# Relative symmetric polynomials and money change problem

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ABSTRACT. This article is devoted to the number of non-negative solutions of the linear Diophantine equation

$$a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = d,$$

where $a_1, \ldots, a_n$, and $d$ are positive integers. We obtain a relation between the number of solutions of this equation and characters of the symmetric group, using *relative symmetric polynomials*. As an application, we give a necessary and sufficient condition for the space of the relative symmetric polynomials to be non-zero.

Suppose $a_1, \ldots, a_n$, and $d$ are positive integers, and consider the following linear Diophantine equation:

$$a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = d.$$

Let $Q_d(a_1, \ldots, a_n)$ be the number of non-negative integer solutions of this equation. Computing the exact values of the function $Q_d$ is the well-known *money change problem*. It is easy to see that a generating function for $Q_d(a_1, \ldots, a_n)$ is

$$\prod_{i=1}^{n} \frac{1}{1 - t^{a_i}}.$$

This article is devoted for an interesting relation between $Q_d$ and irreducible complex characters of the symmetric group $S_m$, where $m = a_1 + \cdots + a_n$. In fact, we will show that $Q_d$ is a permutation character of

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$S_m$, and then we will find its irreducible constituents. Our main tool, in the investigation of $Q_d$, is the notion of *relative symmetric polynomials*, which is introduced by the author in [3]. Once, we find the irreducible constituents of $Q_d$, we can also obtain a necessary and sufficient condition for vanishing of the space of relative symmetric polynomials. A similar result was obtained in [2], for vanishing of *symmetry classes of tensors*, using the same method.

We need a survey of results about relative symmetric polynomials in this article. For a detailed exposition, one can see [3].

Let $G$ be a subgroup of the full symmetric group $S_m$ of degree $m$ and suppose $\chi$ is an irreducible complex character of $G$. Let $H_d[x_1, \ldots, x_m]$ be the complex space of homogenous polynomials of degree $d$ with the independent commuting variables $x_1, \ldots, x_m$. Suppose $\Gamma_{m,d}^+$ is the set of all $m$-tuples of non-negative integers, $\alpha = (\alpha_1, \ldots, \alpha_m)$, such that $\sum_i \alpha_i = d$. For any $\alpha \in \Gamma_{m,d}^+$, define $X^\alpha$ to be the monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$. So the set $\{X^\alpha : \alpha \in \Gamma_{m,d}^+\}$ is a basis of $H_d[x_1, \ldots, x_m]$. We define also an inner product on $H_d[x_1, \ldots, x_m]$ by

$$< X^\alpha, X^\beta > = \delta_{\alpha,\beta}.$$

The group $G$ acts on $H_d[x_1, \ldots, x_m]$ via

$$q^\sigma(x_1, \ldots, x_m) = q(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)}).$$

It also acts on $\Gamma_{m,d}^+$ by

$$\sigma\alpha = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(m)}).$$

Let $\Delta$ be a set of representatives of orbits of $\Gamma_{m,d}^+$ under the action of $G$.

Now consider the idempotent

$$T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma)\sigma$$

in the group algebra $\mathbb{C}G$. Define the space of *relative symmetric polynomials of degree $d$* with respect to $G$ and $\chi$ to be

$$H_d(G, \chi) = T(G, \chi)(H_d[x_1, \ldots, x_m]).$$

Let $q \in H_d[x_1, \ldots, x_m]$. Then we set

$$q^* = T(G, \chi)(q)$$

and we call it a *symmetrized polynomial* with respect to $G$ and $\chi$. Clearly

$$H_d(G, \chi) = < X^{\alpha,*} : \alpha \in \Gamma^+_{m,d} >,$$

where $< set \ of \ vectors >$ denotes the subspace generated by a given set of vectors.

Recall that the inner product of two characters of an arbitrary group $K$ is defined as follows,

$$[\phi, \psi]_K = \frac{1}{|K|} \sum_{\sigma \in K} \phi(\sigma)\psi(\sigma^{-1}).$$

In the special case where $K$ is a subgroup of $G$ and $\phi$ and $\psi$ are characters of $G$, the notation $[\phi, \psi]_K$ will denote the inner product of the restrictions of $\phi$ and $\psi$ to $K$.

