an *$F$-hollow* module if every proper submodule containing $F$ is $F$-small in $M$; and *$F$-local* if $\mathrm{Rad}_F(M)$ is the largest submodule of $M$ containing $F$ (i.e. a proper submodule which contains all other proper submodules containing $F$). In this case, $\mathrm{Rad}_F(M) \ll_F M$.

**Remark 3.** Since $M$ is hollow if and only if every proper submodule is $F$-small in $M$ (see [1, Proposition 2.21]), it follows easily that a hollow module is $F$-hollow, and the converse is true when $F=0$. Moreover, local modules are always $F$-local (because, in this case, $F \subseteq \mathrm{Rad}(M)$ and so $\mathrm{Rad}(M)=\mathrm{Rad}_F(M)$), and the converse is also true for $F=0$.

In general, $F$-hollow (respectively $F$-local) modules need not be hollow (respectively local).

**Example 2.** Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z}$, and let $F=4\mathbb{Z}$. Then we have $\mathrm{Rad}_F(M)=2\mathbb{Z}$, and so $M$ is $F$-local which implies that $M$ is $F$-hollow. But, since $\mathrm{Rad}(M)=0$, $M$ is not local, and so it is not hollow as a cyclic module.

Now we give some results which are needed to characterize finitely generated $F$-supplemented modules.

**Proposition 10.** *A nonzero module $M$ is $F$-local if and only if it is $F$-hollow and $\mathrm{Rad}_F(M) \neq M$.*

*Proof.* ($\Rightarrow$) Let $K$ be a proper submodule of $M$ containing $F$. Since $M$ is $F$-local, $K \subseteq \mathrm{Rad}_F(M)$. Now since $\mathrm{Rad}_F(M) \ll_F M$, it follows that $K \ll_F M$ by Lemma 1-(2), and that $\mathrm{Rad}_F(M) \neq M$.

($\Leftarrow$) First, we claim that $\overline{M} = M/\operatorname{Rad}_F(M)$ is simple. By assumption, $\overline{M} \neq 0$. Now, if $N/\operatorname{Rad}_F(M) \subsetneq \overline{M}$, then we have $F \subseteq N \subsetneq M$. Since $M$ is $F$-hollow, then $N \ll_F M$, and so $N \subseteq \operatorname{Rad}_F(M)$. Hence $N = \operatorname{Rad}_F(M)$. Next, if $F \subseteq X \subsetneq M$, then $(X + \operatorname{Rad}_F(M))/\operatorname{Rad}_F(M) \subsetneq \overline{M}$. So, $X + \operatorname{Rad}_F(M) = \operatorname{Rad}_F(M)$ since $\overline{M}$ is simple, which implies that $X \subseteq \operatorname{Rad}_F(M)$. Hence $\operatorname{Rad}_F(M)$ is the largest submodule of $M$ that contains $F$, that is, $M$ is $F$-local. □

**Proposition 11.** *Let $M$ be a module. If $M/\operatorname{Rad}_F(M)$ is semisimple and $\operatorname{Rad}_F(M) \ll_F M$, then every proper submodule of $M$ containing $F$ is contained in a maximal submodule.*

*Proof.* Consider the natural epimorphism $\sigma : M \to M/\operatorname{Rad}_F(M) = \overline{M}$. Let $F \subseteq U \subsetneq M$. Since $\operatorname{Rad}_F(M) \ll_F M$, we have $\sigma(U) \neq \overline{M}$. Thus $\sigma(U)$ is contained in a maximal submodule $\overline{N}$ of $\overline{M}$ (as $\overline{M}$ is semisimple). Hence $U$ is contained in the maximal submodule $\sigma^{-1}(\overline{N})$ of $M$. □

**Corollary 6.** *If $M$ is $F$-supplemented and $\operatorname{Rad}_F(M) \ll_F M$, then every proper submodule of $M$ containing $F$ is contained in a maximal submodule.*

*Proof.* It follows from Propositions 8 and 11. □

**Proposition 12.** *Every $F$-hollow module $M$ is $F$-supplemented.*

*Proof.* Let $U$ be any submodule of $M$. Since $F$ is always an $F$-supplement of $M$ in $M$ by Remark 2, we may assume that $U \neq M$. There are two cases to consider. First, if $F \subseteq U$, then $U \ll_F M$ since $M$ is $F$-hollow by assumption. So, $M$ is an $F$-supplement of $U$ in $M$, because $U + M = M$ and $U \cap M = U \ll_F M$. Next, assume that $F \nsubseteq U$. If $U + F = M$, then $F$ is an $F$-supplement of $U$ in $M$ since $U \cap F \subseteq F \ll_F F$. Otherwise, $U + F \neq M$. Therefore, $U + F \ll_F M$ by assumption, from which we get $U \ll_F M$. So, $M$ is an $F$-supplement of $U$ in $M$ as in the first case. Thus, in each case, $U$ has an $F$-supplement in $M$. Hence $M$ is $F$-supplemented. □

