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RESEARCH ARTICLE

Classifying cubic s-regular graphs of orders $22p$ and $22p^2$

A. A. Talebi and N. Mehdipoor

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Abstract. A graph is s-regular if its automorphism group acts regularly on the set of s-arcs. In this study, we classify the connected cubic s-regular graphs of orders $22p$ and $22p^2$ for each $s \geq 1$, and each prime p.

1. Introduction

In this study, all graphs considered are assumed to be undirected, finite, simple, and connected, unless stated otherwise. For a graph X, $V(X)$, $E(X)$, $Arc(X)$, and $Aut(X)$ denote its vertex set, edge set, arc set, and full automorphism group, respectively. Let G be a subgroup of $Aut(X)$. For $u, v \in V(X)$, uv denotes the edge incident to u and v in X, and $N_X(u)$ denotes the neighborhood of u in X, that is, the set of vertices adjacent to u in X.

A graph $\widetilde{X}$ is called a covering of a graph X with projection $p : \widetilde{X} \to X$ if there is a surjection $p : V(\widetilde{X}) \to V(X)$ such that $p|_{N_{\widetilde{X}}(\widetilde{v})} : N_{\widetilde{X}}(\widetilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\widetilde{v} \in p^{-1}(v)$.

A permutation group G on a set Ω is said to be semiregular if the stabilizer G_v of v in G is trivial for each $v \in \Omega$, and is regular if G is transitive, and semiregular.

Let K be a subgroup of $Aut(X)$ such that K is intransitive on $V(X)$. The quotient graph X/K induced by K is defined as the graph such that

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the set Ω of K-orbits in $V(X)$ is the vertex set of X/K and B, $C \in \Omega$ are adjacent if and only if there exist a $u \in B$ and $v \in C$ such that $\{u, v\} \in E(X)$.

A covering $\widetilde{X}$ of X with a projection p is said to be regular (or N-covering) if there is a semiregular subgroup N of the automorphism group $Aut(\widetilde{X})$ such that graph X is isomorphic to the quotient graph $\widetilde{X}/N$, say by h, and the cubic map $\widetilde{X} \to \widetilde{X}/N$ is the composition ph of p and h (in this paper, all functions are composed from left to right). If N is a cyclic or an elementary Abelian, then, $\widetilde{X}$ is called a cyclic or an elementary Abelian covering of X, and if $\widetilde{X}$ is connected, N becomes the covering transformation group.

An s-arc in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i < s$; in other words, a directed walk of length s that never includes a backtracking. For a graph X and a subgroup G of $Aut(X)$, X is said to be G-vertex-transitive, G-edge-transitive, or G-s-arc-transitive if G is transitive on the sets of vertices, edges, or s-arcs of X, respectively, and G-s-regular if G acts regularly on the set of s-arcs of X. A graph X is said to be vertex-transitive, edge-transitive, s-arc-transitive, or s-regular if X is $Aut(X)$-vertex-transitive, $Aut(X)$-edge-transitive, $Aut(X)$-s-arc-transitive, or $Aut(X)$-s-regular, respectively. In particular, 1-arc-transitive means arc-transitive, or symmetric.

Covering techniques have long been known as a powerful tool in topology, and graph theory. Regular covering of a graph is currently an active topic in algebraic graph theory. Tutte [17, 18] showed that every finite cubic symmetric graph is s-regular for some $s \geq 1$, and this s is at most five. It follows that every cubic symmetric graph has an order of the form $2mp$ for a positive integer m and a prime number p. In order to know all cubic symmetric graphs, we need to classify the cubic s-regular graphs of order $2mp$ for a fixed positive integer m and each prime p. Conder and Dobcsányi [4, 5] classified the cubic s-regular graphs up to order 2048 with the help of the "Low index normal subgroups" routine in MAGMA system [2]. Cheng and Oxley [3] classified the cubic s-regular graphs of order $2p$. Recently, by using the covering technique, cubic s-regular graphs with order $2p^2$, $2p^3$, $4p$, $4p^2$, $6p$, $6p^2$, $8p$, $8p^2$, $10p$, $10p^2$, $12p$, $12p^2$, $14p$ and $16p$ were classified in $[1, 7-12, 15, 16]$.

