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RESEARCH ARTICLE

Free abelian dimonoids

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Abstract. We construct a free abelian dimonoid and describe the least abelian congruence on a free dimonoid. Also we show that free abelian dimonoids are determined by their endomorphism semigroups.

1. Introduction

The notion of a dimonoid was introduced by Jean-Louis Loday in [1]. An algebra $(D, \dashv, \vdash)$ with two binary associative operations $\dashv$ and $\vdash$ is called a *dimonoid* if for all $x, y, z \in D$ the following conditions hold:

$$(D_1) \qquad (x \dashv y) \dashv z = x \dashv (y \vdash z),$$
$$(D_2) \qquad (x \vdash y) \dashv z = x \vdash (y \dashv z),$$
$$(D_3) \qquad (x \dashv y) \vdash z = x \vdash (y \vdash z).$$

If operations of a dimonoid coincide, the dimonoid becomes a semigroup.

Dimonoids and in particular dialgebras have been studied by many authors (see, e.g., [2]–[5]), they play a prominent role in problems from the theory of Leibniz algebras. The first result about dimonoids is the description of a free dimonoid [1]. T. Pirashvili [4] introduced the notion of a duplex which generalizes the notion of a dimonoid and constructed a free duplex. Free dimonoids and free commutative dimonoids were

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investigated in [6] and [7] respectively. Free normal dibands and other relatively free dimonoids were described in [8], [9]. In this paper we study free abelian dimonoids.

The paper is organized as follows. In Section 2 we give necessary definitions and examples of abelian dimonoids. In Section 3 we construct a free abelian dimonoid and, in particular, consider a free abelian dimonoid of rank 1. In Section 4 we define the least congruence on a free dimonoid such that the corresponding quotient-dimonoid is isomorphic to the free abelian dimonoid. In Section 5 we prove that free abelian dimonoids are determined by their endomorphisms.

2. Examples of abelian dimonoids

It is well-known that a non-empty class H of algebraic systems is a variety if the Cartesian product of any sequence of H-systems is a H-system, every subsystem of an arbitrary H-system is a H-system and any homomorphic image of an arbitrary H-system is a H-system (Birkhoff [10]).

A dimonoid $(D, \dashv, \vdash)$ we call *abelian* (in the same way as a digroup in [11]) if for all $x, y \in D$,

$$x \dashv y = y \vdash x.$$

The class of all abelian dimonoids satisfies the conditions of Birkhoff's theorem and therefore it is a variety. A dimonoid which is free in the variety of abelian dimonoids will be called a *free abelian dimonoid*.

It should be noted that the class of all abelian dimonoids does not coincide with the class of all commutative dimonoids [7] (both operations of such dimonoids are commutative). For example, a non-singleton left zero and right zero dimonoid [9] is abelian but not commutative.

Let Z be the set of all integers, $E = \{\lambda, \mu\}$ be an arbitrary two-element set. Define two binary operations $\dashv$ and $\vdash$ on $Z \times E$ as follows:

$$(m, x) \dashv (n, y) = \begin{cases} (m+n+1, x), & y = \lambda, \\ (m+n-1, x), & y = \mu, \end{cases}$$

$$(m, x) \vdash (n, y) = \begin{cases} (m+n+1, y), & x = \lambda, \\ (m+n-1, y), & x = \mu. \end{cases}$$

Proposition 1. *The algebra $(Z \times E, \dashv, \vdash)$ is an abelian dimonoid.*

Proof. Let $(m,x),(n,y),(s,z)\in Z\times E$. If $y=z=\lambda$ or $y=z=\mu$, we obtain

$$\begin{aligned}((m,x)\dashv(n,\lambda))\dashv(s,\lambda)&=(m+n+s+2,x)\\&=(m,x)\dashv((n,\lambda)\dashv(s,\lambda))\quad\text{or}\\((m,x)\dashv(n,\mu))\dashv(s,\mu)&=(m+n+s-2,x)\\&=(m,x)\dashv((n,\mu)\dashv(s,\mu))\end{aligned}$$

respectively.

For $y=\lambda$, $z=\mu$ or $y=\mu$, $z=\lambda$, we have

$$\begin{aligned}((m,x)\dashv(n,y))\dashv(s,z)&=(m+n+s,x)\\&=(m,x)\dashv((n,y)\dashv(s,z)).\end{aligned}$$

Therefore, the operation $\dashv$ is associative. Analogously we can show that $\vdash$ is an associative operation too.

