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Form of filters of semisimple modules and direct sums

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Abstract. Some collections of submodules of a module defined by certain conditions are studied. A generalization of the notion of radical (preradical) filter is considered. We study the form of filters of semisimple modules and direct sums.

All rings are considered to be associative with unit $1 \neq 0$ and all modules are left and unitary.

Let R be a ring. Put

$$(N : f)_M = \{x \in M | f(x) \in N\},$$
$$End(M)_N = \{f \in End(M) | f(M) \subseteq N\}.$$

Let E be some non-empty collection of submodules of a left R-module M. We consider the following conditions:

(1) $L \in E, L \leq N \leq M \Rightarrow N \in E$;

(2) $L \in E, f \in End(M) \Rightarrow (L : f)_M \in E$;

(3) $N, L \in E \Rightarrow N \cap L \in E$;

(4) $N \in E, N \in Gen(M), L \leq N \leq M \wedge$
$\wedge \forall g \in End(M)_N : (L : g)_M \in E \Rightarrow L \in E$;

(5) $N, L \in E, N \in Gen(M) \Rightarrow N \cap L \in E$.

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Consider a generalization of the notion of radical (preradical) filter (see [3, 4]).

A non-empty collection E of submodules of a left R-module M satisfying ((1)), ((2)), ((3)) is called a preradical filter of M (see [4]).

A non-empty collection E of submodules of a left R-module M satisfying ((1)), ((2)), ((4)) is called a radical filter of M (see [4]). It is easy to see that for every radical filter of M ((5)) is held.

A preradical (radical) filter E of a left R-module M is said to be trivial if either $E = \{L | L \leq M\}$ or $E = \{M\}$.

Let M be a semisimple left R-module with a unique homogeneous component and let $M = \bigoplus\limits_{i \in I} M_i$, where M_i is simple for each $i \in I$.

If $N = \bigoplus\limits_{i \in J} N_i$, where N_i is simple for each $i \in J$ and $M \cong N$, then $Card(I) = Card(J)$.

Put

$$Card_s(M) := Card(I).$$

Let M be a semisimple R-module with a unique homogeneous component. If $Card_s(M)$ is infinite, then we set

$$E_p(M) := \{L | L \leq M, Card_s(M/L) < p\},$$

where p is an infinite cardinal number.

Theorem 1. *Let M be a semisimple R-module with a unique homogeneous component. If $Card_s(M)$ is infinite, then every non-trivial radical [preradical] filter of M is of the form*

$$E_p(M)$$

for some infinite cardinal number $p \leq Card_s(M)$.

Proof. Let M be a semisimple R-module with a unique homogeneous component, $Card_s(M) = \infty$, and E a non-trivial radical [preradical] filter of M. Put

$$q := Card_s(M).$$

It is obvious that for each $L \in E$ there exists $H \leq M$ such that $M = L \oplus H$. Hence $Card_s H \leq q$.

We claim that $Card_s H \neq q$. Indeed, suppose, contrary to our claim, that $Card_s H = q$. Since M is a semisimple R-module with a unique homogeneous component, for some set I we have that $M = \bigoplus\limits_{i \in I} M_i$, where M_i

is simple for each $i \in I$ and for every $i, j \in I$ there exists an isomorphism $f_{ij} : M_i \to M_j$. Hence $CardI = Card_s(M) = q$. Taking into account that $Card_s(M)$ is infinite, by (2.1) [5, p. 417],

$$q + q = q.$$

Consider a set X such that $CardX = q$ and $X \cap I = \emptyset$. Since $q+q = q$, there exists a bijection $w : X \cup I \to I$. Put

$$Y := w(X), Z := w(I).$$

Therefore, $I = Y \cup Z, Y \cap Z = \emptyset, q = CardI = CardY = CardZ$. Now we obtain $M = A \oplus B$, where $A = \underset{i\in Y}{\oplus} M_i$,$B = \underset{i\in Z}{\oplus} M_i$. Since $H \leq M$, there exists an isomorphism $u : H \to \underset{i\in T}{\oplus} M_i$ for some $T \subseteq I$ (see Proposition 9.4 [1]). It is clear that $Card_s H = CardT = q$. Whence $q = CardY = CardZ = CardT$. Let $g : Y \to T, c : Z \to T$ be bijections. Consider the following maps:

