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On some linear groups, having a big family of G-invariant subspaces

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Abstract. Let F be a field, A a vector space over F, $GL(F, A)$ be the group of all automorphisms of the vector space A. If B is a subspace of A, then denote by BFG the G-invariant subspace, generated by B. A subspace B is called nearly G-invariant, if $dim_F(BFG/B)$ is finite. In this paper we described the situation when every subspace of A is nearly G-invariant.

Introduction

Let F be a field, A a vector space over F and $GL(F, A)$ a group of all F-automorphisms of A. If G is a subgroup of $GL(F, A)$ then, as usual, a subspace B of A is called G-invariant, if $bx \in B$ for every $b \in B$ and $x \in G$. The theory of linear groups over finite dimensional space is very well developed. This is one of the most developed group-theoretical theories (see, for example, the books [1, 10, 11]). However, in the case when A has infinite dimension over F, the situation became totally different. This case is much more complicated and its consideration requires some additional restrictions. Imposing classical finiteness conditions is one of the most efficient and natural approaches here. The study of infinite dimensional linear groups satisfying some finiteness conditions proved to be very promising. Many valuable results have been obtained in this way (see, for example, the surveys [9, 5]).

Recently began to study another approach in studying of infinite dimensional linear groups. This approach is based on the notion of invariance

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of action of a group G. We have the following simple fact: if every subspace of A is G-invariant, then G must be abelian. Consequently the study of infinite dimensional linear groups having very big family of G-invariant subspaces could be fruitful. This has been shown in the papers [2, 3, 7, 8]. In this paper this approach continues to be implemented.

If B is a subspace of A, then $BFG = \sum_{g \in G} Bg$ is a G-invariant subspace. It follows that BFG is the least G-invariant subspace of A, including B. The dimension $dim_F(BFG/B)$ is called the upper measure of non G-invariance. If $dim_F(BFG/B) = 0$, then $B = BFG$. In other words, every subspace of A is G-invariant. Therefore it is natural to consider a situation, when the upper measures of non G-invariance of all subspaces of A are finite. A subspace B of a vector space A is called nearly G-invariant if the upper measure of non G-invariance of B is finite.

1. Preliminary results

We will need some concepts and results of modules theory.

Let R be a ring, A a module over R. Put

$$Tor_R(A) = \{a \in A \mid Ann_R(a) \neq \langle 0 \rangle\}.$$

We observe that if R is an integral domain, then $Tor_R(A)$ is an R-submodule. The submodule $Tor_R(A)$ is called the R-periodic part of A. An R-module A is called R-periodic if $A = Tor_R(A)$. An R-module A is called R-torsion-free, if $Tor_R(A) = \langle 0 \rangle$.
It is easy to see that a factor-module $A/Tor_R(A)$ is R-torsion-free.

Let now G be a subgroup of $GL(F, A)$, $g \in G$. Denote by J a group ring FX where $X = \langle x \rangle$ is an infinite cyclic group. Define the action of x on A by the rule: $ax = ag$ for each element $a \in A$. This action can be continue in a natural way to the action of a ring J on A, thus A become an J-module. For this case we will say that A is an $J(g)$ module.

Lemma 1. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is nearly G-invariant. Then A is periodic as $J(g)$-module for every element $g \in G$.*

Proof. If g has finite order, say n, then $g^n - 1 \in Ann_J(A)$. Let now g be an element of G, having infinite order. Suppose that the result is false; that is, there exists some element $a \in A$ such that $Ann_{F\langle g \rangle}(a) = \{0\}$. Then $aF\langle g \rangle \cong F\langle g \rangle$, so that $aF\langle g \rangle = \bigoplus_{n \in N} aF\langle g^n \rangle$. Put $h = g^2$ and $D = aF\langle h \rangle$. Then $aF\langle g \rangle = D \oplus Dg$. We remark that $dim_F(D)$ and $dim_F(A/D)$

are infinite. Put $C = DFG$. By our conditions $dim_F(C/D)$ is finite. It follows that $dim_F((C \cap aF\langle g\rangle)/D)$ is finite, and hence $dim_F(aF\langle g\rangle/(C \cap aF\langle g\rangle))$ is infinite. On the other hand, $C \cap aF\langle g\rangle = D \oplus (C \cap Dg)$. Since C is G-invariant, $Dg \leq C$, so that $Dg = C \cap Dg$. It follows that $C = C \cap aF\langle g\rangle$, and we obtain a contradiction. This contradiction proves a result. □