Show that the axiom (D_1) holds. If $y=z=\lambda$ or $y=z=\mu$,

$$\begin{aligned}(m,x)\dashv((n,\lambda)\vdash(s,\lambda))&=(m+n+s+2,x)\\&=((m,x)\dashv(n,\lambda))\dashv(s,\lambda)\quad\text{or}\\(m,x)\dashv((n,\mu)\vdash(s,\mu))&=(m+n+s-2,x)\\&=((m,x)\dashv(n,\mu))\dashv(s,\mu).\end{aligned}$$

For $y=\lambda$, $z=\mu$ or $y=\mu$, $z=\lambda$, we obtain

$$\begin{aligned}(m,x)\dashv((n,y)\vdash(s,z))&=(m+n+s,x)\\&=((m,x)\dashv(n,y))\dashv(s,z).\end{aligned}$$

The axiom (D_3) is checked similarly. Now we consider the axiom (D_2). Let $x=z=\lambda$ or $x=z=\mu$. Then

$$\begin{aligned}(m,\lambda)\vdash((n,y)\dashv(s,\lambda))&=(m+n+s+2,y)\\&=((m,\lambda)\vdash(n,y))\dashv(s,\lambda)\quad\text{or}\\(m,\mu)\vdash((n,y)\dashv(s,\mu))&=(m+n+s-2,y)\\&=((m,\mu)\vdash(n,y))\dashv(s,\mu).\end{aligned}$$

If $x=\lambda,z=\mu$ or $x=\mu,z=\lambda$, then

$$\begin{aligned}(m,x)\vdash((n,y)\dashv(s,z))&=(m+n+s,y)\\&=((m,x)\vdash(n,y))\dashv(s,z),\end{aligned}$$

which completes the verification of (D_2).

The fact that $(Z\times E,\dashv,\vdash)$ is abelian can be checked immediately. $\square$

An element e of an arbitrary dimonoid $(D, \dashv, \vdash)$ is called a *bar-unit* (see, e.g., [1]) if for all $g \in D$,

$$e \vdash g = g = g \dashv e.$$

In contrast to monoids a dimonoid may have many bar-units. For example, for the dimonoid from Proposition 1 we have

$$\begin{aligned}(-1, \lambda) \vdash (m, x) = (m, x) = (m, x) \dashv (-1, \lambda),\\ (1, \mu) \vdash (m, x) = (m, x) = (m, x) \dashv (1, \mu)\end{aligned}$$

for any $(m, x) \in Z \times E$. Thus, $(-1, \lambda)$ and $(1, \mu)$ are bar-units. Moreover, another bar-units of $(Z \times E, \dashv, \vdash)$ do not exist.

Let G be an arbitrary additive abelian group, $X_1, X_2, \ldots, X_n$ $(n \geqslant 2)$ be non-empty subsets of G and $X_\alpha = G$ for some $\alpha \in \{1, 2, \ldots, n\}$. For all $t = (t_1, t_2, \ldots, t_n) \in \prod_{i=1}^n X_i$ we put $t^+ = t_1 + t_2 + \ldots + t_n$.

Take arbitrary $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \prod_{i=1}^n X_i$ and define two binary operations $\dashv_\alpha$ and $\vdash_\alpha$ on $\prod_{i=1}^n X_i$ by

$$\begin{aligned}x \dashv_\alpha y &= (x_1, \ldots, x_\alpha + y^+, \ldots, x_n),\\ x \vdash_\alpha y &= (y_1, \ldots, y_\alpha + x^+, \ldots, y_n).\end{aligned}$$

Proposition 2. *For every* $\alpha \in \{1, 2, \ldots, n\}$ *the algebra* $(\prod_{i=1}^n X_i, \dashv_\alpha, \vdash_\alpha)$ *is an abelian dimonoid.*

Proof. Let $x, y, z \in \prod_{i=1}^n X_i$. Then

$$\begin{aligned}(x \dashv_\alpha y) \dashv_\alpha z &= (x_1, \ldots, x_\alpha + y^+, \ldots, x_n) \dashv_\alpha (z_1, z_2, \ldots, z_n)\\ &= (x_1, \ldots, x_\alpha + y^+ + z^+, \ldots, x_n)\\ &= (x_1, x_2 \ldots, x_n) \dashv_\alpha (y_1, \ldots, y_\alpha + z^+, \ldots, y_n)\\ &= x \dashv_\alpha (y \dashv_\alpha z),\\ (x \vdash_\alpha y) \vdash_\alpha z &= (y_1, \ldots, y_\alpha + x^+, \ldots, y_n) \vdash_\alpha (z_1, z_2, \ldots, z_n)\\ &= (z_1, \ldots, z_\alpha + x^+ + y^+, \ldots, z_n)\\ &= (x_1, x_2 \ldots, x_n) \vdash_\alpha (z_1, \ldots, z_\alpha + y^+, \ldots, z_n)\\ &= x \vdash_\alpha (y \vdash_\alpha z).\end{aligned}$$

Thus, operations $\dashv_\alpha$ and $\vdash_\alpha$ are associative.

