Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume 16 **(2013)**. Number 2. pp. 299 – 310
© Journal "Algebra and Discrete Mathematics"

Free n-nilpotent dimonoids

Anatolii V. Zhuchok

Communicated by V. I. Sushchansky

Abstract. We construct a free n-nilpotent dimonoid and describe its structure. We also characterize the least n-nilpotent congruence on a free dimonoid, construct a new class of dimonoids with zero and give examples of nilpotent dimonoids of nilpotency index 2.

1. Introduction

The notion of a dialgebra is based on the notion of a dimonoid [1]. Therefore all results obtained for dimonoids can be applied to dialgebras. This connection between dimonoids and dialgebras shows that dimonoids are very natural objects to study. Another reason for our interest in dimonoids is their connection with the notions of interassociativity [2], strong interassociativity [3], related semigroups [4] and doppelalgebras [5]. Note also that the notion of an n-tuple semigroup, which was used in [6] to study properties of n-tuple algebras of associative type, is related to commutative dimonoids [7] in the case $n = 2$.

The notion of a nilpotent semigroup was introduced by Malcev [8] and independently by Neuman and Taylor [9]. They showed that nilpotent groups can be defined by using semigroup identities. Further, the nilpotency in semigroups has been extensively studied by many authors. In particular, properties of nilpotent semigroups have been investigated by Lallement [10]. The relationships between nilpotent semigroups and

2010 MSC: 08B20, 20M10, 20M50, 17A30, 17A32.

Key words and phrases: n-nilpotent dimonoid, free n-nilpotent dimonoid, 0-diband of subdimonoids, dimonoid, semigroup.

semigroup algebras were studied by Jespers and Okninski [11]. The nilpotency in algebras with two binary associative operations was considered too (see, e.g., [12]).

In this paper we continue researches from [13 – 16] developing the variety theory of dimonoids. The main focus of our paper is to study nilpotent dimonoids.

In Section 2 we present a new class of dimonoids with zero and give examples of nilpotent dimonoids of nilpotency index 2.

In Section 3 we construct a free n-nilpotent dimonoid of an arbitrary rank and consider separately free n-nilpotent dimonoids of rank 1.

In Section 4 we introduce the notion of a 0-diband of subdimonoids and in terms of 0-dibands of subdimonoids describe the structure of free n-nilpotent dimonoids.

In the final section we characterize the least n-nilpotent congruence on a free dimonoid.

2. Dimonoids with zero

In this section we construct a new class of dimonoids with zero and give examples of nilpotent dimonoids of nilpotency index 2.

An element 0 of a dimonoid $(D, \dashv, \vdash)$ (see, e.g., [17]) will be called zero, if $x * 0 = 0 = 0 * x$ for all $x \in D$ and $* \in \{\dashv, \vdash\}$.

Let $\overline{D} = (D, \dashv, \vdash)$ be an arbitrary dimonoid and I be an arbitrary nonempty set. Define operations $\dashv^{'}$ and $\vdash^{'}$ on $D' = (I \times D \times I) \cup \{0\}$ by

$$(i, a, j) *^{'} (k, b, t) = \begin{cases} (i, a * b, t), & j = k, \\ 0, & j \neq k, \end{cases}$$

$$(i, a, j) *^{'} 0 = 0 *^{'} (i, a, j) = 0 *^{'} 0 = 0$$

for all $(i, a, j), (k, b, t) \in D' \backslash \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra $(D', \dashv^{'}, \vdash^{'})$ will be denoted by $B[\overline{D}, I]$.

Proposition 1. *$B[\overline{D}, I]$ is a dimonoid with zero.*

Proof. By [18] operations $\dashv'$ and $\vdash'$ are associative. Let $(i, a, j), (k, b, t)$, $(m, c, n) \in D' \backslash \{0\}$. If $j \neq k$ or $t \neq m$, then, obviously, all axioms of a dimonoid hold. If $j = k$ and $t = m$, then

$$((i, a, j) \dashv' (k, b, t)) \dashv' (m, c, n) = (i, a \dashv b, t) \dashv' (m, c, n) =$$

