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RESEARCH ARTICLE

On stable finiteness of group rings

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Abstract. For an arbitrary field or division ring K and an arbitrary group G, stable finiteness of $K[G]$ is equivalent to direct finiteness of $K[G \times H]$ for all finite groups H.

1. Introduction

Kaplansky's Direct Finiteness Conjecture (DFC) asks whether for every field (or division ring) K and every group G, the group ring $K[G]$ is directly finite, namely, whether $a, b \in K[G]$ and $ab = 1$ implies $ba = 1$ (where 1 denotes the identity element of $K[G]$. Kaplansky showed that the answer is positive when K has characteristic 0 — see [4], p. 122. One can also ask for more: one can ask for all matrix algebras $M_n(K[G])$ over the group ring to be directly finite. If this holds, the group ring $K[G]$ is said to be stably finite. Stable finiteness of $K[G]$ was shown (at the same time as direct finiteness) by Ara, O'Meara and Perera [1] when G is residually amenable and by Elek and Szabó [3] when G is (more generally) sofic. In this note, we show that stable finiteness of $K[G]$ is equivalent to direct finiteness of $K[G \times H]$ for all finite groups H.

Though the argument is not difficult, we believe the result is of interest because it may offer a short-cut for proving stable finiteness for certain classes of groups. Our investigation was motivated by the approach of [2]

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to the testing the DFC by testing certain finitely presented groups – the Universal Left Invertible Element, or ULIE, groups. At first we wondered whether an analogous effort should be made for stable finiteness, but Corollary 2.3 shows that there would be no advantage in doing so.

2. Stable finiteness

Lemma 2.1. *Given a field F and a positive integer n, there is a finite group H such that the group ring $F[H]$ has a nonunital subalgebra isomorphic to $M_n(F)$.*

Proof. We prove first the case $n = 2$. Let p be the characteristic of the field F. Consider the symmetric group $S_3 = \langle a, b \mid a^3 = b^2 = 1,\, bab = a^{-1}\rangle$. Consider the representation π of S_3 on F^2 given by

$$\pi(a) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \pi(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

extended by linearity to a representation of $F[S_3]$. We have

$$\pi(a^2) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \pi(ab) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \pi(a^2b) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

An easy row reduction computation shows that when $p \neq 3$, we have $\operatorname{span} \pi(S_3) = M_2(F)$. Let us assume $p \neq 3$. In the case $p > 3$, the desired conclusion of the lemma will follow from Maschke's theorem, but by performing the actual computations, we will now see that the conclusion holds also for $p = 2$. Let $Q = \frac{1}{3}(2 - a - a^2) \in F[S_3]$. Then $Q^2 = Q$, and $Q(F[S_3])Q$ is a subalgebra of $F[S_3]$. An easy computation shows that $Q(F[S_3])Q$ has dimension 4 over F, and $\pi(Q) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$; this implies that the restriction of π to $Q(F[S_3])Q$ is an isomorphism onto $M_2(F)$ and $Q(F[S_3])Q \cong M_2(F)$ as algebras. The lemma is proved in the case of $n = 2$ and $p \neq 3$.

We now suppose $p > 2$ and consider the dihedral group of order 8

$$\mathrm{Dih}_4 = \langle c, d \mid c^4 = d^2 = 1,\, dcd = c^{-1}\rangle$$

and its representation on F^2 given by

$$\sigma(c) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma(d) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which gives

$$\sigma(c^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma(c^3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\sigma(cd) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma(c^2 d) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma(c^3 d) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

We easily see $\operatorname{span} \sigma(\mathrm{Dih}_4) = M_2(F)$. Now the result follows by Maschke's theorem, but let us perform the easy calculation. Letting $Q = \frac{1}{2}(1 - c^2) \in F[\mathrm{Dih}_4]$, we have $Q^2 = Q$ and $Q(F[\mathrm{Dih}_4])Q$ is a subalgebra of $F[\mathrm{Dih}_4]$. We have $\sigma(Q) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\dim(Q(F[\mathrm{Dih}_4])Q) = 4$ and the restriction of σ to $Q(F[\mathrm{Dih}_4])Q$ is an isomorphism onto $M_2(F)$. Thus, the lemma is proved in the case $n = 2$ and $p > 2$. Taken together, these considerations prove the lemma in the case of $n = 2$.

For groups H_1 and H_2 we have the natural identification $F[H_1 \times H_2] \cong F[H_1] \otimes_F F[H_2]$, and for positive integers m and n we have $M_m(F) \otimes_F M_n(F) \cong M_{mn}(F)$. Therefore, starting from the case $n = 2$ of the lemma and taking cartesian products of an appropriate group, arguing by induction we prove the lemma in the case when n is a power of 2. Now taking corners of the matrix algebras $M_{2^k}(F)$ proves the lemma for arbitrary n. □

Theorem 2.2. *Let K be a field or a division ring and let Γ be a group. Then the following are equivalent:*

(a) *$K[\Gamma]$ is stably finite*

(b) *$K[\Gamma \times H]$ is directly finite for every finite group H.*

Proof. Suppose (a) holds and let H be a finite group. Let F be the base field of K. Then H acting on itself by permutations yields an injective, unital algebra homomorphism $F[H] \to M_n(F)$, where n is the order of H, and we may regard $F[H]$ as a unital subalgebra of $M_n(F)$. Then

$$K[\Gamma \times H] \cong K[\Gamma] \otimes_F F[H] \subseteq K[\Gamma] \otimes_F M_n(F) \cong M_n(K[\Gamma]),$$

with the inclusion being unital, and direct finiteness of $K[\Gamma \times H]$ follows from that of $M_n(K[\Gamma])$.

Suppose (b) holds. Let n be a positive integer. By Lemma 2.1, choose a finite group H so that $F[H]$ contains $M_n(F)$ as a nonunital subalgebra. Then $F[H]$ contains a copy of $M_n(F) \oplus F$ as a unital subalgebra, and we have

$$\begin{aligned} K[\Gamma \times H] &\cong K[\Gamma] \otimes_F F[H] \supseteq K[\Gamma] \otimes_F (M_n(F) \oplus F) \\ &\cong (K[\Gamma] \otimes_F M_n(F)) \oplus K[\Gamma] \cong M_n(K[\Gamma]) \oplus K[\Gamma] \supseteq M_n(K[\Gamma]) \oplus K, \end{aligned}$$

where all inclusions are as unital subalgebras. Now given $c, d \in M_n(K[\Gamma])$ such that $cd = 1$, take $a = c \oplus 1$ and $b = d \oplus 1$ in $M_n(K[\Gamma]) \oplus K$. We have $ab = 1$, and by the above inclusions and the direct finiteness of $K[\Gamma \times H]$, we must have $ba = 1$, so $dc = 1$. □

Consequently, truth of the Direct Finiteness Conjecture implies truth of the stronger looking Stable Direct Finiteness Conjecture.

Corollary 2.3. *For K is a division ring, if $K[\Gamma]$ is directly finite for all groups Γ, then $K[\Gamma]$ is stably finite for all groups Γ.*

Remark 2.4. From the proofs of Lemma 2.1 and Theorem 2.2, we see that the conditions (a) and (b) of Theorem 2.2 are also equivalent to the following:

(c) *Let $H_1 = \mathrm{Dih}_4$ if the characteristic of K is 3 and otherwise let $H_1 = S_3$. Then $K[\Gamma \times H]$ is directly finite whenever H is a Cartesian product of finitely many copies of H_1.*

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