In this paper, by employing the covering technique, and group-theoretical construction, we investigate connected cubic s-regular graphs of orders $22p$ and $22p^2$ for each $s \geq 1$, and each prime p.

2. Preliminaries

We start by introducing two propositions for later applications in this paper.

Proposition 2.1. [14, *Theorem* 9] Let X be a connected symmetric graph of prime valency and G a s-regular subgroup of $Aut(X)$ for some $s \geq 1$. If a normal subgroup N of G has more than two orbits, then it is semiregular and G/N is an s-regular subgroup of $Aut(X_N)$, where X_N is the quotient graph of X corresponding to the orbits of N. Furthermore, X is a N-regular covering of X_N.

Proposition 2.2. [18] If X is an s-arc regular cubic graph, then $s \leq 5$.

Remark. If X is a regular graph with valency k on n vertices and $s \geq 1$, then there exactly $nk(k-1)^{s-1}$ s-arcs. It follows that if X is s-arc transitive then $|Aut(X)|$ must be divisible by $nk(k-1)^{s-1}$, and if X is s-regular, then $|Aut(X)| = nk(k-1)^{s-1}$. In particular, a cubic arc-transitive graph X is s-regular if and only if $|Aut(X)| = (3n)2^{s-1}$.

3. Cubic s-regular graphs of orders $22p$ and $22p^2$

In this section, we investigate the connected cubic s-regular graphs of orders $22p$ and $22p^2$, where p is a prime. We have the following lemma, by [4, 5].

Lemma 3.1. Let p be a prime. Let X be a connected cubic symmetric graph. If X has order $22p$, and $p \leq 89$, then X is isomorphic to one of the 2-regular graph $F242$ with order 242, the 3-regular graphs $F110$, $F506A$ with orders 110, 506 respectively, or the 4-regular graph $F506B$ with order 506.

Lemma 3.2. Let p be a prime. Then, there is no cubic symmetric graph of order $22p$ for $p > 89$.

Proof. Suppose that X is a connected cubic symmetric graph of the order $22p$, where $p > 89$ is a prime. Set $A := Aut(X)$. By proposition 2.2, and [18], X is at most 5-regular. Then, $|A| = 2^s.\ 3.\ 11.\ p$, where $1 \leq s \leq 5$. Then we deduce that solvable. Because if not, then by the classification of finite simple groups, its composition factors would have to be one of the following non-abelian simple groups

$$\begin{gathered} M_{11}, M_{12}, PSL(2,11), PSL(2,19), PSL(2,23), \\ PSL(2,32),\ \text{or}\ PSU(5,2). \end{gathered} \tag{3.1}$$

Since $p > 89$, this contradicts the order of A. Therefore, A is solvable, and hence, elementary Abelian. Let N is a minimal normal subgroup of A. Then, N is an elementary Abelian. Hence, N is 2, 3, 11, or p-group. Then, N has more than two orbits on $V(X)$, and hence it is semiregular, by proposition 2.1. Thus, $|N| \mid 22p$. Therefore $|N| = 2, 11$, or p. In each case, we get a contradiction.

case I): $|N| = p$

If $|N| = p$, then the quotient graph X_N of X relative to N is an A/N-symmetric graph of the order 22, by Proposition 2.1. It is impossible by [4]. Suppose that $Q := O_p(A)$ be the maximal normal p-subgroup of A. Therefore, $O_p(A) = 1$.

case II): $|N| = 2$

If $|N| = 2$, then Proposition 2.1, implies that the quotient graph X_N corresponding to orbits of N has odd number of vertices and valency 3, which is impossible.

case III): $|N| = 11$

Now, we consider the quotient graph $X_N = X/N$ of X relative to N, where A/N is a s-regular of $Aut(X_N)$. Let K/N be a minimal normal subgroup of A/N. By the same argument as above K/N is solvable, and elementary Abelian. Then, we must have $|K/N| = 2$, or p. Consequently $|K| = 22$, or $11p$. If $|K| = 22$, we consider the quotient graph $X_K = X/K$ of X relative to K, where A/K is a s-regular of $Aut(X_K)$. By Proposition 2.1, X_K is an A/K-symmetric graph of the order p. Then, with the same reasoning as case II, we get a contradiction. Now, suppose that $|K| = 11p$. Since p > 89, K has a normal subgroup of order p, which is characteristic in K and hence is normal in A, contradicting to $O_p(A) = 1$. □