$$= (i, (a \dashv b) \dashv c, n) = (i, a \dashv (b \vdash c), n) =$$

$$= (i,a,j) \dashv' (k, b\vdash c, n) = (i,a,j) \dashv' \left((k,b,t) \vdash' (m,c,n)\right),$$
$$\left((i,a,j) \vdash' (k,b,t)\right) \dashv' (m,c,n) = (i, a\vdash b, t) \dashv' (m,c,n) =$$
$$= (i, (a\vdash b) \dashv c, n) = (i, a\vdash (b \dashv c), n) = (i,a,j) \vdash' (k, b \dashv c, n) =$$
$$= (i,a,j) \vdash' ((k,b,t) \dashv' (m,c,n)),$$
$$\left((i,a,j) \dashv' (k,b,t)\right) \vdash' (m,c,n) = (i, a\dashv b, t) \vdash' (m,c,n) =$$
$$= (i, (a\dashv b) \vdash c, n) = (i, a\vdash (b\vdash c), n) = (i,a,j) \vdash' (k, b \vdash c, n) =$$
$$= (i,a,j) \vdash' ((k,b,t) \vdash' (m,c,n))$$

according to the axioms of a dimonoid $\overline{D}$.

The proofs of the remaining cases are obvious. Thus, $B[\overline{D}, I]$ is a dimonoid with zero 0. □

Observe that if operations of a dimonoid $\overline{D}$ coincides and it is a group G, then any Brandt semigroup [18] is isomorphic to some semigroup $B[G, I]$. So, $B[\overline{D}, I]$ generalizes the semigroup $B[G, I]$. We call the dimonoid $B[\overline{D}, I]$ a Brandt dimonoid.

As usual, $\mathbb{N}$ denotes the set of all positive integers.

A dimonoid $(D, \dashv, \vdash)$ with zero will be called nilpotent, if for some $n \in \mathbb{N}$ and any $x_i \in D$, $1 \leq i \leq n+1$, and $*_j \in \{\dashv, \vdash\}$, $1 \leq j \leq n$, any parenthesizing of

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1} \tag{$*$}$$

gives $0 \in D$. The least such n we shall call the nilpotency index of $(D, \dashv, \vdash)$. For $k \in \mathbb{N}$ a nilpotent dimonoid of nilpotency index $\leq k$ is said to be k-nilpotent.

Note that from $(*)$ it follows that operations of any 1-nilpotent dimonoid coincide and it is a zero semigroup.

We finish this section with the consideration of examples of nilpotent dimonoids of nilpotency index 2.

Let X and Y be arbitrary disjoint sets, $0 \in X$, and let

$$\varphi : Y \times Y \to X, \ \psi : Y \times Y \to X$$

be arbitrary different maps. Define operations $\dashv$ and $\vdash$ on $X \cup Y$ by

$$x \dashv y = \begin{cases} (x,y)\varphi, \ x,y \in Y, \\ \quad 0 \quad \text{otherwise}, \end{cases}$$

$$x \vdash y = \begin{cases} (x,y)\psi, \ x,y \in Y, \\ \quad 0 \quad \text{otherwise} \end{cases}$$

for all $x, y \in X \cup Y$.

Proposition 2. $(X \cup Y, \dashv, \vdash)$ *is a nilpotent dimonoid of nilpotency index* 2.

Proof. By [7] $(X \cup Y, \dashv, \vdash)$ is a dimonoid. From the definition of operations $\dashv$ and $\vdash$ it follows that $(X \cup Y, \dashv, \vdash)$ is not a 1-nilpotent dimonoid. As

$$(x *_1 y) *_2 z = 0 = x *_1 (y *_2 z)$$

for all $x, y, z \in X \cup Y$, $*_1, *_2 \in \{\dashv, \vdash\}$, then the dimonoid $(X \cup Y, \dashv, \vdash)$ is nilpotent of nilpotency index 2. $\square$

Recall that a dimonoid is called commutative, if its both operations are commutative.

Let X be an arbitrary set such that $0,\ a,\ b,\ c,\ d \in X$ and $a \neq b$, $b \neq c$, $c \neq d$, $d \neq a$. Define operations $\dashv$ and $\vdash$ on X, assuming

$$x \dashv y = \begin{cases} b, & x = y = a, \\ 0 & \text{otherwise}, \end{cases}$$

$$x \vdash y = \begin{cases} d, & x = y = c, \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in X$.

Proposition 3. *If* $b \neq 0$ *or* $d \neq 0$*, then* $(X, \dashv, \vdash)$ *is a nilpotent commutative dimonoid of nilpotency index* 2.