**Proposition 13.** *Let $M$ be a module. Every $F$-supplement of a maximal submodule of $M$ containing $F$ is $F$-local.*

*Proof.* Let $U$ be a maximal submodule of $M$ containing $F$. Assume that $V$ is an $F$-supplement of $U$ in $M$. Then by Corollary 2, we have $\operatorname{Rad}_F(V) = V \cap U$ is the unique maximal submodule of $V$ containing $F$. In fact, $\operatorname{Rad}_F(V)$ is the largest proper submodule of $V$ containing $F$, because, for any submodule $F \subseteq X \subseteq V$ with $X \nsubseteq \operatorname{Rad}_F(V)$, we have $X + \operatorname{Rad}_F(V) = V$ since $\operatorname{Rad}_F(V)$ is maximal in $V$, and so $X = V$ since $\operatorname{Rad}_F(V) \ll_F V$ as $V$ is an $F$-supplement of $U$. Hence $V$ is $F$-local. □

Let $\{M_i\}_I$ be a family of modules for some index set $I$. The sum $M = \sum_I M_i$ is called *irredundant* if, for every $i_0 \in I$, $\sum_{i \neq i_0} M_i \neq M$ holds.

**Theorem 3.** *For a module $M$, the following are equivalent.*

1) *$M$ is a sum of $F$-hollow submodules and $\mathrm{Rad}_F(M) \ll_F M$.*
2) *Every proper submodule of $M$ containing $F$ is contained in a maximal submodule, and*
   (a) *every maximal submodule containing $F$ has an $F$-supplement in $M$, or*
   (b) *every submodule $K$ of $M$, with $M/K$ is finitely generated, has an $F$-supplement in $M$.*
3) *$M$ is an irredundant sum of $F$-local modules and $\mathrm{Rad}_F(M) \ll_F M$.*

*Proof.* (1) $\Leftrightarrow$ (3) Let $M = \sum_I L_i$ with $F$-hollow submodules $L_i$ of $M$ for some index set $I$. Then $M/\mathrm{Rad}_F(M) = \sum_I (L_i + \mathrm{Rad}_F(M))/\mathrm{Rad}_F(M)$. Since $(L_i + \mathrm{Rad}_F(M))/\mathrm{Rad}_F(M) \cong L_i/(L_i \cap \mathrm{Rad}_F(M))$, we claim that these factors are simple or zero. If $L_i \cap \mathrm{Rad}_F(M) \neq L_i$ and $X/(L_i \cap \mathrm{Rad}_F(M))$ is any proper submodule of $L_i/(L_i \cap \mathrm{Rad}_F(M))$, then we have $F \subseteq X \subsetneqq L_i$ as $F \subseteq L_i \cap \mathrm{Rad}_F(M)$. But, then $X \ll_F L_i$ as $L_i$ is $F$-hollow, and so $X \subseteq \mathrm{Rad}_F(L_i) \subseteq L_i \cap \mathrm{Rad}_F(M)$. This implies that $X = L_i \cap \mathrm{Rad}_F(M)$, and $L_i/(L_i \cap \mathrm{Rad}_F(M))$ is simple as claimed. Therefore, we obtain that $M/\mathrm{Rad}_F(M) = \oplus_J (L_i + \mathrm{Rad}_F(M))/\mathrm{Rad}_F(M)$ for some subset $J \subseteq I$. Since $\mathrm{Rad}_F(M) \ll_F M$, it follows that $M = \sum_J L_i$ with $F$-local modules $L_i$ by Proposition 10 (since $\mathrm{Rad}_F(L_i) \neq L_i$).

Since $(b) \Rightarrow (a)$ is clear, it suffices to prove the following implications:

(3) $\Rightarrow$ (2)$(b)$ Clearly, $M/\mathrm{Rad}_F(M)$ is semisimple (see (1) $\Leftrightarrow$ (3)). Since $\mathrm{Rad}_F(M) \ll_F M$, it follows by Proposition 11 that every proper submodule of $M$ containing $F$ is contained in a maximal submodule.

