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RESEARCH ARTICLE

On representations of permutations groups as isometry groups of n-semimetric spaces

Oleg Gerdiy and Bogdana Oliynyk

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Abstract. We prove that every finite permutation group can be represented as the isometry group of some n-semimetric space. We show that if a finite permutation group can be realized as the isometry group of some n-semimetric space then this permutation group can be represented as the isometry group of some $(n+1)$-semimetric space. The notion of the semimetric rank of a permutation group is introduced.

Introduction

Permutation groups occur as groups of symmetries of various mathematical objects that preserve basic structure of those objects. The automorphism groups of graphs, boolean functions, ordered sets, isometry groups of metric or semimetric spaces can be considered as permutation groups. The following well-known problem is called sometimes the generalized Koenig's problem. Let X be a discrete structure. Does exist a permutation group (G, X) that the automorphism group $Aut(X)$ is isomorphic to (G, X) as a permutation group (see g.e. [1])? For example, not every permutation group can be represented as the automorphism group of a graphs (see [1], [2]) or as the isometry group of a metric space [3]. There is no complete answers on the questions about characterization of permutation groups that can be realized as a automorphism

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groups of graphs or isometry groups of metric spaces. The case of regular permutation groups is the most studied in this direction. Regular permutation groups that arise as automorphism groups of simple graphs are well-known [4]-[9]. In [10] for some classes of metric spaces it is obtained similar results. But every permutation group (G, X) can be represented as the automorphism group of a system of k-ary relations on X [11] for some k or the automorphism group of a supergraph defined on X [12].

In this paper we continue research in this direction. We prove that every finite permutation group can be represented as the isometry group of some n-semimetric space. Over proof is self-contained. We show that if a finite permutation group can be realized as the isometry group of some n-semimetric space then this permutation group can be represented as the isometry group of some $(n+1)$-semimetric space. It follows from the last statement that we can introduce the notion of the semimetric rank of a permutation group. Additionaly we give some examles.

1. Preliminaries

We will need the following definitions.

Let X be a nonempty set, n be a positive integer, S_n be the symmetric group of degree n. Denote by $\mathbb{R}^+$ the set of all nonnegative integers.

An *n-semimetric* (see [13]) defined on the set X is a function $d^n : X^{n+1} \to \mathbb{R}^+$ satisfying the folowing conditions:

1) the function d^n is fully symmetric, i.e. for arbitrary x_1, x_2, …, $x_{n+1} \in X^{n+1}$ and for arbitrary permutations $\pi \in S_{n+1}$ the equality

$$d^n(x_{1^\pi}, x_{2^\pi}, \ldots, x_{(n+1)^\pi}) = d^n(x_1, x_2, \ldots, x_{n+1}).$$

holds.

2) d^n satisfies the following simplex-type inequality, i.e. for arbitrary $x_1, x_2, \ldots, x_{n+2} \in X$ the inequality

$$d^n(x_1, x_2, \ldots, x_{n+1}) \leqslant d^n(x_2, \ldots, x_{n+2}) + \sum_{i=2}^{n+1} d^n(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+2}) \quad (1)$$

holds.

The set X with an n-semimetric d^n is called *n-semimetric space* and denoted by (X, d^n).

A function $f : X \to X$ is said to be an *isometry*, if for arbitrary $a_1, a_2, \ldots, a_{n+1} \in X$ the equality

$$d_X^n(f(a_1), f(a_2), \ldots, f(a_{n+1})) = d_X(a_1, a_2, \ldots, a_{n+1})$$

holds.

As for metric spaces, all isometries of the space (X, d^n) form a group with respect to superposition. We call this group the isometry group of the n-semimetric space (X, d^n). We can consider this group as a permutation group on the set X and denote it by $(\operatorname{Isom} X, X)$.

2. Representations of permutations groups

The main result of this paper is as follows.