Show that axioms $(D_1) - (D_3)$ hold:

$$\begin{aligned}
(x \dashv_\alpha y) \dashv_\alpha z &= (x_1, \ldots, x_\alpha + y^+ + z^+, \ldots, x_n) \\
&= (x_1, x_2, \ldots, x_n) \dashv_\alpha (z_1, \ldots, z_\alpha + y^+, \ldots, z_n) \\
&= x \dashv_\alpha (y \vdash_\alpha z), \\
(x \vdash_\alpha y) \dashv_\alpha z &= (y_1, \ldots, y_\alpha + x^+, \ldots, y_n) \dashv_\alpha (z_1, z_2, \ldots, z_n) \\
&= (y_1, \ldots, y_\alpha + z^+ + x^+, \ldots, y_n) \\
&= (x_1, x_2 \ldots, x_n) \vdash_\alpha (y_1, \ldots, y_\alpha + z^+, \ldots, y_n) \\
&= x \vdash_\alpha (y \dashv_\alpha z), \\
(x \dashv_\alpha y) \vdash_\alpha z &= (x_1, \ldots, x_\alpha + y^+, \ldots, x_n) \vdash_\alpha (z_1, z_2, \ldots, z_n) \\
&= (z_1, \ldots, z_\alpha + x^+ + y^+, \ldots, z_n) \\
&= x \vdash_\alpha (y \vdash_\alpha z).
\end{aligned}$$

Therefore, $(\prod_{i=1}^n X_i, \dashv_\alpha, \vdash_\alpha)$ is a dimonoid. Moreover,

$$x \dashv_\alpha y = (x_1, \ldots, x_\alpha + y^+, \ldots, x_n) = y \vdash_\alpha x$$

for all $x, y \in \prod_{i=1}^n X_i$. □

Let $(S, \circ)$ be an arbitrary semigroup. A semigroup $(S, *)$, where $x * y = y \circ x$ for all $x, y \in S$, is called a *dual semigroup* to $(S, \circ)$.

A semigroup $(S, \circ)$ is called *left commutative* (respectively, *right commutative*) if it satisfies the identity $x \circ y \circ a = y \circ x \circ a$ (respectively, $a \circ x \circ y = a \circ y \circ x$).

Proposition 3. *Let $(S, \circ)$ be an arbitrary right commutative semigroup and $(S, *)$ be a dual semigroup to $(S, \circ)$. Then the algebra $(S, \circ, *)$ is an abelian dimonoid.*

Proof. The proof follows from Lemma 3 of [9]. □

Proposition 4. *Let $(S, *)$ be an arbitrary left commutative semigroup and $(S, \circ)$ be a dual semigroup to $(S, *)$. Then the algebra $(S, \circ, *)$ is an abelian dimonoid.*

Proof. The proof follows from Lemma 4 of [9]. □

An important example of abelian dimonoids is the class of abelian digroups (see [11]). The idea of the notion of a digroup first appeared in the work of Jean-Louis Loday [1].

3. The free abelian dimonoid

Let X be an arbitrary set and N be the set of all natural numbers. Denote by $FCm(X)$ the free commutative monoid on X with the identity ε. Words of $FCm(X)$ we write as $w = w_1^{\alpha_1} w_2^{\alpha_2} \ldots w_n^{\alpha_n}$, where $w_1, w_2, \ldots, w_n \in X$ are pairwise distinct, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in N \cup \{0\}$. Here $w_i^0, 1 \leqslant i \leqslant n$, is the empty word ε and $w^1 = w$ for all $w \in X$.

