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RESEARCH ARTICLE

# On the $m$-ary partition numbers

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Abstract. We give explicit formulas and recurrences for the $m$-ary partition numbers and relate them to Toeplitz-Hessenberg determinants. Some of these results are direct analogues of similar statements for the classical (unrestricted) partition numbers.

## 1. Introduction

Let $m$ be a fixed integer such that $m \geqslant 2$. For a positive integer $n$, let $b_m(n)$ denote the number of $m$-ary partitions of $n$, i.e. the number of ways of writing $n$ as a sum of powers of $m$ using non-negative exponents with repetitions allowed and the order of the summands not being taken into account. We also set $b_m(0) = 1$.

The special case of binary partitions, i.e. the case $m = 2$, was apparently first studied by Euler in 1750. The recurrence formulas

$$b_2(2n+1) = b_2(2n), \qquad b_2(2n) = b_2(2n-1) + b_2(n),$$

for $n \geqslant 0$ are well-known. The sequence $\{b_2(n) : n \in \mathbb{N}\}$ is labelled A018819 in the On-Line Encyclopedia of Integer Sequences (available at http://oeis.org/AO18819).

Various number-theoretic properties as well as the asymptotic behavior of the $m$-ary partition sequence $\{b_m(n) : n \in \mathbb{N}\}$ and some of its variants

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(for the binary and for the general case) have been extensively studied in the literature. We refer the reader to the work of Alkauskas [1,2], Andrews [4], Calkin and Wilf [8], Churchhouse [10], de Bruijn [12], Flajolet and Sedgewick [15], Gupta [17–19], Hirschhorn and Loxton [21], Hirschhorn and Sellers [22], Mahler [23–25], Meinardus [27], Murty and Murty [29, Ch. 7], Pennington [30], Richmond [33], Rödseth [34], Rödseth and Sellers [35,36] for more information, emphasizing the fact that the latter list is by no means exhaustive.

The generating function for the sequence $\{b_m(n) : n \in \mathbb{N}\}$ is

$$f_m(x) = \sum_{n=0}^{\infty} b_m(n)x^n = \prod_{n=0}^{\infty} \sum_{i=0}^{\infty} x^{im^n} = \prod_{n=0}^{\infty} (1 - x^{m^n})^{-1},$$

which implies the functional equation $f_m(x^m) = (1-x)f_m(x)$. Besides its combinatorial importance, the generating function $f_m(x)$ is of special interest in transcendental number theory. For example, as Mahler has shown in [25] (see also its corrigendum [24]), the values of $f_2(x)$ are always transcendental for every algebraic number $x$ such that $0 < |x| < 1$.

In this paper we give explicit formulas and recurrence relations for the sequence $\{b_m(n) : n \in \mathbb{N}\}$. We also relate these formulas to Toeplitz-Hessenberg determinants (first introduced in [37] and [20]). Some of these results are direct analogues of similar statements for the classical (unrestricted) partition numbers.

For a positive integer $n$, let $\xi_m(n)$ denote the sum of all divisors of $n$ that are powers of $m$ with non-negative exponent. Note that $\xi_m(1) = 1$ and $\xi_m(n) \geqslant 1$, for all $n \geqslant 1$. We also define $\xi_m(0) = 1$. The sequence $\{\xi_m(n) : n \in \mathbb{N}\}$ is the $m$-ary analogue of the classical sum-of-divisors sequence.

Our first result in this paper is an elementary yet explicit formula for $b_m(n)$ given below. We are not aware of any other formula in the existing literature that concretely expresses the integer $b_m(n)$ as a finite sum of rational numbers. For more profound analogous results for the classical partition function we refer the reader to the formulas of Rademacher in [31] and [32] and of Bruinier and Ono in [7]. It would be interesting to look for $m$-ary analogues of the latter formulas, if such analogues exist.

**Theorem 1.** *For every positive integer $n$, the $m$-ary partition number $b_m(n)$ is expressed as the finite sum*

$$b_m(n) = \sum_{n_1+2n_2+3n_3+\cdots=n} \frac{\xi_m(1)^{n_1}\xi_m(2)^{n_2}\xi_m(3)^{n_3}\cdots}{n_1!\ n_2!\ n_3!\cdots\ 1^{n_1}2^{n_2}3^{n_3}\cdots}.$$

**Remark 1.** It is a well-known and easy to prove fact that the number of summands on the right-hand side of the latter formula equals $p(n)$, i.e. the number of partitions of the integer $n$.