Theorem 1. *Let (G, X) be a finite permutation group. Then there exist a positive integer n and an n-semimetric d^n on X such that the isometry group $(\operatorname{Isom} X, X)$ of the n-semimetric space (X, d^n) is isomorphic as a permutation group to (G, X).*

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$, where m is some positive integer. Denote by n the number

$$2 + 3 + \ldots + m = \frac{m(m+1)}{2} - 1.$$

Fix an element $\bar{x}$ from the set X^{n+1}:

$$\bar{x} = (x_1, \underbrace{x_2, x_2}_{2}, \underbrace{x_3, x_3, x_3}_{3}, \ldots, \underbrace{x_m, \ldots, x_m}_{m}). \tag{2}$$

Define a function $d^n : X^{n+1} \to \mathbb{R}^+$ by the rule:

$$d^n(\bar{y}) = \begin{cases} 1, & \text{if } y_i = (x_{i^\sigma})^g \text{ for some } g \in G \text{ and } \sigma \in S_{n+1},\ 1 \leqslant i \leqslant n, \\ 2, & \text{in other cases,} \end{cases} \tag{3}$$

where $\bar{y}$ is arbitrary element of X^{n+1}.

It is not difficult to verify that the function d^n is fully symmetric and satisfies simplex-type inequality (1). Then d^n is an n-semimetric. We shall show that the groups $(\operatorname{Isom} X, X)$ and (G, X) are isomorphic as permutation groups.

Let g be an element G. The element g acts as one-to-one correspondence on X. Therefore, it is sufficient to show that g preserves the n-semimetric d^n. Indeed, for any $\bar{y} \in X^{n+1}$

$$d^n(y_1^g, y_2^g, \ldots, y_{n+1}^g) = \\ = \begin{cases} 1, & \text{if there exist } g_1 \in G,\ \sigma \in S_{n+1}, \text{ such that } x_{i^\sigma} = (y_i^g)^{g_1}, \\ & 1 \leqslant i \leqslant n+1, \\ 2, & \text{in other cases.} \end{cases}$$

Assume that $d^n(y_1^g, y_2^g, \ldots, y_{n+1}^g) = 1$. Since $g \in G$, the elements g_1 and σ exist if and only if there exist $g_2 \in G$, such that $g_2 = gg_1$ and the equality

$$x_{i^\sigma} = y_i^{g_2}$$

holds for all i, $1 \leqslant i \leqslant n+1$. This means that $d^n(y_1, \ldots, y_{n+1}) = 1$. Hence

$$d^n(y_1^g, y_2^g, \ldots, y_{n+1}^g) = d^n(y_1, \ldots, y_{n+1}),$$

and g is an isometry of (X, d^n).

Let now f be some isometry of (X, d^n). Assume that $\bar{y}$ is an element of X^{n+1} such that $d^n(\bar{y}) = 1$. By the definition the semimetric d^n (see (3)) we obtain that there exist $g \in G$, $\pi \in S_{n+1}$ such that

$$x_i = y_{i^\pi}^g \tag{4}$$

for all i, $1 \leqslant i \leqslant n+1$. Hence, from (2) we obtaine

$$y_{2^\pi} = y_{3^\pi}, y_{4^\pi} = y_{5^\pi} = y_{6^\pi}, \ldots \tag{5}$$

and $y_{i^\pi} \neq y_{1^\pi}$ for all i, $2 \leqslant i \leqslant n+1$. Moreover,

$$d^n(y_{1^\pi}, y_{2^\pi}, \ldots, y_{n+1^\pi}) = 1.$$

Since the function f is an isometry of (X, d^n), the equality

$$d^n(f(y_{1^\pi}), f(y_{2^\pi}), \ldots, f(y_{n+1^\pi})) = 1$$

holds. By (3) there exist $g_1 \in G$, $\sigma \in S_{n+1}$ such that

$$x_{i^\sigma} = f(y_{i^\pi})^{g_1} \tag{6}$$

for all i, $1 \leqslant i \leqslant n+1$. But $f(y_{i^\pi}) \neq f(y_{1^\pi})$ for all i, $2 \leqslant i \leqslant n+1$. Therefore, using definition (2) of the element $\bar{x}$ and (6), we obtain that

$$f(y_{1^\pi}) = x_1.$$

Analogously, from (5) and (6) we obtain

$$f(y_{i^\pi}) = x_i \qquad \text{for all } i,\ 2 \leqslant i \leqslant n+1.$$

Therefore, from (4) it follows that the following equalities

$$f(y_{i^\pi}) = y^g_{i^\pi}, \qquad 1 \leqslant i \leqslant n+1,$$

hold. This means that the isometry f acts on X as the element g from the permutation group (G, X). □

Definition 1. A permutation group (G, X) is called *n-semimetric realizable* if there exist a positive integer n and an n-semimetric space (Y, d^n_Y) such that the group (G, X) and the isometry group $(\operatorname{Isom} Y, Y)$ are isomorphic as permutation groups. In this case we say that the group (G, X) is *realized* on the n-semimetric space (Y, d^n_Y).