Proof. By [7] $(X, \dashv, \vdash)$ is a commutative dimonoid. If $b \neq 0$ or $d \neq 0$, then, similar to Proposition 2, the fact that $(X, \dashv, \vdash)$ is nilpotent of nilpotency index 2 can be proved. $\square$

3. Constructions

In this section we construct a free n-nilpotent dimonoid of an arbitrary rank and consider separately free n-nilpotent dimonoids of rank 1.

Note that the class of all n-nilpotent dimonoids is a subvariety of the variety of all dimonoids. Indeed, this class is a subclass of the variety of all dimonoids which is closed under taking of homomorphic images, subdimonoids and Cartesian products. A dimonoid which is free in the variety of n-nilpotent dimonoids will be called a free n-nilpotent dimonoid.

The necessary information about varieties of dimonoids can be found in [13].

Let X be an alphabet, $F[X]$ be the free semigroup over X. Denote the length of a word $w \in F[X]$ by l_w. Fix $n \in \mathbb{N}$ and assume

$$FN_n = \{(w, m) \in F[X] \times \mathbb{N} \,|\, m \leq l_w \leq n\} \cup \{0\}.$$

Define operations $\dashv$ and $\vdash$ on FN_n by

$$(w_1, m_1) \dashv (w_2, m_2) = \begin{cases} (w_1w_2, m_1), & l_{w_1w_2} \leq n, \\ 0, & l_{w_1w_2} > n, \end{cases}$$

$$(w_1, m_1) \vdash (w_2, m_2) = \begin{cases} (w_1w_2, l_{w_1} + m_2), & l_{w_1w_2} \leq n, \\ 0, & l_{w_1w_2} > n, \end{cases}$$

$$(w_1, m_1) * 0 = 0 * (w_1, m_1) = 0 * 0 = 0$$

for all $(w_1, m_1), (w_2, m_2) \in FN_n \backslash \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra $(FN_n, \dashv, \vdash)$ will be denoted by $FN_n(X)$.

Theorem 1. *$FN_n(X)$ is the free n-nilpotent dimonoid.*

Proof. Let (w_1, m_1), $(w_2, m_2), (w_3, m_3) \in FN_n \backslash \{0\}$. In order to prove that $FN_n(X)$ is a dimonoid we consider the following cases.

If $l_{w_1w_2} > n$ or $l_{w_2w_3} > n$, then the proof is straightforward. Similar to [19], the case $l_{w_1w_2w_3} \leq n$ can be considered. In the case $l_{w_1w_2} \leq n$, $l_{w_2w_3} \leq n$ and $l_{w_1w_2w_3} > n$ we have

$$((w_1, m_1) *_1 (w_2, m_2)) *_2 (w_3, m_3) = 0 = (w_1, m_1) *_1 ((w_2, m_2) *_2 (w_3, m_3))$$

for $*_1, *_2 \in \{\dashv, \vdash\}$. The proofs of the remaining cases are obvious. Thus, $FN_n(X)$ is a dimonoid.

As for any $(w_i, m_i) \in FN_n \backslash \{0\}$, $1 \leq i \leq n+1$, and $*_j \in \{\dashv, \vdash\}$, $1 \leq j \leq n$, any parenthesizing of

$$(w_1, m_1) *_1 (w_2, m_2) *_2 \ldots *_n (w_{n+1}, m_{n+1})$$

gives 0, then $FN_n(X)$ is nilpotent. At the same time for any $(x_i, 1) \in FN_n \backslash \{0\}$, where $x_i \in X$, $1 \leq i \leq n$,

$$(x_1, 1) \dashv (x_2, 1) \dashv \ldots \dashv (x_n, 1) = (x_1x_2 \ldots x_n, 1) \neq 0.$$

It means that $FN_n(X)$ has nilpotency index n.

Let us show that $FN_n(X)$ is free.