We put

$$FAd(X) = X \times FCm(X)$$

and define two binary operations $\dashv$ and $\vdash$ on $FAd(X)$ as follows:

$$(x, u) \dashv (y, v) = (x, uyv),$$
$$(x, u) \vdash (y, v) = (y, xuv).$$

Note that for every element t of an arbitrary abelian dimonoid $(D, \prec, \succ)$ the degrees

$$t_{\prec}^n = \underbrace{t \prec t \prec \ldots \prec t}_{n}, \qquad t_{\succ}^n = \underbrace{t \succ t \succ \ldots \succ t}_{n}$$

coincide. Therefore, we will write t^n instead of $t_{\prec}^n$ $(= t_{\succ}^n)$.

Theorem 1. *The algebra $(FAd(X), \dashv, \vdash)$ is the free abelian dimonoid.*

Proof. Let $(x, u), (y, v), (z, w) \in FAd(X)$. Then

$$\begin{aligned}((x, u) \dashv (y, v)) \dashv (z, w) &= (x, uyv) \dashv (z, w) \\ &= (x, uyvzw) = (x, u) \dashv ((y, v) \dashv (z, w)), \\ ((x, u) \vdash (y, v)) \vdash (z, w) &= (y, xuv) \vdash (z, w) \\ &= (z, yxuvw) = (x, u) \vdash ((y, v) \vdash (z, w)).\end{aligned}$$

Thus, operations $\dashv$ and $\vdash$ are associative. In addition,

$$\begin{aligned}((x, u) \dashv (y, v)) \dashv (z, w) &= (x, uyvzw) \\ &= (x, u) \dashv (z, yvw) = (x, u) \dashv ((y, v) \vdash (z, w)), \\ ((x, u) \vdash (y, v)) \dashv (z, w) &= (y, xuvzw) \\ &= (x, u) \vdash (y, vzw) = (x, u) \vdash ((y, v) \dashv (z, w)), \\ ((x, u) \dashv (y, v)) \vdash (z, w) &= (x, uyv) \vdash (z, w) \\ &= (z, yxuvw) = (x, u) \vdash ((y, v) \vdash (z, w)).\end{aligned}$$

So, $(FAd(X), \dashv, \vdash)$ is a dimonoid and, obviously, it is abelian.

For all $(t, w) \in FAd(X)$, where $w = w_1^{\alpha_1} w_2^{\alpha_2} \ldots w_n^{\alpha_n}$, we obtain the following representation:

$$(t, w) = (t, \varepsilon) \dashv (w_1, \varepsilon)^{\alpha_1} \dashv \ldots \dashv (w_n, \varepsilon)^{\alpha_n}.$$

This representation we call a canonical form of elements of the dimonoid $(FAd(X), \dashv, \vdash)$. It is clear that such representation is unique up to an order of $(w_i, \varepsilon), 1 \leqslant i \leqslant n$. Moreover, $\langle X \times \varepsilon \rangle = (FAd(X), \dashv, \vdash)$.

Show that the dimonoid $(FAd(X), \dashv, \vdash)$ is free abelian. Let $(D', \dashv', \vdash')$ be an arbitrary abelian dimonoid, ξ be any mapping of $X \times \varepsilon$ into D'. Further, we naturally extend ξ to a mapping Ξ of $FAd(X)$ into D' using the canonical representation of elements of $(FAd(X), \dashv, \vdash)$, that is,

$$(t, w)\Xi = (t, \varepsilon)\xi \dashv' ((w_1, \varepsilon)\xi)^{\alpha_1} \dashv' \ldots \dashv' ((w_n, \varepsilon)\xi)^{\alpha_n}$$

for any $(t, w) \in FAd(X)$, where $w = w_1^{\alpha_1} w_2^{\alpha_2} \ldots w_n^{\alpha_n}$.

It is easy to see that Ξ is a homomorphism of $(FAd(X), \dashv)$ into $(D', \dashv')$. Using that $(D', \dashv', \vdash')$ is an abelian dimonoid too, we obtain

$$\begin{aligned} ((t, u) \vdash (s, v))\Xi &= ((s, v) \dashv (t, u))\Xi \\ &= (s, v)\Xi \dashv' (t, u)\Xi = (t, u)\Xi \vdash' (s, v)\Xi \end{aligned}$$

for all $(t, u), (s, v) \in FAd(X)$. □

Observe that the cardinality of a set X is the *rank* of the constructed free abelian dimonoid $(FAd(X), \dashv, \vdash)$ and this dimonoid is uniquely determined up to an isomorphism by $|X|$.

Now we consider the structure of a free abelian dimonoid of rank 1.

Lemma 1. *Operations of the free abelian dimonoid* $(FAd(X), \dashv, \vdash)$ *coincide if and only if* $|X| = 1$.