Let R be a commutative ring, A an R-module. We define the R-assassinator of A as the set

$$Ass_R(A) = \{P \mid P \text{ is a prime ideal of } R \text{ such that } Ann_A(P) \neq \langle 0\rangle\}.$$

Let now D be a Dedekind domain, I be an ideal of D. Put

$$A_I = \{a \in A \mid aI^n = \langle 0\rangle \text{ for some } n \in N\}.$$

Clearly, A_I is a D-submodule of A. The D-submodule A_I is called the I-component of A. If $A = A_I$, then A is called an I-module. If P is a prime ideal, then instead I-submodule we will say sometimes primary submodule.
If A is D-periodic D-module, then $A = \bigoplus_{P\in\pi} A_P$ where $\pi = Ass_D(A)$ (see, for example, [6, Corollary 3.8]). Put

$$\Omega_{I,\,k}(A) = \{a \in A \mid aI^k = \langle 0\rangle\}.$$

It is easy to see that $\Omega_{I,\,k}(A)$ is a D-submodule and

$$\Omega_{I,\,1}(A) \leq \Omega_{I,\,2}(A) \leq \ldots \leq \Omega_{I,\,k}(A) \leq \ldots$$

$$A_I = \bigcup_{k\in N} \Omega_{I,\,k}(A).$$

Further put $A[1] = \bigoplus_{P\in\pi} \Omega_{P,\,1}(A_p)$, where $\pi = Ass_D(A)$. If A is a D-periodic module and B is a non-zero D-submodule of A, then $B \cap A[1] \neq \{0\}$.

Lemma 2. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is nearly G-invariant. Let g be an element of G and consider A as $J(g)$-module. Then $A = A[1] + C$ where C satisfies the following conditions:*

(i) *C is an FG-submodule of A;*

(ii) *C is artinian as an $J(g)$-submodule.*

Proof. Put $L = A[1]$. We have $A = L \oplus B$ for some subspace B. Put $C = BFG$. Since B is nearly G-invariant, $dim_F(C/B)$ is finite. We have $C = B \oplus (C \cap L)$. The finiteness of $dim_F(C/B)$ implies that $dim_F(C \cap L)$ is finite. Clearly $C \cap L = C[1]$. The finiteness of $dim_F(C[1])$ implies that C is artinian $J(g)$-module (see, for example, [6, Corollary 7.12]). □

Let D be a Dedekind domain, A be a simple D-module. Then $A \cong D/P$ for some maximal ideal P. We noted that D/P^k and P/P^{k+1} are isomorphic as D-modules for any $k \in N$ (see, for example, [6, Corollary 1.28]). In particular, the D-module D/P^k is embedded in the D-module D/P^{k+1}, $k \in N$. Therefore we can consider the injective limit of the family of D-modules $\{D/P^k \mid k \in N\}$. Put

$$C_P\infty = liminj\{D/P^k \mid k \in N\}.$$

The D-module $C_P\infty$ is called the Prüfer P-module. It follows from the construction that $C_P\infty$ is a P-module, moreover,

$$\Omega_{P,\ k}(C_P\infty) \cong_D D/P^k,\ k \in N.$$

Furthermore,

$$\Omega_{P,\ k+1}(C_P\infty)/\Omega_{P,\ k}(C_P\infty) \cong (D/P^{k+1})/(P/P^{k+1}) \cong D/P.$$

Hence, if C is a D-submodule of $C_P\infty$ and $C \neq C_P\infty$, then $C = \Omega_{P,\ k}(C_P\infty)$ for some $k \in N$. Similarly, if $b \notin \Omega_{P,\ k-1}(C_P\infty)$, then $C = bD$.

Observe also that a Prüfer P-module is monolithic and its monolith coincides with $\Omega_{P,\ 1}(C_P\infty)$.