Let $(T, \dashv', \vdash')$ be an arbitrary n-nilpotent dimonoid and $\beta : X \to T$ be an arbitrary map. Define a map

$$\pi : FN_n(X) \to (T, \dashv', \vdash') : u \mapsto u\pi,$$

assuming

$$u\pi = \begin{cases} x_1\beta\vdash' \ldots \vdash' x_l\beta\dashv' \ldots \dashv' x_s\beta, \text{ if } \begin{array}{l} u = (x_1 \ldots x_i \ldots x_s, l), \\ x_i \in X,\ 1 \le i \le s, \end{array} \\ \quad 0, \text{ if } u = 0. \end{cases}$$

Show that π is a homomorphism. For arbitrary elements

$$(x_1 \ldots x_i \ldots x_s, l),\ (y_1 \ldots y_j \ldots y_r, t) \in FN_n \backslash \{0\},$$

where $x_i, y_j \in X$, $1 \le i \le s$, $1 \le j \le r$, we obtain

$$((x_1 \ldots x_i \ldots x_s, l) \dashv (y_1 \ldots y_j \ldots y_r, t))\, \pi =$$

$$= \begin{cases} (x_1 \ldots x_s y_1 \ldots y_r, l)\, \pi, & s + r \le n, \\ 0\pi, & s + r > n, \end{cases}$$

$$((x_1 \ldots x_i \ldots x_s, l) \vdash (y_1 \ldots y_j \ldots y_r, t))\, \pi =$$

$$= \begin{cases} (x_1 \ldots x_s y_1 \ldots y_r, s + t)\, \pi, & s + r \le n, \\ 0\pi, & s + r > n. \end{cases}$$

If $s + r \le n$, then, using the axioms of a dimonoid, we have

$$(x_1 \ldots x_s y_1 \ldots y_r, l)\, \pi =$$

$$= x_1\beta\vdash' \ldots \vdash' x_l\beta\dashv' \ldots \dashv' x_s\beta\dashv' y_1\beta\dashv' \ldots \dashv' y_r\beta =$$

$$= (x_1\beta\vdash' \ldots \vdash' x_l\beta\dashv' \ldots \dashv' x_s\beta)\dashv'(y_1\beta\vdash' \ldots \vdash' y_t\beta\dashv' \ldots \dashv' y_r\beta) =$$

$$= (x_1 \ldots x_i \ldots x_s, l)\pi\dashv' (y_1 \ldots y_j \ldots y_r, t)\, \pi,$$

$$(x_1 \ldots x_s y_1 \ldots y_r, s + t)\, \pi =$$

$$= x_1\beta\vdash' \ldots \vdash' x_s\beta\vdash' y_1\beta\vdash' \ldots \vdash' y_t\beta\dashv' \ldots \dashv' y_r\beta =$$

$$= \left(x_1\beta\vdash' \ldots \vdash' x_l\beta\dashv' \ldots \dashv' x_s\beta\right) \vdash' \left(y_1\beta\vdash' \ldots \vdash' y_t\beta\dashv' \ldots \dashv' y_r\beta\right) =$$

$$= (x_1 \ldots x_i \ldots x_s, l)\, \pi\vdash' (y_1 \ldots y_j \ldots y_r, t)\, \pi.$$

In the case $s + r > n$,

$$0\pi = 0 = (x_1 \ldots x_i \ldots x_s, l)\pi *' (y_1 \ldots y_j \ldots y_r, t)\pi$$

for $* \in \{\dashv, \vdash\}$.

The proofs of the remaining cases are obvious. Thus, π is a homomorphism. This completes the proof of Theorem 1. $\square$

Corollary 1. *The free n-nilpotent dimonoid $FN_n(X)$ generated by a finite set X is finite. Specifically, $|FN_n(X)| = \sum_{i=1}^{n} i|X|^i + 1$.*

Now we construct a dimonoid which is isomorphic to the free n-nilpotent dimonoid of rank 1.

Fix $n \in \mathbb{N}$ and let $\widetilde{\mathbb{N}}_n = \{(m,t) \in \mathbb{N} \times \mathbb{N} \,|\, t \le m \le n\} \cup \{0\}$. Define operations $\dashv$ and $\vdash$ on $\widetilde{\mathbb{N}}_n$ by

$$(m_1,t_1) \dashv (m_2,t_2) = \begin{cases} (m_1+m_2, t_1), & m_1+m_2 \le n, \\ 0, & m_1+m_2 > n, \end{cases}$$

$$(m_1,t_1) \vdash (m_2,t_2) = \begin{cases} (m_1+m_2, m_1+t_2), & m_1+m_2 \le n, \\ 0, & m_1+m_2 > n, \end{cases}$$

$$(m_1,t_1) * 0 = 0 * (m_1,t_1) = 0 * 0 = 0$$

for all $(m_1,t_1), (m_2,t_2) \in \widetilde{\mathbb{N}}_n \backslash \{0\}$ and $* \in \{\dashv, \vdash\}$. An immediate verification shows that axioms of a dimonoid hold concerning operations $\dashv$ and $\vdash$. So, $(\widetilde{\mathbb{N}}_n, \dashv, \vdash)$ is a dimonoid. Denote it by $\mathbb{N}_n$.