Now, for a positive integer $n$, let $d_m(n) \in \mathbb{N}$ be defined as follows:

If $n$ is not the sum of distinct powers of $m$, let $d_m(n) = 0$. If $n$ is the sum of an even number of distinct powers of $m$, let $d_m(n) = 1$. If $n$ is the sum of an odd number of distinct powers of $m$, let $d_m(n) = -1$. We also set $d_m(0) = 1$. Note that

$$\sum_{n=0}^{\infty} d_m(n)\; x^n = \prod_{n=0}^{\infty} (1 - x^{m^n}) = \frac{1}{f_m(x)}.$$

Note also that, in the binary case, we have $d_2(n) = (-1)^{t(n)}$, where $t(n)$ is the $n$-th term of the Prouhet-Thue-Morse sequence, which is discussed extensively in [3]. We now express the $m$-ary partition numbers $b_m(n)$ as determinants of certain Toeplitz-Hessenberg matrices with entries in the set $\{-1, 0, 1\}$, specifically with entries given by the numbers $d_m(n)$. This is the $m$-ary analogue of a formula obtained by Malenfant [26] for the classical partition function. Conversely, we can also express the numbers $d_m(n)$ as determinants of certain Toeplitz-Hessenberg matrices with entries given by the numbers $b_m(n)$:

**Theorem 2.** *For every positive integer $n$, we have*

$$b_m(n) = (-1)^n \det \begin{bmatrix} d_m(1) & 1 & 0 & \cdots & 0 \\ d_m(2) & d_m(1) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_m(n-1) & d_m(n-2) & d_m(n-3) & \cdots & 1 \\ d_m(n) & d_m(n-1) & d_m(n-2) & \cdots & d_m(1) \end{bmatrix}.$$

*Similarly,*

$$d_m(n) = (-1)^n \det \begin{bmatrix} b_m(1) & 1 & 0 & \cdots & 0 \\ b_m(2) & b_m(1) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_m(n-1) & b_m(n-2) & b_m(n-3) & \cdots & 1 \\ b_m(n) & b_m(n-1) & b_m(n-2) & \cdots & b_m(1) \end{bmatrix}.$$

**Remark 2.** The proof of Theorem 1.2 is very simple and based on Cramer's rule for solving linear systems. To be more specific, the fact that the coefficients of the reciprocal of a formal power series can be expressed as determinants of certain Toeplitz-Hessenberg matrices whose entries are functions of the coefficients of the original formal power series is well-known and simple to prove (e.g. page 157 in Comtet's book [11]). The same fact has recently been applied in a number of similar contexts and settings by Chen [9], Malenfant [26] and Vella [38]. On the other hand, what is far from simple is an ingenious and quite general method employed by Brioschi [6] relating Toeplitz-Hessenberg determinants to complete homogeneous forms in the roots of a given polynomial and leading to a general formula expressing these determinants in terms of multinomial coefficients. The same formula was apparently discovered earlier by Fergola [14] using a different method. We refer the reader to Muir's book [28] for an extensive and substantiated survey of the historical development of the theory of determinants. An excellent (albeit significantly shorter) account of the Brioschi-Fergola general formula is also described in the preprint [13] by Fera and Talamini, where two different proofs of the formula are presented. In light of what appears to be a resurgence of recent interest around this circle of ideas, we believe that the Brioschi-Fergola formula deseves to be more widely known. We would also like to point out that, to the best of our knowledge and ability, the Brioschi-Fergola formula applied to the particular setting of Theorem 1.2 does not immediately reduce to the formula given in Theorem 1.1 in any obvious way; for one thing, the summands in the formula given in Theorem 1.1 are not necessarily integers, while those given by the Brioschi-Fergola formula are.

