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RESEARCH ARTICLE

On the finite state automorphism group of a rooted tree

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ABSTRACT. The normalizer of the finite state automorphism group of a rooted homogeneous tree in the full automorphism group of this tree was investigated. General form of elements in the normalizer was obtained and countability of the normalizer was proved.

1. Introduction

Automorphism groups of rooted trees are studied strongly last years in connection with their application in geometric group theory, theory of dynamic systems, ergodic and spectral theory, and that they also contain various interesting subgroups with extremal properties. In particular, there are free constructions among them, various constructions of groups of intermediate growth, etc (see [GNS] and its bibliography).

Among subgroups of automorphism group of a rooted tree the finite state automorphism group arise the big interest [Su].

In the paper [NS] the number of problems on the finite state automorphism group of a rooted tree was posed. This work partially solves one of these problems. In the paper the normalizer of the finite state automorphism group of a rooted tree in the full automorphism group of this tree was investigated. General form of elements in normalizer was obtained and countability of normalizer was proved. According to

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[L, LN] this normalizer is isomorphic to the automorphism group of the finite state automorphism group of a rooted tree.

2. Preliminary

Definition 1. *A synchronous automaton is a set* $A = \langle X_I, X_O, Q, \pi, \lambda \rangle$, *where*

1. X_I *and* X_O *are finite sets (respectively, the input and the output alphabets),*
2. Q *is a set (the set of internal states of the automaton),*
3. $\pi : X_I \times Q \longrightarrow Q$ *is a mapping (transition function), and*
4. $\lambda : X_I \times Q \longrightarrow X_O$ *is a mapping (output function).*

Automaton A is finite if $|Q| < \infty$.

Henceforth, we will consider the automata whose input and output alphabets coincide. Let $X = X_I = X_O$ be a finite alphabet, X^* be the set of all words over X, X^ω be the set of all ω-words (infinite words) over X.

A permutation of the set X^* or X^ω is called a *(finitely) automatic* if it is caused by a (finite) automaton over alphabet X. All finitely automatic permutations form subgroup of the group $GA(X)$ of all automatic permutations over X. Let us denote this subgroup by $FGA(X)$.

For the alphabet X we can construct the word tree T_X (see also [GNS]). The vertices of the tree T_X are the elements of the set X^*. Two vertices u and v are incident if and only if $u = vx$ or $v = ux$ for a certain $x \in X$. The vertex $\emptyset$ is the root of the tree.

The group $\mathrm{Aut}T_X$ of all automorphisms T_X is isomorphic to the group $GA(X)$ of all automatic permutations over X.

For every two vertices u, v of the tree T_X (i. e. $u, v \in V(T_X)$) we define the *distance* between u and v, denoted by $d(u, v)$, to be equal to the length of the path connecting them.

For rooted tree T_X with the root $v_0 = \emptyset$ and an integer $n \geq 0$ we define the level number n (the sphere of the radius n) as the set

$$V_n = \{v \in V(T_X) : d(v_0, v) = n\}.$$

Let us say that vertex v of rooted tree T_X lies under vertex w, if path, that connects vertice v and v_0, contains vertex w.

Let us denote by T_v the full subtree consisting of all vertices, that lie under the vertex v with the root v.

Let $G \leq \mathrm{Aut}T_X$ be an automorphism group of the rooted tree T_X. Then for every vertex v of the tree T_X and a nonnegative integer n:

1. The group of all automorphisms $g \in G$ fixing every vertex outside the subtree T_v is called *the vertex group* (or *the rigid stabilizer* of the vertex) and is denoted by $\mathsf{rist}\, v$.

2. The group of all automorphisms fixing all vertices of the level number n is denoted by $\mathsf{stab}_G(n)$ or just $\mathsf{stab}(n)$ and is called *the level stabilizer.*

An automorphism group G is said to be *level-transitive* if it acts transitively on all the levels of the rooted tree T_X.

An automorphism group G is said to be *weakly branch* if it is level-transitive and for every vertex v of the tree the vertex group is nontrivial.

Statement 1. *[LN] If G is a weakly branch group then the centralizer $C_{\mathrm{Aut}\,T_X}(G)$ of G in the automorphism group $\mathrm{Aut}\,T_X$ is trivial.*

In the word tree T_X every subtree T_v, where $v \in V(T_X)$, can be naturally identified with the whole tree T_X by the map:

$$\pi_v : x_1x_2 \dots x_nx_{n+1} \dots x_m \mapsto x_{n+1}x_{n+2} \dots x_m,$$

where $x_1x_2 \dots x_n = v$.