Lemma 1. *If $|X| = 1$, then $FN_n(X) \cong \mathbb{N}_n$.*

Proof. Let $X = \{a\}$. Define a map

$$\delta : FN_n(X) \to \mathbb{N}_n,$$

assuming

$$u\delta = \begin{cases} (k,l), & u = (a^k, l), \\ 0, & u = 0. \end{cases}$$

An easy verification shows that δ is an isomorphism. □

4. 0-diband decompositions of $FN_n(X)$

In this section we introduce the notion of a 0-diband of subdimonoids and in terms of 0-dibands of subdimonoids describe the structure of free n-nilpotent dimonoids.

For dimonoids with zero there exists a natural analog of the notion of a diband of subdimonoids [17].

A dimonoid S with zero 0 (see Sect. 2) will be called a 0-diband of subdimonoids S_β, $\beta \in B$, where B is an idempotent dimonoid [17], if $S = \bigcup_{\beta \in B} S_\beta$, $S_\beta \cap S_\gamma = \{0\}$ for $\beta \neq \gamma$ and $S_\beta \dashv S_\gamma \subseteq S_{\beta \dashv \gamma}$, $S_\beta \vdash S_\gamma \subseteq S_{\beta \vdash \gamma}$

for any $\beta, \gamma \in B$. If B is an idempotent semigroup (band), then we say that S is a 0-band of subdimonoids S_β, $\beta \in B$.

Observe that the notion of a 0-diband of subdimonoids generalizes the notion of a 0-band of semigroups [20] which plays an important role in the structural theory of semigroups.

Let $w = x_1 \dots x_i \dots x_s \in F[X]$, where $x_i \in X$, $1 \le i \le s$ (see Sect. 3). Denote the set of all letters occurring in w by $c\,(w)$.

Consider the semigroups $X_{rb}, X_{\ell z}, X_{rz}, B(X), B_{\ell z}(X), B_{rz}(X)$ and the dimonoids $FRct(X)$, $X_{\ell z,rb}$, $X_{rb,rz}$, $X_{\ell z,rz}$, $B_{\ell z,rz}(X)$ which were defined in [14] and [15]. It is well-known that the first six semigroups are relatively free bands. In [14] and [15] it was shown that the second five dimonoids are relatively free dibands.

Let

$$P_{(a,b,c)} = \{(x_1 ... x_i ... x_s, m) \in FN_n(X) \,|\, (x_1, x_m, x_s) = (a, b, c)\} \cup \{0\}$$

for $(a, b, c) \in FRct(X)$, $n > 2$ and $|X| > 1$;

$$P_{(a,b]} = \{(x_1 ... x_i ... x_s, m) \in FN_n(X) \,|\, (x_1, x_m) = (a, b)\} \cup \{0\}$$

for $(a, b) \in X_{\ell z,rb}$, $n > 1$ and $|X| > 1$;

$$P_{[b,c)} = \{(x_1 ... x_i ... x_s, m) \in FN_n(X) \,|\, (x_m, x_s) = (b, c)\} \cup \{0\}$$

for $(b, c) \in X_{rb,rz}$, $n > 1$ and $|X| > 1$;

$$P_{(b]} = \{(x_1 ... x_i ... x_s, m) \in FN_n(X) \,|\, x_m = b\} \cup \{0\}$$

for $b \in X_{\ell z,rz}$, $n > 1$ and $|X| > 1$;

$$P^Y_{(b]} = \{(x_1 ... x_i ... x_s, m) \in FN_n(X) | (x_m, c(x_1 ... x_i ... x_s)) = (b, Y)\} \cup \{0\}$$

for $(b, Y) \in B_{\ell z,rz}(X)$, $n > 1$ and $1 < |X| \le n$.

Further we will deal with 0-diband decompositions and 0-band decompositions of free n-nilpotent dimonoids.