Now, assume that $K$ is a submodule of $M$ with $M/K$ is finitely generated. By assumption, there are finitely many $F$-local (and so $F$-hollow) submodules $L_1, L_2, \ldots, L_n$ with $M = K + L_1 + L_2 + \cdots + L_n$. Then by Proposition 12 and Corollary 4, it follows that $L_1 + L_2 + \cdots + L_n$ is $F$-supplemented. Moreover, since $M$ has trivially an $F$-supplement $F$ in $M$, by Proposition 6, $K$ also has an $F$-supplement in $M$.

(2)$(a)$ $\Rightarrow$ (1) Let $H = \sum_I L_i$ with $F$-hollow submodules $L_i$ of $M$ for some index set $I$. Observe that $F \subseteq L_i$ for each $i \in I$. We show that $H = M$. Suppose to the contrary that $H \neq M$. Since $F \subseteq H$, it follows by assumption that $H$ is contained in a maximal submodule $N$ of $M$. By assumption, $N$ has an $F$-supplement in $M$, say $L$. Since $F \subseteq H \subseteq N$, by Proposition 13, we obtain that $L$ is $F$-local, and so it is $F$-hollow. Thus we get $L \subseteq H \subseteq N$ by the choice of $H$, which implies that $N = M$ as $N + L = M$. This contradiction shows that $H = M$. □

The following result gives a characterization for finitely generated $F$-supplemented modules.

**Corollary 7.** *For a finitely generated module $M$, the following statements are equivalent.*

1) *$M$ is $F$-supplemented.*
2) *Every maximal submodule containing $F$ has an $F$-supplement in $M$.*
3) *$M$ is a sum of $F$-hollow submodules.*
4) *$M$ is an irredundant sum of $F$-local submodules.*

*Proof.* Since $M$ is finitely generated, first, we have $\mathrm{Rad}_F(M) \ll_F M$. Indeed, assume that $\mathrm{Rad}_F(M) + X = M$ with $F \subseteq X$, and that $X \neq M$. Then $X$ is contained in a maximal submodule $N$ of $M$. So, we have $\mathrm{Rad}_F(M) + N = M$. Since $F \subseteq N$, it follows that $\mathrm{Rad}_F(M) \subseteq N$, and so $N = M$. It is a contradiction. Next, for every submodule $K$ of $M$, we have $M/K$ is finitely generated. Thus the proof follows immediately by Theorem 3. □

## 3. Amply $F$-supplemented modules

In this section, we introduce and characterize amply $F$-supplemented modules.

Let $M$ be a module. We say a submodule $U \subseteq M$ has *ample* (or *enough*) *$F$-supplements* in $M$ if, for every $V \subseteq M$ with $U + V = M$, there is an $F$-supplement $V'$ of $U$ with $V' \subseteq V$. If every submodule of $M$ has ample $F$-supplements in $M$, then we call $M$ *amply $F$-supplemented.*

Since for each submodule $U \subseteq M$, we have $U + M = M$, it follows that every amply $F$-supplemented module is $F$-supplemented.

**Proposition 14.** *Let $M$ be an amply $F$-supplemented module. Then*

1) *Every $F$-supplement submodule of $M$ is amply $F$-supplemented.*
2) *Every direct summand of $M$ containing $F$ is amply $F$-supplemented.*

*Proof.* 1) Let $V \subseteq M$ be an $F$-supplement of $U \subseteq M$. For $X \subseteq V$, assume that $V = X + Y$. Then $M = U + V = (U + X) + Y$, and so there is an $F$-supplement $Y'$ of $U + X$ in $M$ with $Y' \subseteq Y$ by assumption. We claim that $Y'$ is an $F$-supplement of $X$ in $V$. Since $X \cap Y' \subseteq (U + X) \cap Y' \ll_F Y'$, we have $X \cap Y' \ll_F Y'$ by Lemma 1-(2). Now, since $F \subseteq X + Y'$, $M = U + X + Y'$ implies that $V = X + Y'$ by the minimality of $V$.

2) Since any direct summand of $M$ containing $F$ is an $F$-supplement in $M$ in the obvious way, it is amply $F$-supplemented by (1). □

**Proposition 15.** *Let $M$ be a module with $M = U_1 + U_2$. If the submodules $U_1, U_2$ have ample $F$-supplements in $M$, then so does $U_1 \cap U_2$.*