$$G : A \to H, C : B \to H,$$

where

$$G(\sum_{i\in Y} m_i) = u^{-1}(\sum_{i\in Y} f_{i,g(i)}(m_i)),$$
$$(m_i \in M_i(i \in I), Card\{i \in Y | m_i \neq 0\} < \infty),$$

$$C(\sum_{i\in Z} m_i) = u^{-1}(\sum_{i\in Z} f_{i,c(i)}(m_i)),$$
$$(m_i \in M_i(i \in I), Card\{i \in Z | m_i \neq 0\} < \infty).$$

It is easily seen that these maps are isomorphisms. Let $n, r : M \to M$ are maps such that $n(a + b) = G(a), (a \in A, b \in B)$ and $r(a + b) = C(b), (a \in A, b \in B)$. It is clear that $n, r : M \to M$ are endomorphisms. Since $L \cap H = 0$ and G, C are isomorphisms, $(L : n)_M = B$ and $(L : r)_M = A$. As $L \in E$, by ((2)), we get $B \in E$ and $A \in E$. By ((3)) or ((5)) , $0 = A \cap B \in E$. Consequently, E is trivial. This contradicts our assumption. Hence $Card_s H < q$. The natural isomorphism $H \cong M/L$ implies that $Card_s(M/L) < q$. Now we consider the set Ω of all cardinal numbers v such that

$$v \leq q \forall L \in E : Card_s(M/L) < v.$$

$\Omega \neq \emptyset$, because $q \in \Omega$. By [2, p. 82] , there exists the least element p belonging to Ω. Thus $\forall L \in E : Card_s(M/L) < p$. It means that $E \subseteq E_p(M)$.

Let $L \in E_p(M)$. Whence $Card_s(M/L) < p$. We claim that there exists $D \in E$ such that $Card_s(M/L) \leq Card_s(M/D)$. Conversely, suppose that

$$\forall D \in E : Card_s(M/D) < Card_s(M/L).$$

But $Card_s(M/L) < p \leq q$. Hence $Card_s(M/L) \in \Omega$. Since p is the least element belonging to Ω, $p \leq Card_s(M/L)$, contrary to $Card_s(M/L) < p$.

Now we have that there exists $D \in E$ such that $Card_s(M/L) \leq Card_s(M/D)$. It is easily seen that for L, D there exist $H, K \leq M$ such that $M = L \oplus H, M = D \oplus K$. Since $M/L \cong H, M/D \cong K$, $Card_s(H) \leq Card_s(K)$. Since $H \leq M$ and $K \leq M$, there exist isomorphisms $u : H \to \underset{i \in T}{\oplus} M_i$ for some $T \subseteq I$ and $w : K \to \underset{i \in S}{\oplus} M_i$ for some $S \subseteq I$. Therefore $CardT \leq CardS$. From this we have that there exists an injective map $\gamma : T \to S$.

Consider the following map:

$$\psi : \underset{i \in T}{\oplus} M_i \to \underset{i \in S}{\oplus} M_i,$$

where

$$\psi(\sum_{i \in T} m_i) = \sum_{i \in Y} f_{i,\gamma(i)}(m_i),$$

$$(m_i \in M_i (i \in I),\, Card\{i \in T | m_i \neq 0\} < \infty).$$

It is obvious that ψ is a monomorphism. Now consider the following map:

$$\eta : M \to M,$$

where

$$\eta(l + h) = w^{-1}\psi u(h), \quad (l \in L, h \in H).$$

It is clear that $\eta \in End(M)$. Since $D \cap K = 0$ and im $\eta \subseteq K$, for every $l \in K, h \in H$: $\eta(l + h) \in D \Leftrightarrow w^{-1}\psi u(h) \in D \Leftrightarrow w^{-1}\psi u(h) = 0$. Since u, w are isomorphisms and ψ is monomorphism, for every $h \in H$: $w^{-1}\psi u(h) = 0 \Leftrightarrow h = 0$. From the above it follows that $(D : \eta)_M = L$. Since E is a radical [preradical] filter of M and $D \in E$, $(D : \eta)_M = L$ shows that $L \in E$, by ((2)). It means that $E_p(M) \subseteq E$. But $E \subseteq E_p(M)$. Hence $E = E_p(M)$