The following structure theorem gives decompositions of free n-nilpotent dimonoids into 0-dibands of subdimonoids.

Theorem 2. *The free n-nilpotent dimonoid $FN_n(X)$ is a* 0*-diband of subdimonoids*

(i) $P_{(a,b,c)}$, $(a, b, c) \in FRct(X)$, *if* $n > 2$ *and* $|X| > 1$;
(ii) $P_{(a,b]}$, $(a, b) \in X_{\ell z,rb}$, *if* $n > 1$ *and* $|X| > 1$;
(iii) $P_{[b,c)}$, $(b, c) \in X_{rb,rz}$, *if* $n > 1$ *and* $|X| > 1$;
(iv) $P_{(b]}$, $b \in X_{\ell z,rz}$, *if* $n > 1$ *and* $|X| > 1$;
(v) $P^Y_{(b]}$, $(b, Y) \in B_{\ell z,rz}(X)$, *if* $n > 1$ *and* $1 < |X| \le n$.

Proof. We prove (v). It is clear that in the case $n > 1$ and $1 < |X| \leq n$, $P^Y_{(b]} \backslash \{0\} \neq \varnothing$ for all $(b, Y) \in B_{\ell z, rz}(X)$. Moreover, $P^Y_{(b]}$, $(b, Y) \in B_{\ell z, rz}(X)$, is a subdimonoid of $FN_n(X)$. Obviously,

$$FN_n(X) = \bigcup_{(b,Y)\in B_{\ell z,rz}(X)} P^Y_{(b]} \text{ and } P^Y_{(b]} \cap P^\Lambda_{(y]} = \{0\}$$

for $(b, Y) \neq (y, \Lambda)$. It is immediate to check that

$$P^Y_{(b]} \dashv P^\Lambda_{(y]} \subseteq P^{Y\cup\Lambda}_{(b]} \text{ and } P^Y_{(b]} \vdash P^\Lambda_{(y]} \subseteq P^{Y\cup\Lambda}_{(y]}$$

for any (b, Y), $(y, \Lambda) \in B_{\ell z, rz}(X)$. Thus, $FN_n(X)$ is a 0-diband of subdimonoids $P^Y_{(b]}$, $(b, Y) \in B_{\ell z, rz}(X)$.

The proofs of the remaining cases are similar. □

Assume

$$P_{(a,c)} = \{(x_1...x_i...x_s, m) \in FN_n(X) \,|\, (x_1, x_s) = (a, c)\} \cup \{0\}$$

for $(a, c) \in X_{rb}$, $n > 1$ and $|X| > 1$;

$$P_{(a)} = \{(x_1...x_i...x_s, m) \in FN_n(X) \,|\, x_1 = a\} \cup \{0\}$$

for $a \in X_{\ell z}$, $n > 1$ and $|X| > 1$;

$$P_{[c]} = \{(x_1...x_i...x_s, m) \in FN_n(X) \,|\, x_s = c\} \cup \{0\}$$

for $c \in X_{rz}$, $n > 1$ and $|X| > 1$;

$$P_Y = \{(x_1...x_i...x_s, m) \in FN_n(X) \,|\, c(x_1...x_i...x_s) = Y\} \cup \{0\}$$

for $Y \in B(X)$, $n > 1$ and $1 < |X| \leq n$;

$$P^Y_{(a)} = \{(x_1...x_i...x_s, m) \in FN_n(X) \,|\, (x_1, c(x_1...x_i...x_s)) = (a, Y)\} \cup \{0\}$$

for $(a, Y) \in B_{\ell z}(X)$, $n > 1$ and $1 < |X| \leq n$;

$$P^Y_{[c]} = \{(x_1...x_i...x_s, m) \in FN_n(X) \,|\, (x_s, c(x_1...x_i...x_s)) = (c, Y)\} \cup \{0\}$$

for $(c, Y) \in B_{rz}(X)$, $n > 1$ and $1 < |X| \leq n$.

The following structure theorem gives decompositions of free n-nilpotent dimonoids into 0-bands of subdimonoids.