So, if $g \in \mathsf{stab}(n)$ then the action of g on T_v for every $v \in V_n$ can be identified by π_v with the isometry g_v of T_X defined by the equality

$$\pi_v\left(u^g\right) = \left(\pi_v\left(u\right)\right)^{g_v}.$$

The isometry g_v is called *the state of g in v* or *the restriction of g on v.*

When $g \in \mathsf{stab}(n)$, we write $g = (g_{v_1}, g_{v_2}, \dots, g_{v_{r^n}})^{(n)}$, where

$$\{v_1, v_2, \dots, v_{r^n}\} = V_n,\ r = |X|.$$

Let T_X^n be the subtree of the rooted tree T_X, that consists of all vertices on a distance no greater than n from the root. Then the group $\mathrm{Aut}T_X^n$ is naturally embedded in the group $\mathrm{Aut}T_X$ and the latter is decomposed into semidirect product

$$\mathrm{Aut}T_X = \mathsf{stab}(n) \leftthreetimes \mathrm{Aut}T_X^n.$$

So for each $g \in \mathrm{Aut}T_X$ we can write

$$g = g_na_g = (g_{v_1}, g_{v_2}, \dots, g_{v_{r^n}})^{(n)}a_g, \tag{1}$$

where $g_n \in \mathsf{stab}(n)$, and $a_g \in \mathrm{Aut}T_X^n$.

By the state of an element g_n in the vertex $v \in V_n$ we mean the state of an element $g \in \mathrm{Aut}T_X$ in the vertex v.

An automorphism $g \in \mathrm{Aut}T_X$ is called *finite state* automorphism if the set of all its states is finite.

All finite state automorphisms form a subgroup of the group $\mathrm{Aut}T_X$. The group $FGA(T_X)$ of all finite state automorphisms T_X is isomorphic to the group $FGA(X)$ of all finitely automatic permutations.

End is an infinite sequence of vertices $(v_0, v_1, v_2, \ldots)$, $v_k \in V_k$ such that $d(v_k, v_{k+1}) = 1$ for every nonnegative integer k. Every ω-word determines an end of the tree T_X. Conversely every end of the tree T_X determines some ω-word.

An ω-word (end) w is called *periodic* if there exists the word $v \in X^*$ such that $w = v \cdot v \cdot v \cdot \ldots = v^\omega$. We say that w is *ultimately periodic* if there exist words $u, v \in X^*$ such that $w = u \cdot v^\omega$.

Let X^{up} be the set of all ultimately periodic words over alphabet X (of the ends of the tree T_X).

Lemma 2. *[Su]*

1. *The set X^{up} is an orbit of the group $FGA(X)$.*

2. *The action of the group $FGA(T_X)$ is faithful on this orbit.*

3. *The permutation group $(FGA(T_X), X^{up})$ is an imprimitive group and its domain of imprimitivity are intersections of domains of imprimitivity of permutation group $(\mathrm{Aut}T_X, X^\omega)$ with the set X^{up}.*

3. Main results

In the paper the normalizer $N = N_{\mathrm{Aut}T_X}(FGA(T_X))$ of the group $FGA(T_X)$ in the group $\mathrm{Aut}T_X$ of all automorphisms of rooted tree T_X, $|X| \geq 2$ is investigated.

As it was shown in [L] (see also [LN]) the normalizer N is isomorphic to the automorphism group of the group $A = FGA(T_X)$.

In the paper the next results on the structure of normalizer (of automorphism group) have been obtained:

Theorem 3. *Let $g \in N$. For every ultimately periodic end u the sequence of states $\{g_{(n)} \mid n \in \mathbb{N}\}$ that correspond to the end u (i.e. states in vertices pertinent to this end) is ultimately periodic.*

Theorem 4. *For an element $g \in N$ there exist $m, k \in \mathbb{N}$, $a, b \in FGA(T_X)$ and $h \in N$ such that*

$$h = (h, ..., h)^{(m)} a,$$

$$g = (h, ..., h)^{(k)} b.$$

Corollary 1. *The normalizer $N = N_{\mathrm{Aut}T_X}(FGA(T_X))$, $|X| \geq 2$, is countable.*

4. Proofs

Let $|X| = r \geq 2$, and let $u_0 = 00\dots$ be an end of the tree T_r.

Lemma 5. *An element of the group N turn an ultimately periodic end to an ultimately periodic one. That is $N : X^{up} \to X^{up}$.*

Proof. Since X^{up} is an orbit of the group A, it is sufficient to prove the statement for one ultimately periodic end. Let us consider, for example, the end u_0.