It is proved it [3] that for any $\alpha$, we have

$$||X^{\alpha,*}||^2 = \chi(1)\frac{[\chi, 1]_{G_\alpha}}{[G : G_\alpha]},$$

where $G_\alpha$ is the stabilizer subgroup of $\alpha$ under the action of $G$. Hence, $X^{\alpha,*} \neq 0$, if and only if $[\chi, 1]_{G_\alpha} \neq 0$. According to this result, let $\Omega$ be the set of all $\alpha \in \Gamma^+_{m,d}$, with $[\chi, 1]_{G_\alpha} \neq 0$ and suppose $\bar{\Delta} = \Delta \cap \Omega$.

We proved in [3], the following formula for the dimension of $H_d(G, \chi)$

$$\dim H_d(G, \chi) = \chi(1) \sum_{\alpha \in \bar{\Delta}} [\chi, 1]_{G_\alpha}.$$

Note that, $\bar{\Delta}$ depends on $\chi$, but $\Delta$ depends only on $G$. Since, $[\chi, 1]_{G_\alpha} = 0$, for all $\alpha \in \Delta - \bar{\Delta}$, we can re-write the above formula, as

$$\dim H_d(G, \chi) = \chi(1) \sum_{\alpha \in \Delta} [\chi, 1]_{G_\alpha}.$$

There is also another interesting formula for the dimension of $H_d(G, \chi)$. This is the formula which employs the function $Q_d$ and so it connects the money change problem to relative symmetric polynomials. Let $\sigma \in G$ be any element with the cycle structure $[a_1, \ldots, a_n]$, ( i.e. $\sigma$ is equal to a product of $n$ disjoint cycles of lengths $a_1, \ldots, a_n$, respectively). Define $Q_d(\sigma)$ to be the number of non-negative integer solutions of the equation

$$a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = d,$$

so, we have $Q_d(\sigma) = Q_d(a_1, \ldots, a_n)$. If we consider the free vector space $\mathbb{C}[\Gamma^+_{m,d}]$ as a $\mathbb{C}G$-module, then for all $\sigma \in G$, we have

$$Tr\ \sigma = Q_d(\sigma),$$

and hence, $Q_d$ is a permutation character of $G$. It is proved in [3], that we have also

$$\dim H_d(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) Q_d(\sigma).$$

Note that, we can write this result as $\dim H_d(G, \chi) = \chi(1)[\chi, Q_d]_G$. Now, comparing two formulae for the dimension of $H_d(G, \chi)$ and using the *reciprocity relation* for induced characters, we obtain

$$Q_d(a_1, \ldots, a_n) = \sum_{\alpha \in \Delta} (1_{G_\alpha})^G(\sigma),$$

where $\sigma \in S_m$ is any permutation of the cycle structure $[a_1, \ldots, a_n]$, $G$ is any subgroup of $S_m$ containing $\sigma$ and $m = a_1 + \cdots + a_n$. It is clear that, if $\alpha$ and $\beta$ are in the same orbit of $\Gamma^+_{m,d}$, then $(1_{G_\alpha})^G = (1_{G_\beta})^G$, so we have also

$$Q_d(a_1, \ldots, a_n) = \frac{1}{|G|} \sum_{\alpha \in \Gamma^+_{m,d}} |G_\alpha| (1_{G_\alpha})^G(\sigma).$$

As our main result in this section, we have,

**Theorem A.**

$$Q_d = \frac{1}{|G|} \sum_{\alpha \in \Gamma^+_{m,d}} |G_\alpha| (1_{G_\alpha})^G.$$

For a fixed equation

$$a_1 t_1 + a_2 t_2 + \cdots + a_n t_n = d,$$

suppose $\sigma$ is a permutation of the cycle structure $[a_1, \ldots, a_n]$ and let $G$ be the cyclic group generated by $\sigma$. Clearly this $G$ is the smallest subgroup of $S_m$ which we can use to obtain the number of non-negative solutions of the above equation. By Theorem A, the required number is

$$Q_d(\sigma) = \frac{1}{|G|} \sum_{\alpha \in \Gamma^+_{m,d}} |G_\alpha| (1_{G_\alpha})^G(\sigma).$$

On the other hand, if we want to have a formula for all values of $Q_d$, we must let $G = S_m$, which contains any possible cycle structure. So in the remaining part of this article, we will assume that, $G = S_m$, then

using representation theory of symmetric groups, we will find, irreducible constituents of $Q_d$.