Theorem 3. *The free n-nilpotent dimonoid $FN_n(X)$ is a 0-band of subdimonoids*

(i) $P_{(a,c)}$, $(a,c) \in X_{rb}$, *if* $n > 1$ *and* $|X| > 1$;
(ii) $P_{(a)}$, $a \in X_{\ell z}$, *if* $n > 1$ *and* $|X| > 1$;
(iii) $P_{[c]}$, $c \in X_{rz}$, *if* $n > 1$ *and* $|X| > 1$;
(iv) P_Y, $Y \in B(X)$, *if* $n > 1$ *and* $1 < |X| \leq n$;
(v) $P_{(a)}^Y$, $(a,Y) \in B_{\ell z}(X)$, *if* $n > 1$ *and* $1 < |X| \leq n$;
(vi) $P_{[c]}^Y$, $(c,Y) \in B_{rz}(X)$, *if* $n > 1$ *and* $1 < |X| \leq n$.

Proof. We prove (v). It is easy to see that in the case $n > 1$ and $1 < |X| \leq n$, $P_{(a)}^Y \backslash \{0\} \neq \varnothing$ for all $(a,Y) \in B_{\ell z}(X)$. Besides, $P_{(a)}^Y$, $(a,Y) \in B_{\ell z}(X)$, is a subdimonoid of $FN_n(X)$. Clearly,

$$FN_n(X) = \bigcup_{(a,Y)\in B_{\ell z}(X)} P_{(a)}^Y \text{ and } P_{(a)}^Y \cap P_{(x)}^\Lambda = \{0\}$$

for $(a,Y) \neq (x,\Lambda)$. One can check that

$$P_{(a)}^Y \dashv P_{(x)}^\Lambda \subseteq P_{(a)}^{Y\cup\Lambda} \text{ and } P_{(a)}^Y \vdash P_{(x)}^\Lambda \subseteq P_{(a)}^{Y\cup\Lambda}$$

for any $(a,Y),(x,\Lambda) \in B_{\ell z}(X)$. Consequently, $FN_n(X)$ is a 0-band of subdimonoids $P_{(a)}^Y$, $(a,Y) \in B_{\ell z}(X)$.

The proofs of the remaining cases are similar. □

5. The least n-nilpotent congruence on a free dimonoid

In this section we present the least n-nilpotent congruence on a free dimonoid.

If $f : D_1 \to D_2$ is a homomorphism of dimonoids, then the corresponding congruence on D_1 will be denoted by Δ_f. If ρ is a congruence on a dimonoid $(D,\dashv,\vdash)$ such that $(D,\dashv,\vdash)/\rho$ is an n-nilpotent dimonoid (see Sect. 2), then we say that ρ is an n-nilpotent congruence.

Let $\breve{F}[X]$ be the free dimonoid over X (see [14], [19]). Fix $n \in \mathbb{N}$ and define a relation ξ_n on $\breve{F}[X]$ by

$$(w_1,m_1)\xi_n(w_2,m_2) \text{ if and only if}$$
$$(w_1,m_1) = (w_2,m_2) \text{ or } l_{w_1} > n,\ l_{w_2} > n.$$

Theorem 4. *The relation ξ_n on the free dimonoid $\breve{F}[X]$ is the least n-nilpotent congruence.*

Proof. Define a map $\psi : \breve{F}[X] \to FN_n(X)$ by

$$(w,m)\,\psi = \begin{cases} (w,m)\,, & l_w \leq n, \\ 0, & l_w > n, \end{cases} \qquad (w,m) \in \breve{F}[X]\,.$$

Let $(w_1,m_1)\,,(w_2,m_2) \in \breve{F}[X]$ and $l_{w_1w_2} \leq n$. From $l_{w_1w_2} \leq n$ it follows that $l_{w_1} < n$ and $l_{w_2} < n$. Then