Theorem 2. *If M is a left R-module such that $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where $M_i = Tr_M(M_i)$ for each $i \in \{1, 2, \ldots, n\}$ and $\forall S : S \leq M \Rightarrow S \in Gen(M)$, then every radical [preradical] filter E of M is of the form*

$$E = \{J_1 + J_2 + \ldots + J_n | J_i \in E_i(i \in \{1, 2, \ldots, n\})\},$$

where E_i is a radical [preradical] filter of M_i for each $i \in \{1, 2, \ldots, n\}$.

Proof. Let E be a radical [preradical] filter of M and $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$, where $M_i = Tr_M(M_i)$ for each $i \in \{1, 2, \ldots, n\}$. Put

$$E_i := \{f_i(K) | K \in E\}$$

for each $i \in \{1, 2, \ldots, n\}$, where $f_i : M \to M, f_i(m_1 + m_2 + \ldots + m_n) = m_i, (m_1 \in M_1, m_2 \in M_2, \ldots, m_n \in M_n)$ for each $i \in \{1, 2, \ldots, n\}$.

(1) Let $L \in E_i, L \leq N \leq M_i$. Hence there exists $P \in E$ such that $L = f_i(P)$. Since $L \leq N$, $P \leq f_i^{-1}(N)$. By (1), $f_i^{-1}(N) \in E$, because $P \in E$. Therefore $N = f_i(f_i^{-1}(N)) \in E_i$.

(2) Let $L \in E_i, f \in End(M_i)$. Hence there exists $P \in E$ such that $L = f_i(P)$. Consider

$$F : M \to M,$$

where $F : m_1 + m_2 + \ldots + m_i + \ldots + m_n \mapsto f(m_i), (m_1 \in M_1, \ldots, m_n \in M_n)$. Thus $F \in End(M)$.

We claim that $f_i((P : F)_M) \leq (L : f)_{M_i}$. Indeed, let $x_i \in f_i((P : F)_M)$. We have that $x_i \in M_i$. Thus there exists $x \in (P : F)_M$ such that $f_i(x) = x_i$. Hence $f(x_i) = F(x) \in P$. It is clear that $f(x_i) \in M_i$. Therefore $f(x_i) = f_i(f(x_i)) \in f_i(P) = L$. Whence $x_i \in (L : f)_{M_i}$. We obtain $f_i((P : F)_M) \leq (L : f)_{M_i}$.

Since $P \in E$ and $F \in End(M)$, $(P : F)_M \in E$, by (2). $(P : F)_M \in E$ implies $f_i((P : F)_M) \in E_i$. Since $f_i((P : F)_M) \leq (L : f)_{M_i}$, (1) implies $(L : f)_{M_i} \in E_i$.

(3) Let $L, N \in E_i$. Hence there exist $P, T \in E$ such that $L = f_i(P)$ and $N = f_i(T)$. By (3) (for the preradical filter E), $P \cap T \in E$. Therefore $f_i(P \cap T) \in E_i$. Since $f_i(P \cap T) \subseteq f_i(P) \cap f_i(T) = L \cap N$ and $f_i(P \cap T) \in E_i$, we obtain $L \cap N \in E_i$, by (1).

(4) Let $N \in E_i, N \in Gen(M_i), L \leq N \leq M_i \wedge \forall g \in End(M_i)_N : (L : g)_{M_i} \in E_i$.

Hence $N = f_i(T)$ for some $T \in E$. Since $T \subseteq f_i^{-1}(N)$, $f_i^{-1}(N) \in E$, by (1). And $f_i^{-1}(N) \in Gen(M)$. $L \leq N$ implies $f_i^{-1}(L) \leq f_i^{-1}(N)$.