We also give recurrence formulas for the numbers $b_m(n)$ and $d_m(n)$. The recurrence relation for $b_m(n)$ is the $m$-ary analogue of a rather well known recurrence relation for the classical partition function (see [5], Entry 52, page 108):

**Theorem 3.** *For $n \geqslant 1$, we have*

$$b_m(n) = \frac{1}{n} \sum_{j=0}^{n-1} \xi_m(n-j)\, b_m(j) \quad \text{and} \quad d_m(n) = \frac{-1}{n} \sum_{j=0}^{n-1} \xi_m(n-j)\, d_m(j).$$

## 2. Proofs

*Proof of Theorem 1.* As usual, let $\log(\cdot)$ denote the natural logarithm. As formal power series, we have

$$\log(f_m(x)) = \log\left(\prod_{n=0}^{\infty} (1 - x^{m^n})^{-1}\right) = -\sum_{n=0}^{\infty} \log(1 - x^{m^n})$$

$$= \sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\frac{x^{im^n}}{i} = \sum_{n=0}^{\infty}\sum_{i=1}^{\infty}\frac{x^{im^n}}{im^n}m^n.$$

Define integers $a_k$, for $k \in \{1, 2, \ldots\}$, by setting $a_k = k$, if $k$ is a power of $m$, and $a_k = 0$, if $k$ is not a power of $m$. Then

$$\log(f_m(x)) = \sum_{k=1}^{\infty}\frac{x^k}{k}\sum_{d|k}a_d = \sum_{k=1}^{\infty}\frac{x^k}{k}\xi_m(k).$$

Therefore,

$$f_m(x) = e^{\log(f_m(x))} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=1}^{\infty}\frac{x^k}{k}\xi_m(k)\right)^n.$$

For a sequence of natural numbers $n_1$, $n_2, \ldots$, with only finitely many non-zero terms, the multinomial coefficient

$$\binom{n_1+n_2+\cdots}{n_1, n_2, \cdots} = \frac{(n_1+n_2+\cdots)!}{n_1!\; n_2!\cdots}$$

makes sense and one can use the “infinite version" of the multinomial theorem

$$\left(\sum_{k=1}^{\infty}y_k\right)^n = \sum_{n_1+n_2+\cdots=n}\binom{n}{n_1, n_2, \ldots}\prod_{k=1}^{\infty}{y_k}^{n_k},$$

to get

$$f_m(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{n_1+n_2+\cdots=n}\binom{n}{n_1, n_2, \ldots}\prod_{k=1}^{\infty}\left(\frac{x^k}{k}\xi_m(k)\right)^{n_k}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{n_1+n_2+\cdots=n}\binom{n}{n_1, n_2, \ldots}\prod_{k=1}^{\infty}\frac{x^{kn_k}}{k^{n_k}}\xi_m(k)^{n_k}$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{n_1+n_2+\cdots=n}\binom{n}{n_1, n_2, \ldots}\frac{x^{n_1+2n_2+\cdots}}{1^{n_1}2^{n_2}\cdots}\xi_m(1)^{n_1}\xi_m(2)^{n_2}\cdots$$

$$= \sum_{r=0}^{\infty}\left(\sum_{n_1+2n_2+\cdots=r}\binom{n_1+n_2+\cdots}{n_1, n_2, \cdots}\frac{\xi_m(1)^{n_1}\xi_m(2)^{n_2}\cdots}{(n_1+n_2+\cdots)!1^{n_1}2^{n_2}\cdots}\right)x^r$$

$$= \sum_{r=0}^{\infty}\left(\sum_{n_1+2n_2+3n_3+\cdots=r}\frac{\xi_m(1)^{n_1}\xi_m(2)^{n_2}\xi_m(3)^{n_3}\cdots}{n_1!\; n_2!\; n_3!\cdots\; 1^{n_1}2^{n_2}3^{n_3}\cdots}\right)x^r.$$

For all $r \geqslant 0$, the coefficient of $x^r$ in the latter series must equal $b_m(r)$ and the theorem follows.

*Proof of Theorem 2.* Comparing coefficients of the same powers of $x$ on both sides of the equality

$$\left(\sum_{i=0}^{\infty} d_m(i)\ x^i\right)\ \left(\sum_{i=0}^{\infty} b_m(i)\ x^i\right) = 1,$$

gives the linear system

$$\sum_{i=0}^{r} d_m(r-i)\ b_m(i) = 0, \text{ for every r} \geqslant 1.$$

Combining the latter equations for $r = 1, \ldots, n$, with $b_m(0) = d_m(0) = 1$, results in the $n \times n$ linear system $AX = B$, where

$$A = \begin{bmatrix} d_m(n-1) & d_m(n-2) & \cdots & 1 \\ d_m(n-2) & d_m(n-3) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix},$$

$$X = \begin{bmatrix} b_m(1) \\ b_m(2) \\ \vdots \\ b_m(n) \end{bmatrix}, \qquad B = \begin{bmatrix} -d_m(n) \\ -d_m(n-1) \\ \vdots \\ -d_m(1) \end{bmatrix}.$$