$$((w_1,m_1) \dashv (w_2,m_2))\,\psi = (w_1w_2,m_1)\,\psi = (w_1w_2,m_1) =$$

$$= (w_1,m_1) \dashv (w_2,m_2) = (w_1,m_1)\,\psi \dashv (w_2,m_2)\,\psi,$$

$$((w_1,m_1) \vdash (w_2,m_2))\,\psi =$$

$$= (w_1w_2, l_{w_1} + m_2)\,\psi = (w_1w_2, l_{w_1} + m_2) =$$

$$= (w_1,m_1) \vdash (w_2,m_2) = (w_1,m_1)\,\psi \vdash (w_2,m_2)\,\psi.$$

If $l_{w_1w_2} > n$, then

$$((w_1,m_1) \dashv (w_2,m_2))\,\psi = (w_1w_2,m_1)\,\psi = 0 =$$

$$= (w_1,m_1)\,\psi \dashv (w_2,m_2)\,\psi,$$

$$((w_1,m_1) \vdash (w_2,m_2))\,\psi = (w_1w_2, l_{w_1} + m_2)\,\psi = 0 =$$

$$= (w_1,m_1)\,\psi \vdash (w_2,m_2)\,\psi.$$

Thus, ψ is a surjective homomorphism. By Theorem 1 $FN_n(X)$ is the free n-nilpotent dimonoid. Then Δ_ψ is the least n-nilpotent congruence on $\breve{F}[X]$. From the definition of ψ it follows that $\Delta_\psi = \xi_n$. $\square$

References

[1] J.-L. Loday, *Dialgebras, In: Dialgebras and related operads, Lect. Notes Math.* **1763**, Springer-Verlag, Berlin (2001), 7–66.

[2] M. Gould, K.A. Linton, A.W. Nelson, *Interassociates of monogenic semigroups*, Semigroup Forum **68** (2004), 186–201.

[3] M. Gould, R.E. Richardson, *Translational hulls of polynomially related semigroups*, Czechoslovak Math. J. **33** (1983), no. 1, 95–100.

[4] E. Hewitt, H.S. Zuckerman, *Ternary operations and semigroups*, Semigroups, Proc. Sympos. Detroit, Michigan 1968. (1969), 95–100.

[5] B. Richter, *Dialgebren, Doppelalgebren und ihre Homologie*, Diplomarbeit, Universitat Bonn. (1997). Available at http://www.math.uni-bonn.de/people/richter/.

[6] N.A. Koreshkov, *n-tuple algebras of associative type*, Izv. Vyssh. Uchebn. Zaved. Mat. **12** (2008), 34–42 (in Russian).

[7] A.V. Zhuchok, *Commutative dimonoids*, Algebra and Discrete Math. **2** (2009), 116–127.

[8] A.I. Malcev, *Nilpotent semigroups*, Uchen. Zap. Ivanov. Gos. Ped. Inst. **4** (1953), 107–111 (in Russian).

[9] B.H. Neumann, T. Taylor, *Subsemigroups of nilpotent groups*, Proc. Royal Soc. London, Ser. A **274** (1963), 1–4.

[10] G. Lallement, *On nilpotency and residual finiteness in semigroups*, Pacific J. Math. **42** (1972), no. 3, 693–700.

[11] E. Jespers, J. Okninski, *Nilpotent semigroups and semigroup algebras*, Journal of Algebra **169** (1994), 984–1011.

[12] R.S. Kruse, D.T. Price, *On the classification of nilpotent rings*, Mathematische Zeitschrift **113** (1970), no. 3, 215–223.

[13] A.V. Zhuchok, *Free commutative dimonoids*, Algebra and Discrete Math. **9** (2010), no. 1, 109–119.

[14] A.V. Zhuchok, *Free rectangular dibands and free dimonoids*, Algebra and Discrete Math. **11** (2011), no. 2, 92–111.

[15] A.V. Zhuchok, *Free normal dibands*, Algebra and Discrete Math. **12** (2011), no. 2, 112–127.

[16] A.V. Zhuchok, *Free* $(\ell r, rr)$*-dibands*, Algebra and Discrete Math. **15** (2013), no. 2, 295–304.

[17] A.V. Zhuchok, *Dimonoids*, Algebra and Logic **50** (2011), no. 4, 323–340.

[18] A.H. Clifford, G.B. Preston, *The algebraic theory of semigroups*, American Mathematical Society V. **1, 2** (1964), (1967).

[19] A.V. Zhuchok, *Free dimonoids*, Ukr. Math. J. **63** (2011), no. 2, 196–208.

[20] L.N. Shevrin, *Semigroups*, In the book: V. Artamonov, V. Salii, L. Skornyakov and others, General algebra V. **2**, Sect. IV (1991), 11–191.

Contact information

A. V. Zhuchok Department of Mathematical Analysis and Algebra, Luhansk Taras Shevchenko National University, Oboronna str., 2, Luhansk, 91011, Ukraine
E-Mail: zhuchok_a@mail.ru

Received by the editors: 02.07.2013
and in final form 27.08.2013.