Proof. Assume that operations of $(FAd(X), \dashv, \vdash)$ coincide and $x, y \in X$ are distinct. Then for all $u, v \in FCm(X)$,

$$(x, u) \dashv (y, v) = (x, uyv) \neq (y, xuv) = (x, u) \vdash (y, v),$$

which contradicts the fact that $\dashv = \vdash$.

Let $X = \{x\}$, then for all $(x, u), (x, v) \in FAd(X)$ we have

$$(x, u) \dashv (x, v) = (x, uxv) = (x, u) \vdash (x, v).$$ □

Let $(S, \circ)$ be an arbitrary semigroup and $a \in S$. Define on S a new binary operation $\circ_a$ by

$$x \circ_a y = x \circ a \circ y$$

for all $x, y \in S$.

Clearly, $(S, \circ_a)$ is a semigroup, it is called a *variant* of $(S, \circ)$.

Proposition 5. *The free abelian dimonoid $(FAd(X), \dashv, \vdash)$ of rank 1 is isomorphic to the variant $(N^0, +_1)$ of the additive semigroup of all non-negative integers.*

Proof. Let $X = \{x\}$, then $FAd(X) = \{(x, x^n) | n \in N^0\}$. By Lemma 1, for $(FAd(X), \dashv, \vdash)$ we have $\dashv = \vdash$. Define a mapping φ of $(FAd(X), \dashv, \vdash)$ into $(N^0, +_1)$ by

$$\varphi : (x, x^n) \mapsto n$$

for any $(x, x^n) \in FAd(X)$.

It is clear that φ is a bijection. In addition, for all (x, x^n), $(x, x^m) \in FAd(X)$ we obtain

$$\begin{aligned} ((x, x^n) \dashv (x, x^m))\varphi &= (x, x^{n+m+1})\varphi = n + m + 1 \\ &= n +_1 m = (x, x^n)\varphi +_1 (x, x^m)\varphi. \end{aligned}$$ □

4. The least abelian congruence

Let $(D, \dashv, \vdash)$ be an arbitrary dimonoid, ρ be an equivalence relation on D which is stable on the left and on the right with respect to each of operations $\dashv, \vdash$. In this case ρ is called a *congruence* on $(D, \dashv, \vdash)$.

If $f : D_1 \to D_2$ is a homomorphism of dimonoids, then the corresponding congruence on D_1 will be denoted by $\triangle_f$. For a congruence ρ on a dimonoid $(D, \dashv, \vdash)$ the corresponding quotient-dimonoid is denoted by $(D, \dashv, \vdash)/\rho$. A congruence ρ on a dimonoid $(D, \dashv, \vdash)$ is called *abelian* if $(D, \dashv, \vdash)/\rho$ is an abelian dimonoid.

As usual N denotes the set of all positive integers, and let $n \in N$. For an arbitrary set X by $\widetilde{X}$ we denote the copy of X, that is, $\widetilde{X} = \{\widetilde{x} \mid x \in X\}$ and put

$$Y_n^{(1)} = \underbrace{\widetilde{X} \times X \times \ldots \times X}_{n}, \quad Y_n^{(2)} = \underbrace{X \times \widetilde{X} \times X \times \ldots \times X}_{n},$$

$$Y_n^{(3)} = \underbrace{X \times X \times \widetilde{X} \times \ldots \times X}_{n}, \quad \ldots, \quad Y_n^{(n)} = \underbrace{X \times X \times \ldots \times \widetilde{X}}_{n}.$$

We denote the union of n different copies $Y_n^{(i)}, 1 \leqslant i \leqslant n$, of X^n by Y_n and assume $Fd(X) = \bigcup_{n \geqslant 1} Y_n$. Define operations $\prec$ and $\succ$ on $Fd(X)$ as follows:

$$(x_1, \ldots, \widetilde{x}_i, \ldots, x_m) \prec (y_1, \ldots, \widetilde{y}_j, \ldots, y_n) = (x_1, \ldots, \widetilde{x}_i, \ldots, x_m, y_1, \ldots, y_n),$$

$$(x_1, \ldots, \widetilde{x}_i, \ldots, x_m) \succ (y_1, \ldots, \widetilde{y}_j, \ldots, y_n) = (x_1, \ldots, x_m, y_1, \ldots, \widetilde{y}_j, \ldots, y_n)$$

for all $(x_1, \ldots, \widetilde{x}_i, \ldots, x_m), (y_1, \ldots, \widetilde{y}_j, \ldots, y_n) \in Fd(X)$.