Theorem 3.3. Let p be a prime. Let X be a connected cubic symmetric graph. Let p be a prime. Let X be a connected cubic symmetric graph. If X has order $22p$ then, X is isomorphic to one of the 2-regular graph $F242$ with order 242, the 3-regular graphs $F110$, $F506A$ with orders 110, 506 respectively, or the 4-regular graph $F506B$ with order 506.

Proof. By Lemmas 3.1 and 3.2, the proof is complete. □

Theorem 3.4. Let p be a prime. Then, there is no cubic symmetric graph of order $22p^2$.

Proof. For $p \leq 7$, by [3], there is no connected cubic symmetric graph of the order $22p^2$. Thus we may assume that $p \geq 11$. Suppose that X is a connected cubic symmetric graph of the order $22p^2$, where $p > 7$ is a prime. Set $A := Aut(X)$. Then, $|A| = 2^s.\ 3.\ 11.\ p^2$, where $1 \leq s \leq 5$. First, suppose that A is nonsolvable. Then, A is a product of isomorphic non-abelian simple groups. By the classification of finite simple groups, its composition factors would have to be a non-abelian simple $\{2, 3, 11\}$-group, or $\{2, 3, 11, p\}$-group. Let q be a prime. Then, by [13, pp. 12-14], and [6], a non-abelian simple $\{2, q, p\}$-group is one of the groups

$$\begin{gathered} A_5, A_6, PSL(2,7), PSL(2,8), PSL(2,17), PSL(3,3), \\ PSU(3,3), \text{ or } PSU(4,2). \end{gathered} \tag{3.2}$$

But, this is contradiction to the order of A. Then, composition factors is a $\{2, 3, 11, p\}$-group. By the classification of finite simple groups, its composition factors would have to be one of the following non-abelian simple groups listed in (3.1). However, this contradicts the order of A. Therefore, A is solvable, and hence, elementary Abelian. Let N is a minimal normal subgroup of A. Then, N is an elementary Abelian. Hence, N is 2, 3, 11, or p-group. Then, N has more than two orbits on $V(X)$, and hence it is semiregular, by proposition 2.1. Thus, $|N| \mid 22p^2$. Therefore $|N| = 2, 11, p$, or p^2. In each case, we get a contradiction.

case I): $|N| = p^2$

If $|N| = p^2$, then the quotient graph X_N of X relative to N is an A/N-symmetric graph of the order 22, by Proposition 2.1. It is impossible by [4]. Suppose that $Q := O_p(A)$ be the maximal normal p-subgroup of A.

case II): $|N| = p$

Now, we consider the quotient graph $X_N = X/N$ of X relative to N, where A/N is a s-regular of $Aut(X_N)$. Let K/N be a minimal normal subgroup of A/N. By the same argument as above K/N is solvable, and elementary Abelian. Then, we must have $|K/N| = 2, 11$, or p. Now, by considering the quotient graph X_K with the same reasoning as lemma 3.2, a contradiction can be obtained.

case III): $|N| = 11$

Now, we consider the quotient graph $X_N = X/N$ of X relative to N, where A/N is a s-regular of $Aut(X_N)$. Let K/N be a minimal normal subgroup of A/N. By the same argument as above K/N is solvable, and elementary Abelian. Then, we must have $|K/N| = 2, p$, or p^2. Consequently $|K| = 22, 11p$, or $11p^2$. Then, with the same reasoning as case III of lemma 3.2 , we arrive at a contradiction.

case IV): $|N| = 2$

In this case by the argument as in the case II of Lemma 3.2 a similar contradiction is obtained. □

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CONTACT INFORMATION

A. A. Talebi, N. Mehdipoor
Department of Mathematics, University of Mazandaran, Babolsar, Iran
E-Mail: a.talebi@umz.ac.ir, nargesmehdipoor@yahoo.com

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