We need some standard notions from representation theory of symmetric groups. Ordinary representations of $S_m$ are in one to one correspondence with *partitions* of $m$. Let

$$\pi = (\pi_1, \ldots, \pi_s)$$

be any partition of $m$. The irreducible character of $S_m$, corresponding to a partition $\pi$ is denoted by $\chi^\pi$. There is also a subgroup of $S_m$, associated to $\pi$, which is called the *Young subgroup* and it is defined as,

$$S_\pi = S_{\{1,\ldots,\pi_1\}} \times S_{\{\pi_1+1,\ldots,\pi_1+\pi_2\}} \times \cdots.$$

Therefore, we have $S_\pi \cong S_{\pi_1} \times \cdots \times S_{\pi_s}$.

Let $\pi = (\pi_1, \ldots, \pi_s)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be two partitions of $m$. We say that $\mu$ majorizes $\pi$, iff for any $1 \le i \le min\{s, l\}$, the inequality

$$\pi_1 + \cdots + \pi_i \le \mu_1 + \cdots + \mu_i$$

holds. In this case we write $\lambda \trianglelefteq \mu$. This is clearly a partial ordering on the set of all partitions of $m$. A generalized $\mu$-tableau of type $\pi$ is a function

$$T : \{(i, j) : 1 \le i \le h(\mu), 1 \le j \le \mu_i\} \to \{1, 2, \ldots, m\}$$

such that for any $1 \le i \le m$, we have $|t^{-1}(i)| = \pi_i$. This generalized tableau is called semi-standard if for each $i$, $j_1 < j_2$ implies $T(i, j_1) \le T(i, j_2)$ and for any $j$, $i_1 < i_2$ implies $T(i_1, j) < T(i_2, j)$. In other words, $T$ is semi-standard, iff every row of $T$ is non-descending and every column of $T$ is ascending. The number of all such semi-standard tableaux is denoted by $K_{\mu\pi}$ and it is called the *Kostka number*. It is well known that $K_{\mu\pi} \ne 0$ iff $\mu$ majorizes $\pi$, see [1], for example. We have also,

$$
\begin{aligned}
K_{\mu\pi} &= [(1_{S_\pi})^{S_m}, \chi^\mu]_{S_m} \\
&= [1, \chi_\mu]_{S^\pi}.
\end{aligned}
$$

For any $\alpha \in \Gamma_{m,d}^+$, the multiplicity partition is denoted by $M(\alpha)$, so, to obtain $M(\alpha)$, we must arrange the multiplicities of numbers $0, 1, \ldots, d$ in $\alpha$ in the descending order. It is clear that $(S_m)_\alpha \cong S_{M(\alpha)}$, the Young subgroup. If $M(\alpha) = (k_1, \ldots, k_s)$, then we have

$$|(S_m)_\alpha| = k_1! k_2! \ldots k_s!.$$

In what follows, we use the notation $M(\alpha)!$ for the number $k_1!k_2!\ldots k_s!$. On the other hand, we have

$$
\begin{aligned}
(1_{G_\alpha})^G &= (1_{S_{M(\alpha)}})^{S_m} \\
&= \sum_{M(\alpha)\trianglelefteq\pi} K_{\pi,M(\alpha)}\chi^\pi.
\end{aligned}
$$

Now, using Theorem A, we obtain,

**Theorem B.**

$$
Q_d = \frac{1}{m!}\sum_{\alpha\in\Gamma^+_{m,d}}\sum_{M(\alpha)\trianglelefteq\pi} M(\alpha)!K_{\pi,M(\alpha)}\chi^\pi.
$$

As a result, we can compute the dimension of $H_d(S_m,\chi^\pi)$, in a new fashion. We have,

$$
\begin{aligned}
\dim H_d(S_m,\chi^\pi) &= \chi^\pi(1)[\chi^\pi,Q_d]_{S_m} \\
&= \frac{\chi^\pi(1)}{m!}\sum_{\alpha\in\Gamma^+_{m,d},M(\alpha)\trianglelefteq\pi} M(\alpha)!K_{\pi,M(\alpha)}.
\end{aligned}
$$

Note that, this generalizes the similar formulae in the final part of the second section of [3]. Now, as a final result, we have also, a necessary and sufficient condition for $H_d(S_m,\chi^\pi)$ to be non-zero.

**Theorem C.** *We have $H_d(S_m,\chi^\pi)\neq 0$, if and only if there exists $\alpha\in\Gamma^+_{m,d}$, such that $M(\alpha)\trianglelefteq\pi$.*

### References

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