According to [1], $(Fd(X), \prec, \succ)$ is the *free dimonoid* on X. Elements of $Fd(X)$ are called *words*, $\widetilde{X}$ is the *generating set* of $(Fd(X), \prec, \succ)$.

Let $(Fd(X), \prec, \succ)$ be the free dimonoid on X and $w \in Fd(X)$. The canonical form of $w = (w_1, \ldots, \widetilde{w}_l, \ldots, w_k)$ is its representation in the shape:

$$w = \widetilde{w}_1 \succ \ldots \succ \widetilde{w}_l \prec \ldots \prec \widetilde{w}_k.$$

We call k as the *length* of w and denote it by $l(w)$. For any $x \in X$ by $q_{\widetilde{x}}(w)$ we denote the quantity of all elements $\widetilde{x} \in \widetilde{X}$ that are included in the canonical form $\widetilde{w}_1 \succ \ldots \succ \widetilde{w}_l \prec \ldots \prec \widetilde{w}_k$ of w.

Define a binary relation σ on $Fd(X)$ as follows: $u = (u_1, \ldots, \widetilde{u}_i, \ldots, u_n)$ and $v = (v_1, \ldots, \widetilde{v}_j, \ldots, v_m)$ of $Fd(X)$ are σ-*equivalent* if for all $x \in X$,

$$q_{\widetilde{x}}(u) = q_{\widetilde{x}}(v) \ \textit{ and } \ u_i = v_j.$$

We note that $q_{\widetilde{x}}(u) = q_{\widetilde{x}}(v)$ for all $x \in X$ implies $l(u) = l(v)$.

For example, for $u = (a, \widetilde{b}, a, c)$, $v = (a, \widetilde{a})$ and $w = (c, a, a, \widetilde{b})$ we have $q_{\tilde{a}}(p) = 2$ for all $p \in \{u, v, w\}$, $l(v) = 2$ and $(u, w) \in \sigma$.

Theorem 2. *The binary relation σ is the least abelian congruence on the free dimonoid $(Fd(X), \prec, \succ)$.*

Proof. It is easy to see that σ is an equivalence relation. Assume that $u = (u_1, \ldots, \widetilde{u}_i, \ldots, u_n), v = (v_1, \ldots, \widetilde{v}_j, \ldots, v_m) \in Fd(X)$ such that $u \sigma v$ and $w = (w_1, \ldots, \widetilde{w}_k, \ldots, w_l) \in Fd(X)$. Then

$$\begin{aligned} u \prec w &= (u_1, \ldots, \widetilde{u}_i, \ldots, u_n, w_1, \ldots, w_l), \\ v \prec w &= (v_1, \ldots, \widetilde{v}_j, \ldots, v_m, w_1, \ldots, w_l), \\ u \succ w &= (u_1, \ldots, u_n, w_1, \ldots, \widetilde{w}_k, \ldots, w_l), \\ v \succ w &= (v_1, \ldots, v_m, w_1, \ldots, \widetilde{w}_k, \ldots, w_l). \end{aligned}$$

Since $u_i = v_j$ and

$$q_{\widetilde{x}}(u \prec w) = q_{\widetilde{x}}(v \prec w), \qquad q_{\widetilde{x}}(u \succ w) = q_{\widetilde{x}}(v \succ w)$$

for any $x \in X$, we have $(u \prec w)\sigma(v \prec w)$ and $(u \succ w)\sigma(v \succ w)$. Analogously we can show that $(w \prec u)\sigma(w \prec v)$ and $(w \succ u)\sigma(w \succ v)$. Thus, σ is a congruence.

In addition, we note that $(u \prec v)\sigma(v \succ u)$ for all $u, v \in Fd(X)$, therefore $(Fd(X), \prec, \succ)/\sigma$ is abelian. A class of $(Fd(X), \prec, \succ)/\sigma$ which contains w we denote by $[w]$.

Further, we show that the quotient-dimonoid $(Fd(X), \prec, \succ)/\sigma$ is isomorphic to the free abelian dimonoid $(FAd(X), \dashv, \vdash)$ (see Theorem 1).