Note, that if a permutation group (G, X) is n-semimetric realizable, then there exists an n-semimetric d^n on X such that the permutation group (G, X) is realized on the space (X, d^n).

The proof of the next statement is not difficult to verify.

Lemma 1. *Let (G, X) be a permutation group, n be a positive integer and (X, d^n) be an n-semimetric space. Assume that (G, X) is realized on the n-semimetric space (X, d^n). Then (G, X) is realized on an n-semimetric space $(X, \hat{d^n})$, where $\hat{d^n}$ takes only positive integer values.*

Theorem 2. *Let (G, X) be a permutaion group, n be a positive integer. If the permutaion group (G, X) is n-semimetric realizable, then it is $(n+1)$-semimetric realizable.*

Proof. Let d^n be some n-semimetric on X, such that the permutation group (G, X) is realized on the space (X, d^n). From Lemma 1 is follows, that we can consider the case when d^n takes only positive integer values. The maximum

$$M = \max_{x_i \in X} d^n(x_1, x_2, \ldots, x_{n+1})$$

exists for X is finite. Denote by b the positive integer

$$b = n + 1 + M.$$

Define a function $d^{n+1} : X^{n+2} \to \mathbb{R}^+$ by the rule:

$$d^{n+1}(x_1, x_2, \ldots, x_{n+2}) = b^{d^n(x_2,\ldots,x_{n+2})} + \\ + b^{d^n(x_1,x_3\ldots,x_{n+2})} \ldots + b^{d^n(x_1,x_2\ldots,x_{n+1})}, \qquad x_i \in X, \quad 1 \leqslant i \leqslant n+2. \quad (7)$$

Obviously, d^{n+1} is fully symmetric. We need to veryfy simplex-type inequality (1):

$$b^{d^n(x_2,\dots,x_{n+2})} + b^{d^n(x_1,x_3\dots,x_{n+2})} \dots + b^{d^n(x_1,x_2\dots,x_{n+1})} \leqslant \\ \leqslant \sum_{i=1}^{n+3} d^{n+1}(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+3}). \quad (8)$$

But the sum

$$d^{n+1}(x_2,\dots,x_{n+3}) = \sum_{i=2}^{n+3} b^{d^n(x_2,\dots,x_{i-1},x_{i+1},\dots,x_{n+3})}$$

contains the term $b^{d^n(x_2,\dots,x_{n+2})}$, the sum

$$d^{n+1}(x_1,x_3,\dots,x_{n+3}) = \sum_{i=1,i\neq 2}^{n+3} b^{d^n(x_1,\dots,x_{i-1},x_{i+1},\dots,x_{n+3})}$$

contains the term $b^{d^n(x_1,x_3,\dots,x_{n+2})}$ and so on. Hence, inequality (8) holds. Therefore the function d^{n+1} is an $(n+1)$-semimetric.

From the definition of the function d^{n+1} it follows, that all isometries of the n-semimetric space (X,d^n) are isometries of the $(n+1)$-semimetric space (X,d^{n+1}).

Let $f : X \to X$ be an isometry of (X,d^{n+1}). Assume on the contrary that f is not an isometry of (X,d^n). Then there exist $a_1,\dots,a_{n+1} \in X$ such that

$$d^n(a_1,\dots,a_n,a_{n+1}) \neq d^n(f(a_1),f(a_2),\dots,f(a_n),f(a_{n+1})). \quad (9)$$

As f is an isometry of (X,d^{n+1}), the equality

$$d^{n+1}(a_1,\dots,a_n,a_{n+1},a_{n+1}) = \\ = d^n(f(a_1),f(a_2),\dots,f(a_n),f(a_{n+1}),f(a_{n+1})) \quad (10)$$

holds. From (10) and defininition (7) $(n+1)$-semimetric d^{n+1} we have

$$b^{d^n(a_2,\dots,a_{n+1},a_{n+1})} + b^{d^n(a_1,a_3\dots,a_{n+1},a_{n+1})} \dots + b^{d^n(a_1,a_2\dots,a_{n+1})} = \\ = b^{d^n(f(a_2),\dots,f(a_{n+1}),f(a_{n+1}))} + b^{d^n(f(a_1),f(a_3)\dots,f(a_{n+1}),f(a_{n+1}))} + \dots + \\ + b^{d^n(f(a_1),f(a_2)\dots,f(a_{n+1}))} \quad (11)$$