*Proof.* Let $V \subseteq M$ with $U_1 \cap U_2 + V = M$. Then by modular law, we have $U_1 \cap U_2 + U_2 \cap V = U_2$, and so $U_1 + U_2 \cap V = M$. So, by assumption, there is an $F$-supplement $V_2'$ of $U_1$ with $V_2' \subseteq U_2 \cap V$. Similarly, there is also an $F$-supplement $V_1'$ of $U_2$ with $V_1' \subseteq U_1 \cap V$. Thus, for $V_1' + V_2' \subseteq V$, we obtain that $U_1 \cap U_2 + (V_1' + V_2') = M$, and $(V_1' + V_2') \cap (U_1 \cap U_2) = (V_1' \cap U_2) + (V_2' \cap U_1) \ll_F V_1' + V_2'$ by Lemma 1. Hence $V_1' + V_2'$ is the desired $F$-supplement of $U_1 \cap U_2$ in $M$. □

**Proposition 16.** *Let $M$ be a module, and $U, K \subseteq M$. If $K \ll_F M$ and $U + K$ has ample $F$-supplements, then $U$ has also ample $F$-supplements.*

*Proof.* Let $V \subseteq M$ with $U + V = M$. Then $M = (U + K) + V$, and so there is an $F$-supplement $V' \subseteq V$ of $U + K$ by assumption. Since $K \ll_F M$ and $F \subseteq V' + U$, the equality $M = V' + U + K$ implies that $V' + U = M$. Moreover, $V' \cap U \subseteq V' \cap (U + K) \ll_F V'$ implies that $V' \cap U \ll_F V'$ by Lemma 1-(2). Hence $V'$ is an $F$-supplement of $U$ in $M$. □

Now we give a characterization for amply $F$-supplemented modules.

**Theorem 4.** *For a module $M$, the following statements are equivalent.*

1) *$M$ is amply $F$-supplemented.*
2) *Every submodule $U \subseteq M$ is of the form $U = X + Y$, where $X$ is $F$-supplemented and $Y \ll_F M$.*
3) *For every submodule $U \subseteq M$, there is an $F$-supplemented submodule $X \subseteq U$ such that $U/X \ll M/X$.*

   *If $M$ is finitely generated, then $(1) - (3)$ are equivalent to:*

4) *Every maximal submodule containing $F$ has ample $F$-supplements in $M$.*

*Proof.* (1) ⇒ (2) Clearly, $M$ is $F$-supplemented. So, let $V$ be an $F$-supplement of $U$ in $M$. Then $U + V = M$, and by assumption there is an $F$-supplement $X$ of $V$ in $M$ with $X \subseteq U$. Therefore, $U = U \cap M = U \cap (X + V) = X + U \cap V$, where $U \cap V \ll_F M$ since $U \cap V \ll_F V \subseteq M$, and $X$ is (amply) $F$-supplemented by Proposition 14-(1).

(2) ⇒ (3) Let $U = X + Y$, where $X$ is $F$-supplemented and $Y \ll_F M$. Then $U/X = (Y + X)/X \ll_{(F+X)/X} M/X$ by Lemma 1-(1), that is, $U/X \ll M/X$ since $F \subseteq X$ implies that $(F + X)/X = 0$

(3) ⇒ (1) Let $U \subseteq M$ with $U + V = M$. By assumption, there is an $F$-supplemented submodule $X$ of $V$ in $M$ with $V/X \ll M/X$. So, the equality $(U + X)/X + V/X = M/X$ implies that $U + X = M$. Now, the submodule $U \cap X \subseteq X$ has an $F$-supplement in $X$, say $V'$. Therefore, we get $M = U + (U \cap X) + V' = U + V'$, and $U \cap V' = (U \cap X) \cap V' \ll_F V'$. Thus, $V'$ is an $F$-supplement of $U$ in $M$ with $V' \subseteq V$. Hence $M$ is amply $F$-supplemented.

(1) ⇒ (4) Clear.

(4) ⇒ (1) Now suppose that $M$ is finitely generated, and that all maximal submodules of $M$ containing $F$ have ample $F$-supplements (so $F$-supplements). Then $M$ is $F$-supplemented by Corollary 7, and $M/\operatorname{Rad}_F(M)$ is semisimple by Proposition 8. Therefore, for any submodule $U \subseteq M$, we have $M/(U+\operatorname{Rad}_F(M))$ is semisimple. Thus, $\operatorname{Rad}(M/(U+\operatorname{Rad}_F(M))) = 0$, and so $U+\operatorname{Rad}_F(M) = \cap_{i=1}^{k} N_i$, where $N_i$'s are maximal submodules of $M$ containing $U + \operatorname{Rad}_F(M)$ (and so containing $F$). From assumption and Proposition 15, we obtain that $U + \operatorname{Rad}_F(M)$ has ample $F$-supplements. Since $\operatorname{Rad}_F(M) \ll_F M$, Proposition 16 implies that $U$ also has ample $F$-supplements. □

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