Define a mapping φ of $(Fd(X), \prec, \succ)/\sigma$ into $(FAd(X), \dashv, \vdash)$ by

$$[w]\varphi = (w_k, w_1 \ldots w_{k-1} w_{k+1} \ldots w_l)$$

for all words $w = (w_1, \ldots, \widetilde{w}_k, \ldots, w_l) \in Fd(X)$ with $l(w) \geqslant 2$, and $[w]\varphi = (w_1, \varepsilon)$ for any $w = \widetilde{w}_1 \in Fd(X)$. It is clear that φ is a bijection.

For all $[u], [v] \in (Fd(X), \prec, \succ)/\sigma$, where $u = (u_1, \ldots, \widetilde{u}_i, \ldots, u_n)$, $v = (v_1, \ldots, \widetilde{v}_j, \ldots, v_m)$, we have

$$\begin{aligned}([u] \prec [v])\varphi &= [(u_1, \ldots, \widetilde{u}_i, \ldots, u_n, v_1, \ldots, v_m)]\varphi \\ &= (u_i, u_1 \ldots u_{i-1} u_{i+1} \ldots u_n v_1 \ldots v_m) \\ &= (u_i, u_1 \ldots u_{i-1} u_{i+1} \ldots u_n) \dashv (v_j, v_1 \ldots v_{j-1} v_{j+1} \ldots v_m) \\ &= [u]\varphi \dashv [v]\varphi.\end{aligned}$$

Since dimonoids $(Fd(X), \prec, \succ)/\sigma$ and $(FAd(X), \dashv, \vdash)$ are abelian,

$$([u] \succ [v])\varphi = ([v] \prec [u])\varphi = [v]\varphi \dashv [u]\varphi = [u]\varphi \vdash [v]\varphi$$

for all $[u], [v] \in (Fd(X), \prec, \succ)/\sigma$.

Thus, $(Fd(X), \prec, \succ)/\sigma$ is free abelian and the composition $\eta^\natural \circ \varphi$, where $\eta^\natural : (Fd(X), \prec, \succ) \to (Fd(X), \prec, \succ)/\sigma$ is the natural homomorphism, is an epimorphism of $(Fd(X), \prec, \succ)$ on $(FAd(X), \dashv, \vdash)$ inducing the least abelian congruence on $Fd(X)$. From the definition of $\eta^\natural \circ \varphi$ it follows that $\triangle_{\eta^\natural \circ \varphi} = \sigma$. □

5. Determinability

One of the venerable algebraic problems the first instance of which was considered by E. Galois (see [12]) is the determinability of an algebraic structure by its endomorphism semigroup. The determinability problem for free algebras in a certain variety was raised by B. Plotkin [13]. For free groups this problem was solved by E. Formanek [14]. An analogous problem for free semigroups and free monoids was decided in [15].

Some characteristics for the enomorphism monoid of a free dimonoid of rank 1 were obtained in [16]. Determinability of free trioids by their endomorphism semigroups was proved in [17].

Recall that an algebra A of some class Ω *is determined* by its endomorphism semigroup in the class Ω if for any algebra $B \in \Omega$ the condition $End(A) \cong End(B)$ implies $A \cong B$. Note that the converse implication is obvious.

Let $\mathfrak{F}_X = (FAd(X), \dashv, \vdash)$ be the free abelian dimonoid on X and $(t, u) \in FAd(X)$, $u = u_1^{\alpha_1} u_2^{\alpha_2} \ldots u_n^{\alpha_n}$. From Theorem 1 it follows that an arbitrary endomorphism $\Xi \in End(\mathfrak{F}_X)$ has form:

$$(t, u)\Xi = (t, \varepsilon)\xi \dashv ((u_1, \varepsilon)\xi)^{\alpha_1} \dashv \ldots \dashv ((u_n, \varepsilon)\xi)^{\alpha_n},$$

where $\xi : X \times \varepsilon \to FAd(X)$ is any mapping.

An endomorphism $\theta_{(t,u)} \in End(\mathfrak{F}_X)$ we call *constant* if $(x, \varepsilon)\theta_{(t,u)} = (t, u)$ for all $x \in X$.

Lemma 2.

(i) *An endomorphism f of the free abelian dimonoid $\mathfrak{F}_X$ is constant if and only if $\psi f = f$ for all $\psi \in Aut(\mathfrak{F}_X)$.*

(ii) *An endomorphism f of the free abelian dimonoid $\mathfrak{F}_X$ is constant idempotent if and only if $f = \theta_{(x,\varepsilon)}$ for some $x \in X$.*

Proof. (i) Suppose that an endomorphism $f \in End(\mathfrak{F}_X)$ is constant and $\psi \in Aut(\mathfrak{F}_X)$. Then $f = \theta_{(t,u)}$ for some $(t, u) \in FAd(X)$, in addition,

$$(x, \varepsilon)(\psi\theta_{(t,u)}) = ((x, \varepsilon)\psi)\, \theta_{(t,u)} = (t, u) = (x, \varepsilon)\theta_{(t,u)}$$

for any $x \in X$. Thus, $\psi\theta_{(t,u)} = \theta_{(t,u)}$.