From (9) we obtaine, that the last term on the left is not equal to the last term on the right in (11). Then either

$$b^{d^n(a_2,\ldots,a_{n+1},a_{n+1})} \neq b^{d^n(f(a_2),\ldots,f(a_{n+1}),f(a_{n+1}))}, \tag{12}$$

or there exists a positive integer i, $2 \leqslant i \leqslant n+1$, such that the inequality

$$b^{d^n(a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{n+1},a_{n+1})} \neq b^{d^n(f(a_1),\ldots,f(a_{i-1}),f(a_{i+1}),\ldots,f(a_{n+1}),f(a_{n+1}))}$$

holds. As d^n is fully symmetric, without loss of generality we can assume that inequality (12) holds. As f is an isometry of (X, d^{n+1}), we have

$$d^{n+1}(a_2,\ldots,a_n,a_{n+1},a_{n+1},a_{n+1}) = \\ = d^n(f(a_2),\ldots,f(a_n),f(a_{n+1}),f(a_{n+1}),f(a_{n+1})). \tag{13}$$

Using (13) and (12), by the similar arguments as above we get

$$b^{d^n(a_3,\ldots,a_{n+1},a_{n+1},a_{n+1})} \neq b^{d^n(f(a_3),\ldots,f(a_{n+1}),f(a_{n+1}),f(a_{n+1}))}.$$

Then we consider $d^{n+1}(a_3,\ldots,a_{n+1},a_{n+1},a_{n+1},a_{n+1})$, and so on. Finally we obtain

$$b^{d^n(a_{n+1},\ldots,a_{n+1},a_{n+1},a_{n+1})} \neq b^{d^n(f(a_{n+1}),\ldots,f(a_{n+1}),f(a_{n+1}),f(a_{n+1}))}.$$

From this inequality and the definition of the $(n+1)$-semimetric d^{n+1} (7) we get

$$d^{n+1}(a_{n+1},\ldots,a_{n+1},a_{n+1},a_{n+1}) \neq \\ \neq d^n(f(a_{n+1}),\ldots,f(a_{n+1}),f(a_{n+1}),f(a_{n+1})).$$

But f is an isometry of the space (X, d^{n+1}). Hence, our assumption is incorrect and f is isometry of (X, d^n). □

3. n-semimetric ranks of groups

Theorem 1 and Theorem 2 imply that the following definition is correct.

Definition 2. The semimetric rank of a permutation group (G, X) is the minimal number $k \in \mathbb{N}$ such that there exists a k-semimetric space (X, d^k) such that (G, X) is realized on the space (X, d^k).

We denote the semimetric rank of (G, X) as $sr(G, X)$.

Proposition 1. *Let (G, X) be a finite permutation group, $|X| = n$. Then the following inequality*

$$sr(G, X) \leqslant \frac{n(n+1)}{2} - 1$$

holds.

The proof of this proposition follows directly from the proof of Theorem 1.

A permutation group (G, X) is called *metric realizable* if there exists a metric space (Y, d_Y) such that the group (G, X) and the isometry group $(\operatorname{Isom} Y, Y)$ are isomorphic as permutation groups. In this case we say that the group (G, X) is *realized* on the metric space (Y, d_Y).

As any metric space is an 1-semimetric space, any metric realized permutation group is an 1-semimetric realizable. For example, the symmetric group of degree n, direct sums, direct products and wreath products of symmetric groups of some degree are 1-semimetric realizable and hence

$$sr(S_n) = sr(S_n \wr S_m) = sr(S_n \times S_m) = sr(S_n \oplus S_m) = 1,$$

where $n, m \in \mathbb{N}$ (see [3]).

Open questions. Is semimetric rank unbounded, i.e. is it true that for arbitrary positive integer n there exists a permutation group (G, X) such that $sr(G, X) > n$? If the answer is affirmative then find natural series $(G_n, X_n), n \geqslant 1$, of permutation groups such that their semimetric ranks have a common bound, i.e. there exists $C > 0$ such that $sr(G_n, X_n) < C, n \geqslant 1$.

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Contact information

O. Gerdiy, B. Oliynyk

Department of Computer Sciences, National University of "Kyiv-Mohyla Academy", Skovorody St. 2, Kyiv, 04655, Ukraine

E-Mail(s): lotuseater24@gmail.com, bogdana.oliynyk@gmail.com

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