Let w be not an ultimately periodic end. Suppose there is $g \in N$ which turn the end w to the end u_0.

Let $a = (a, 1, ..., 1)^{(1)}\tau$ lie in A where τ is a cyclic permutation of order $r - 1$ with 0 as fixed point. Therefore, $u_0^a = u_0$, and u_0 is the only fixed end of the element a.

We have $gag^{-1} : w \to w$. Since g acts on set of ends as permutation, we have that the end w is the only fixed end of the element gag^{-1}.

Since $w \notin X^{up}$, among subtrees with roots in the vertices of the end w there are infinitely many different subtrees. That is, $gag^{-1} \notin A$. We have contradiction. □

This lemma implies

Corollary 2. *1. The set X^{up} is an orbit of the group N.*

2. Action of the group N is faithful on this orbit.

Let $g \in N$, and

$$g = g_n a_g = (g_{v_1}, g_{v_2}, \dots, g_{v_{r^n}})^{(n)} a_g$$

be decomposition (1) for g where $\{v_1, v_2, \dots, v_{r^n}\} = V_n$.

Lemma 6. *Let $g \in N$. For each V_n the elements $g_{v_1}, g_{v_2}, \dots, g_{v_{r^n}}$ are contained in the same left (right) coset of A.*

Proof. We can assume $a_g = 1$. Let $v_i, v_j \in V_n$ and $A \ni b : v_i \to v_j$ be such that $b_n = 1$. We have

$$(b^g)_{v_j} = g_{v_i}^{-1} g_{v_j}.$$

Since $b^g \in A$ then $g_{v_i}^{-1} g_{v_j} \in A$ for all $v_i, v_j \in V_n$. □

Corollary 3. *For an element $g \in N$ there exists $a \in A$ such that $ga \in \mathsf{stab}(n)$ and $(ga)_{v_i} = (ga)_{v_1}$ for all $i = 2, ..., r$.*

Let T be a rooted tree. We will denote by $k_n(v)$ the number of vertices belonging to V_{n+1} and adjacent to v for each integer $n \geq 0$ and $v \in V_n$. A tree is *spherically homogeneous* if $k_n(v)$ does not depend on $v \in V_n$. If k_n does not depend on n too then the tree is called *homogeneous*. For example word tree T_X is homogeneous.

For spherically homogeneous tree the sequence $\chi = \langle k_0, k_1, \ldots \rangle$ is called *tree index* and such a tree is denoted by T_χ. We will use denotation $\bar{k} = \{k, k, ...\}$ for homogeneous tree.

For denotation of vertices of the tree T_χ we will use two indices: first one is the number of the level containing this vertex, second one is the number of this vertex (in the lexigraphic ordering) among the all vertices of the given level.

We will need the next fact

Lemma 7. *The group $\operatorname{Aut} T_\chi$ contains finitely generated weakly branch subgroups for all $\chi = \langle k_1, k_2, ... \rangle$ $(k_i \geq 2)$.*

Proof. The group $\operatorname{Aut} T_{\bar{2}}$ contains finitely generated weakly branch subgroups, for example, the Grigorchuk 2-group Gr is a such one [GNS].

The natural embeddings $\{0, 1\}$ in $\{0, ...k_i - 1\}$ define the natural embedding $T_{\bar{2}}$ in T_χ and the group $\operatorname{Aut} T_{\bar{2}}$ is being ebedded in $\operatorname{Aut} T_\chi$.

Let us define $h = h_1 \in \operatorname{Aut} T_\chi$ recurrently

$$h_i = (h_{i+1}, 1, ..., 1)^{(1)} \sigma_i$$

where σ_i is the cyclic permutation $(v_{i2}, ..., v_{ik_i})$.

Let $H = \langle Gr, h \rangle$. The group H acts level-transitively on T_χ. We use induction by level number n. The group Gr acts transitively on $\{v_{11}, v_{12}\} \subset V_1$ and h cyclically permutes the vertices $v_{12}, ..., v_{1k_i}$. Thus H acts transitively on the first level. Let H acts transitively on V_n. It is sufficient to prove that for the level number $n+1$ the group H acts transitively on the vertices that are adjacent to the vertex v_{n1} from level number n. In this case the proof is similar to the proof for the level number one with substitution $h^{k_1 \ldots k_n}$ for h.

Therefore H is a level-transitive subgroup of T_χ.

Since there are vertices with infinite rigid stabilizers in G on each level we conclude that rigid stabilizer in H of each vertex is infinite.