Conversely, let $\psi f = f$ for all $\psi \in Aut(\mathfrak{F}_X)$ and some $f \in End(\mathfrak{F}_X)$. For fixed $x \in X$ we obtain

$$(x, \varepsilon)f = (x, \varepsilon)\,(\psi f) = ((x, \varepsilon)\psi)\, f = (y, \varepsilon)f,$$

where $(y, \varepsilon) = (x, \varepsilon)\psi$. Since $\{(x, \varepsilon)\psi \mid \psi \in Aut(\mathfrak{F}_X)\} = X \times \varepsilon$, we have $(a, \varepsilon)f = (b, \varepsilon)f$ for all $a, b \in X$. From here $f = \theta_{(t,u)}$ for $(t, u) = (x, \varepsilon)f$.

(ii) Let $f \in End(\mathfrak{F}_X)$ be a constant idempotent endomorphism. Then $f = \theta_{(x,u)}$, $(x, u) \in FAd(X)$, and $\theta^2_{(x,u)} = \theta_{(x,u)}$. Since $\theta_{(x,u)\theta_{(x,u)}} = \theta^2_{(x,u)}$, we have

$$\theta_{(x,u)} = \theta_{(x,u)}\theta_{(x,u)} = \theta_{(x,u)\theta_{(x,u)}} = \theta_{(x,u^{l(u)+1}x^{l(u)})}.$$

It means that $(x, u) = (x, u^{l(u)+1}x^{l(u)})$, whence $l(u) = 0$, i.e., $u = \varepsilon$.

Clearly, $\theta^2_{(x,\varepsilon)} = \theta_{(x,\varepsilon)}$ for all $x \in X$. □

Theorem 3. *Let $\mathfrak{F}_X = (FAd(X), \dashv, \vdash)$ and $\mathfrak{F}_Y = (FAd(Y), \dashv, \vdash)$ be free abelian dimonoids such that $End(\mathfrak{F}_X) \cong End(\mathfrak{F}_Y)$. Then $\mathfrak{F}_X$ and $\mathfrak{F}_Y$ are isomorphic.*

Proof. Let Ψ be an arbitrary isomorphism of $End(\mathfrak{F}_X)$ into $End(\mathfrak{F}_Y)$. In according to the statements of Lemma 2 for some constant idempotent endomorphism $\theta_{(x,\varepsilon)}, x \in X$, of the free abelian dimonoid $\mathfrak{F}_X$ and for all $\alpha \in Aut(\mathfrak{F}_X)$, we have $\alpha\theta_{(x,\varepsilon)} = \theta_{(x,\varepsilon)}$. Taking into account that Ψ is a homomorphism, we obtain

$$\theta_{(x,\varepsilon)}\Psi = \left(\alpha\theta_{(x,\varepsilon)}\right)\Psi = \alpha\Psi\ \theta_{(x,\varepsilon)}\Psi.$$

Since $Aut(\mathfrak{F}_X)\Psi = Aut(\mathfrak{F}_Y)$, by the statement (i) of Lemma 2 we have $\theta_{(x,\varepsilon)}\Psi$ is a constant endomorphism of $\mathfrak{F}_Y$. Then $\theta_{(x,\varepsilon)}\Psi = \theta_{(y,v)}$ for some $(y, v) \in FAd(Y)$, in addition, $\theta_{(y,v)}$ is an idempotent of $End(\mathfrak{F}_Y)$. By the statement (ii) of Lemma 2, $v = \varepsilon'$, where ε' is the empty word of $FCm(Y)$ (see Section 3).

Define a map $\xi : X \to Y$ putting $x\xi = y$ if and only if $\theta_{(x,\varepsilon)}\Psi = \theta_{(y,\varepsilon')}$. It is clear that ξ is a bijection. Thus, abelian dimonoids $\mathfrak{F}_X$ and $\mathfrak{F}_Y$ are isomorphic. □

Using similar arguments, the fact that the free dimonoid also is uniquely determined up to an isomorphism by its endomorphism semigroup can be proved.

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