Lemma 3. *Let F be a field, $J = F\langle x\rangle$ be a group ring of infinite cyclic group $\langle x\rangle$ over F and A be an artinian J-module. If $dim_F(A)$ is infinite, then A includes a F-subspace B, which is not nearly $\langle x\rangle$-invariant.*

Proof. Since $dim_F(A)$ is infinite, A includes a Prüfer P-submodule C for some maximal ideal P of J. Then C generated by elements $\{c_n \mid n \in N\}$ such that $c_nJ = C_n \leq C_{n+1} = c_{n+1}J$, $n \in N$ and $C = \bigcup_{n \in N} C_n$. Furthermore, $c_{n+1} \notin C_n$, $n \in N$, so that the elements c_n, $n \in N$, are linearly independent. It follows that an F-subspace E, generated by c_n, $n \in N$, is $\bigoplus_{n \in N} c_nF$. Put $B = \bigoplus_{k \in N} c_{2k}F$, then $dim_F(E/B)$ is infinite, therefore $dim_F(C/B)$ is infinite. Assume that B is nearly $\langle x\rangle$-invariant. Then B has a finite codimension in $D = BF\langle x\rangle = B\langle x\rangle$. It follows that $dim_F(C/D)$ is infinite. On the other hand, the equations

$$c_{2k-1}F\langle x\rangle = C_{2k-1} \leq C_{2k} = c_{2k}F\langle x\rangle = (c_{2k}F)\langle x\rangle,\ k \in N,$$

shows that $C_n \leq D$ for all $n \in N$, so that $C = D$. This contradiction proves a result. □

Corollary 1. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is nearly G-invariant. Let g be an element of G and consider A as $J(g)$-module. Then $A = A[1] + C$ where $dim_F(C)$ is finite.*

Proof. Lemma 2 implies that $A = A[1] + C$ where an FG-submodule C is artinian as an $J(g)$-submodule. Lemma 3 shows that in this case $dim_F(C)$ must be finite. □

In the next Proposition we describe the local structure of vector space, whose subspaces are nearly G-invariant.

Proposition 1. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is nearly G-invariant. Let g be an arbitrary element of G. Then A includes an $F\langle g\rangle$-submodule D satisfying the following conditions:*

(i) *$dim_F(A/D)$ is finite;*

(ii) *every subspace of D is $\langle g\rangle$-invariant.*

Proof. Put $L = A[1]$. Using Corollary 1 we obtain that L has finite codimension. For an $F\langle g\rangle$-submodule L we have a direct decomposition $L = \bigoplus_{\lambda\in\Lambda} A_\lambda$, where A_λ is a simple $F\langle g\rangle$-submodule for every $\lambda \in \Lambda$. Put $M = \{\lambda \in \Lambda \mid dim_F(A_\lambda) > 1\}$ and $B = \bigoplus_{\lambda\in M} A_\lambda$. Since every subspace of A is nearly G-invariant, it is likewise nearly $\langle g\rangle$-invariant. An application of Lemma 2.1 of paper [7] shows that $dim_F(B)$ is finite. Put $\Delta = \Lambda \setminus M$. Since $dim_F(A_\lambda) = 1$ for each $\lambda \in \Delta$, $A_\lambda = a_\lambda F$ for some elements $a_\lambda \in A_\lambda$, $\lambda \in \Delta$. It follows that $a_\lambda g = \alpha_\lambda a_\lambda$ for some elements $\alpha_\lambda \in F$, $\lambda \in \Delta$. Repeating almost word to word the arguments from a proof of Proposition 2.2 of paper [7], we obtain that there exists a subset $\Gamma \subseteq \Delta$ such that $\alpha_\lambda = \alpha_\mu = \alpha$ for all $\lambda,\ \mu \in \Gamma$, and a subset $\Delta \setminus \Gamma$ is finite. It follows that a subspace $D = \bigoplus_{\lambda\in\Gamma} A_\lambda$ has finite codimension. If $a \in D$, then $a = \sum_{1\leq j\leq n} \beta_{\lambda(j)} a_{\lambda(j)}$, where $\beta_{\lambda(j)} \in F$, $\lambda(j) \in \Gamma$, $1 \leq j \leq n$. We have

$$ag = \sum_{1\leq j\leq n} (\beta_{\lambda(j)} a_{\lambda(j)})g = \sum_{1\leq j\leq n} \beta_{\lambda(j)}(a_{\lambda(j)} g) = \sum_{1\leq j\leq n} \beta_{\lambda(j)}(\alpha a_{\lambda(j)}) =$$

$$\sum_{1\leq j\leq n} \alpha\beta_{\lambda(j)} a_{\lambda(j)} = \alpha \sum_{1\leq j\leq n} \beta_{\lambda(j)} a_{\lambda(j)} = \alpha a.$$