Let G be an arbitrary element of $End(M)_{f_i^{-1}(N)}$. By Proposition 8.16 [1] , $M_s = Tr_M(M_s)$ is a fully invariant submodule of M for each $s \in \{1, 2, \ldots, n\}$. Hence $G(M_s) \subseteq M_s$ for each $s \in \{1, 2, \ldots, n\}$. Consider

$$g : M_i \to M_i, m \mapsto G(m), (m \in M_i).$$

Since $\forall g \in End(M_i)_N : (L : g)_{M_i} \in E_i$, there exists $Y_g \in E_i$ such that $g(Y_g) \leq L$. Since $G(M_s) \subseteq M_s$ for each $s \in \{1, 2, \ldots, n\}$,

$$\begin{aligned} G(f_i^{-1}(Y_g)) = G(M_1 \oplus \ldots \oplus M_{i-1} \oplus Y_g \oplus M_{i+1} \oplus \ldots \oplus M_n) \subseteq \\ \subseteq M_1 \oplus \ldots \oplus M_{i-1} \oplus G(Y_g) \oplus M_{i+1} \oplus \ldots \oplus M_n = \\ = M_1 \oplus \ldots \oplus M_{i-1} \oplus g(Y_g) \oplus M_{i+1} \oplus \ldots \oplus M_n \subseteq \\ \subseteq M_1 \oplus \ldots \oplus M_{i-1} \oplus L \oplus M_{i+1} \oplus \ldots \oplus M_n = f_i^{-1}(L). \end{aligned}$$

Hence $f_i^{-1}(Y_g) \subseteq (f_i^{-1}(L) : G)_M$. Since $Y_g \in E_i$, there exists $P \in E$ such that $Y_g = f_i(P)$. Thus $P \subseteq f_i^{-1}(Y_g)$. Hence $P \subseteq (f_i^{-1}(L) : G)_M \& P \in E$. By (1), $(f_i^{-1}(L) : G)_M \in E$. Since $f_i^{-1}(N) \in E$, $f_i^{-1}(N) \in Gen(M), f_i^{-1}(L) \leq f_i^{-1}(N) \leq M$ and $\forall G \in End(M)_{f_i^{-1}(N)} : (f_i^{-1}(L) : G)_M \in E$, obtain $f_i^{-1}(L) \in E$. Therefore $L = f_i(f_i^{-1}(L)) \in E_i$.

Let $J \in E$. Put $J_i := f_i(J), (i \in \{1, 2, \ldots, n\})$. By Proposition 8.20 [1], $Tr_J(M) = Tr_J(M_1 \oplus M_2 \oplus \ldots \oplus M_n) = \sum_{i=1}^{n} Tr_J(M_i)$. Since $J \leq M$, $Tr_J(M_i) \leq Tr_M(M_i) = M_i$ for any $i \in \{1, 2, \ldots, n\}$, by Proposition 8.16 [1]. Hence $Tr_J(M) = \bigoplus_{i=1}^{n} Tr_J(M_i)$, because $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. Since $J \in Gen(M)$, $Tr_J(M) = J$, by Proposition 8.12 [1]. Whence

$$J = \bigoplus_{i=1}^{n} Tr_J(M_i) \& \forall i \in \{1, 2, \ldots, n\} : Tr_J(M_i) \leq M_i.$$

Therefore $Tr_J(M_i) = J_i$ for any $i \in \{1, 2, \ldots, n\}$. Thus $J = J_1 + J_2 + \ldots + J_n$, where $J_1 \in E_1, J_2 \in E_2, \ldots, J_n \in E_n$.

Let $P_i \in E_i$ for each $i \in \{1, 2, \ldots, n\}$. Hence there exists $H_i \in E$ such that $P_i = f_i(H_i)$. Thus $H_i \subseteq f_i^{-1}(P_i)$. By (1), $f_i^{-1}(P_i) \in E$. $f_i^{-1}(P_i) \in Gen(M)$ for any $i \in \{1, 2, \ldots, n\}$. By ((3)) or ((5)), $f_1^{-1}(P_1) \cap f_2^{-1}(P_2) \cap \ldots \cap f_n^{-1}(P_n) \in E$. Since $f_i^{-1}(P_i) = M_1 + \ldots + M_{i-1} + P_i + M_{i+1} + \ldots + M_n$ for any $i \in \{1, 2, \ldots, n\}$, $P_1 + P_2 + \ldots + P_n \in E$.

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