The determinant of $A$ equals $(-1)^{\frac{n(n+3)}{2}}$. Also, the determinant of the matrix obtained from $A$ by replacing its last column by $B$ equals

$$-\det \begin{bmatrix} d_m(n-1) & d_m(n-2) & \cdots & d_m(1) & d_m(n) \\ d_m(n-2) & d_m(n-3) & \cdots & 1 & d_m(n-1) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_m(1) & 1 & \cdots & 0 & d_m(2) \\ 1 & 0 & \cdots & 0 & d_m(1) \end{bmatrix}$$

$$= (-1)^n \det \begin{bmatrix} d_m(n) & d_m(n-1) & d_m(n-2) & \cdots & d_m(1) \\ d_m(n-1) & d_m(n-2) & d_m(n-3) & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_m(2) & d_m(1) & 1 & \cdots & 0 \\ d_m(1) & 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$(-1)^{\frac{n(n+1)}{2}} \det \begin{bmatrix} d_m(1) & 1 & 0 & \cdots & 0 \\ d_m(2) & d_m(1) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_m(n-1) & d_m(n-2) & d_m(n-3) & \cdots & 1 \\ d_m(n) & d_m(n-1) & d_m(n-2) & \cdots & d_m(1) \end{bmatrix}.$$

Hence, by Cramer's rule, the formula for $b_m(n)$ in Theorem 2 follows. A similar argument (interchanging the roles of $b_m(n)$ and $d_m(n)$) establishes the formula for $d_m(n)$ in Theorem 2.

*Proof of Theorem 3.* The idea of the proof is classically known and due to Euler (as explained by Fuchs and Tabachnikov in Section 3.5 of [16]). To prove the first recurrence formula, we argue as follows:

As formal power series, we have

$$-x\frac{d}{dx}\log\left(1-x^{m^k}\right) = x\frac{d}{dx}\left(\sum_{i=1}^{\infty}\frac{x^{im^k}}{i}\right) = m^k\sum_{i=1}^{\infty}x^{im^k}.$$

Therefore,

$$x\frac{d}{dx}\log\left(f_m(x)\right) = -\sum_{k=0}^{\infty}x\frac{d}{dx}\log\left(1-x^{m^k}\right) = \sum_{k=0}^{\infty}\sum_{i=1}^{\infty}m^k x^{im^k}$$

$$= \sum_{n=1}^{\infty}\xi_m(n)x^n.$$

Hence,

$$x\frac{d}{dx}f_m(x) = f_m(x)\sum_{n=1}^{\infty}\xi_m(n)x^n,$$

which implies that

$$x\frac{d}{dx}\sum_{n=0}^{\infty}b_m(n)x^n = f_m(x)\sum_{n=1}^{\infty}\xi_m(n)x^n.$$

Hence,

$$\sum_{n=1}^{\infty}nb_m(n)x^n = \left(\sum_{n=0}^{\infty}b_m(n)x^n\right)\left(\sum_{n=1}^{\infty}\xi_m(n)x^n\right)$$

$$= \sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1}b_m(j)\ \xi_m(n-j)\right)x^n,$$

and the recurrence formula follows by comparing the coefficients of $x^n$ on both sides of the latter equality.

The second recurrence formula can be proven by a similar argument: The relation (proven above)

$$x\frac{d}{dx}f_m(x) = f_m(x)\sum_{n=1}^{\infty}\xi_m(n)x^n,$$

also implies that

$$x\frac{d}{dx}\left(\frac{1}{f_m(x)}\right) = \frac{-1}{f_m(x)}\sum_{n=1}^{\infty}\xi_m(n)x^n.$$

Therefore,

$$x\frac{d}{dx}\sum_{n=0}^{\infty}d_m(n)x^n = \frac{-1}{f_m(x)}\sum_{n=1}^{\infty}\xi_m(n)x^n,$$

which implies that

$$\sum_{n=1}^{\infty}nd_m(n)x^n = -\left(\sum_{n=0}^{\infty}d_m(n)x^n\right)\left(\sum_{n=1}^{\infty}\xi_m(n)x^n\right)$$
$$= -\sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1}d_m(j)\ \xi_m(n-j)\right)x^n,$$

and the recurrence formula follows by comparing of the coefficients of $x^n$ on both sides of the latter equality.

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