This equation shows that every subspace of D is $\langle g\rangle$-invariant. □

Lemma 4. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is nearly G-invariant. Then $dim_F(aFG)$ is finite for each element $a \in A$.*

Proof. Let $g_1 \in G$ and consider $A_1 = aF\langle g_1 \rangle$. By Lemma 1 an $J(g_1)$-module A is periodic, so that $Ann_J(a_1) \neq \{0\}$. Recall that every non-zero ideal of J has finite F-dimension. Hence $A_1 \cong F\langle x \rangle / Ann_J(a_1)$ has finite F-dimension. If $ag \in A_1$ for each element $g \in G$, then $aFG = A_1$ and all is proved. Therefore suppose that there exists an element $g_2 \in G$ such that $ag_2 \notin A_1$. Again a subspace $aF\langle g_2 \rangle$ has finite F-dimension, therefore and $A_2 = A_1 + aF\langle g_2 \rangle$ also has finite F-dimension. By $ag_2 \notin A_1$ we obtain that $ag_1F + ag_2F = ag_1F \oplus ag_2F$. Suppose that aFG has infinite F-dimension. Then using the above arguments we can find an infinite subset $\{g_n \mid n \in N\}$ of elements of G such that $ag_1F + \ldots + ag_nF = ag_1F \oplus \ldots \oplus ag_nF$ for all $n \in N$. It follows that an F-subspace B, generated by ag_nF, $n \in N$, is $\bigoplus_{n \in N} ag_nF$. Put $C = \bigoplus_{k \in N} ag_{2k}F$, $D = \bigoplus_{k \in N} ag_{2k-1}F$, then $B = C \oplus D$ and both $dim_F(B/C)$ and $dim_F(B/D)$ are infinite. Let $E = CFG$. Since C is nearly G-invariant, $dim_F(E/C)$ is finite and hence $dim_F((E \cap B)/C)$ is finite. It follows that $dim_F(B/(E \cap B))$ is infinite. On the other hand, since E is G-invariant, $a = (ag_1)g_1^{-1} \in E$. Then $ag_{2k-1} \in E$ for all $k \in N$, so that $D \leq E$ and hence $B \leq E$. This contradiction proves a result. □

2. Proof of the main results

Now we can prove the following main results of this paper.

Theorem 1. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is nearly G-invariant. Then A includes an FG-submodule C satisfying the following conditions:*

(i) *$dim_F(C)$ is finite;*

(ii) *every subspace of A/C is G-invariant.*

Proof. If every subspace of A is G-invariant, then all is proved. Suppose now that there are the elements $a_1 \in A$ and $g_1 \in G$ such that $a_1g_1 \notin a_1F$. Put $d_1 = a_1(g_1 - 1)$. It readily follows that $dim_F(a_1F + d_1F) = 2$, that is $a_1F + d_1F = a_1F \oplus d_1F$. By Lemma 4 an FG-submodule $A_1 = a_1FG$ has finite dimension and $d_1 \in A_1$. Proposition 1 shows that A includes an $F\langle g_1 \rangle$-submodule D_1 such that $dim_F(A/D_1)$ is finite and every subspace of D_1 is $\langle g_1 \rangle$-invariant. Let Y_1 be a complement to D_1, that is Y_1 is an

F-subspace with a property $A = D_1 \oplus Y_1$. Then $dim_F(Y_1)$ is finite. If $ag \in aF + A_1$ for each element $a \in D_1$, then put $C = A_1 + Y_1FG$. By Lemma 4 $dim_F(Y_1FG)$ is finite, so that an FG-submodule C has finite dimension. Furthermore, by $ag \in aF + A_1$ we can see that every subspace of A/C is G-invariant. Therefore assume that there exist the elements $a_2 \in D_1,\ g_2 \in G$ such that $a_2g_2 \notin a_2F + A_1$. Put $d_2 = a_2(g_2 - 1)$, then $dim_F((a_2F + d_2F + A_1)/A_1) = 2$. It follows that $a_2F + d_2F \cap A_1 = \{0\}$. In particular, $(a_2F + d_2F) \cap (a_1F + d_1F) = \{0\}$. Put $A_2 = A_1 + a_2FG$. By Lemma 4 $dim_F(a_2FG)$ is finite, so that and $dim_F(A_2)$ is finite. By this choice $a_2, d_2, a_1, d_1 \in A_2$. Proposition 1 shows that A includes an $F\langle g_2\rangle$-submodule D_2 such that $dim_F(A/D_2)$ is finite and every subspace of D_2 is $\langle g_2\rangle$-invariant. Then the intersection $D_1 \cap D_2$ has finite codimension. Let Y_2 be a complement to $D_1 \cap D_2$, then $dim_F(Y_2)$ is finite. If $ag \in aF + A_2$ for each element $a \in D_1 \cap D_2$, then put $C = A_2 + Y_2FG$. By Lemma 4 $dim_F(Y_2FG)$ is finite, so that an FG-submodule C has finite dimension. Furthermore, by $ag \in aF + A_2$ we can see that every subspace of A/C is G-invariant. If not, then there exist the elements $a_3 \in D_1 \cap D_2,\ g_3 \in G$ such that $a_3g_3 \notin a_3F + A_2$. Again put $d_3 = a_3(g_3 - 1)$ and repeat the above arguments. Using these arguments, we come to the two possibilities:

(1) this process will finish after finitely many steps, that is we find an FG-submodule C, having finite dimension, such that every subspace of A/C is G-invariant;

(2) For every positive integer n we find the elements $a_1, \dots, a_n \in A$ and $g_1, \dots, g_n \in G$ such that the following conditions hold:

   (a) $a_jF + d_jF = a_jF \oplus d_jF$, where $d_j = a_j(g_j - 1),\ 1 \le j \le n$;
   (b) $(a_nF \oplus d_nF) \bigcap (\bigoplus_{1 \le k \le n-1}(a_kF \oplus d_kF)) = \{0\},\ 1 \le j \le n$.

Consider the second possibility more detail. Put $B = \bigoplus_{j\in N} a_jF$, $K = \bigoplus_{j\in N} d_jF$, $E = B + K$. Then $B \cap K = \langle 0\rangle$. It follows that $dim_F(E/B)$ and $dim_F(E/K)$ are infinite. Since B is nearly G-invariant, B has finite codimension in $V = BFG$. It follows that $dim_F((V \cap E)/B)$ is finite, and hence $dim_F(E/(V \cap E))$ is infinite. An equation $V \cap E = B \oplus (V \cap E \cap K)$ shows that $dim_F(V \cap E \cap K)$ is finite. Then there exists a positive integer t such that $d_t \notin V \cap E$. On the other hand, $d_t = a_t(g_t - 1)$. Then from $a_t \in B \le V$ and the fact that V is G-invariant we obtain an inclusion $d_t \in V$ and hence $d_t \in V \cap E$. This contradiction shows that second possibility can not appear really, which proves a result. □

As corollary we can obtain a description of a structure of a group G.

Theorem 2. *Let G be a subgroup of $GL(F, A)$ and suppose that every subspace of A is nearly G-invariant. Then the following assertions hold:*

(i) *if $char(F) = 0$, then G includes a normal abelian torsion-free subgroup Z such that G/Z is isomorphic to subgroup of $L \times V$, where V is a subgroup of multiplicative group of F and L is a subgroup of $GL_n(F)$ for some positive integer n.*

(ii) *if $char(F) = p$ is a prime, then G includes a normal abelian elementary p-subgroup Z such that G/Z is isomorphic to subgroup of $L \times V$, where V is a subgroup of multiplicative group of F and L is a subgroup of $GL_n(F)$ for some positive integer n.*

Proof. Theorem 1 shows that A includes an FG-submodule C, having finite dimension, such that every subspace of A/C is G-invariant. Put $K = C_G(A/C)$. By Lemma 3.4 of paper [7] $V = G/K$ is isomorphic to a subgroup of a multiplicative group of a field F. Put now $T = C_G(C)$. Since $dim_F(C) = n$ is finite, $L = G/T$ is isomorphic to a subgroup of finite dimensional linear group $GL_n(F)$. Finally, let $Z = T \cap K$, then Z stabilizes the series of

$$\{0\} \leq C \leq A.$$

By a classical result due to Kaluznin (see, for example, [4, Theorem 1.C.1 and Proposition 1.C.3]) Z is either an elementary abelian p-subgroup if $char(F) = p > 0$, or a torsion-free abelian subgroup if $char(F) = 0$. Finally, by the Remak's Theorem, we obtain a new embedding of G/Z in the direct product $G/K \times G/T = V \times L$ and the result is proved. □

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