Thus, H is a finitely generated weakly branch subgroup of group T_χ. □

Remark 1. *For homogeneous tree $T_{\bar{k}}$ group H is contained in the group $FGA(T_{\bar{k}})$.*

Proof of theorem 3. Let $|X| = r$. Since the group $FGA(X)$ acts transitively on X^{up}, it is sufficient to prove the theorem only for one ultimately periodic end u_0, and $g : u_0 \to u_0$.

1. $r = 2$.

   Let $\alpha_i \in A$ $(i = 1, ..., k)$ such that $\alpha_i = (\alpha_i, a_i)^{(1)}$ where $a_1, ..., a_k$ are elements genereting a weakly branch group H (for example, Grigorchuk group). Then

   $$\alpha^g : u_0 \longrightarrow u_0, \tag{2}$$

   $$(\alpha^g)_{v_{n2}} = a_i^{g_{v_{n2}}} \tag{3}$$

   where $v_{n2} \in V_n$ and $v_{n2} = 00\dots01$.

   Since $\alpha_i^g \in A$ and taking into account (2) we conclude that sequences $\{a_i^{g_{v_{n2}}} \mid n \in \mathbb{N}\}$ are ultimately periodic for $i = 1, ..., k$. Therefore there are $p_i, n_0 \in \mathbb{N}$ such that for $i = 1, ..., k$ and $n \geq n_0$ the next equality holds

   $$a_i^{g_{v_{n+p_i,2}}} = a_i^{g_{v_{n2}}}.$$

   Thus

   $$g_{v_{n+p_i,2}} g_{v_{n2}}^{-1} \in C_{\mathrm{Aut}\, T_2}(\langle a_i \rangle).$$

   Taking $p = gcd(p_1, ..., p_k)$ we have

   $$a_i^{g_{v_{n+p,2}}} = a_i^{g_{v_{n2}}},$$

   $$g_{v_{n+p,2}} g_{v_{n2}}^{-1} \in C_{\mathrm{Aut}\, T_2}(\langle a_i \rangle)$$

   for $i = 1, ..., k$ and $n \geq n_0$. Therefore using (1) we have

   $$g_{v_{n+p,2}} g_{v_{n2}}^{-1} \in \bigcap_{i=1}^{k} C_{\mathrm{Aut}\, T_2}(\langle a_i \rangle) = C_{\mathrm{Aut}\, T_2}(\langle a_1, ..., a_k \rangle) = C_{\mathrm{Aut}\, T_2}(H) = 1$$

   for $n \geq n_0$.

   Thus $\{g_{v_{n2}} \mid n \in \mathbb{N}\}$ is ultimately periodic, and, taking into account (2), we have that $\{g_{v_{n1}} \mid n \in \mathbb{N}\}$ is ultimately periodic too.

2. $r > 2$.

   Let $\alpha_i \in A$ $(i = 1, ..., k)$ such that $\alpha_i = (\alpha_i, a_i, ..., a_i)^{(1)}\sigma$ where $a_1, ..., a_k$ are elements genereting a weakly branch group H (such group exists by statement 7), and σ is the permutation on r points: $\sigma = (0)(123...r-1)$.

   Denote $(1, a_i, ..., a_i)^{(1)}\sigma$ by b_i $(i = 1, ..., k)$.

   All elements $g_{v_{n1}}, \alpha_1, b_1, ..., \alpha_k, b_k$ act naturally on T_χ where $\chi = \{r-1, r, r, ...\}$ that is from the tree T_r truncate the subtree $T_{v_{10}}$.

   For α_i, b_i (i=1,...,k) the next equations hold:

$$\alpha^g : u_0 \longrightarrow u_0, \tag{4}$$

$$(\alpha^g)_{v_{n1}}|_{T_\chi} = b_i^{g_{v_{n1}}}|_{T_\chi} \tag{5}$$

   where $v_{n1} \in V_n$ and $v_{n1} = 00\ldots00$.

   Since $\alpha_i^g \in A$ and taking into account (4)we get that sequences $\{b_i^{g_{v_{n1}}}|_{T_\chi} \mid n \in \mathbb{N}\}$ are ultimately periodic for $i = 1, ..., k$. Therefore there are $p_i, n_0 \in \mathbb{N}$ such that for $i = 1, ..., k$ and $n \geq n_0$ the next equality holds

$$b_i^{g_{v_{n+p_i,1}}}|_{T_\chi} = b_i^{g_{v_{n1}}}|_{T_\chi}.$$

   Thus

$$(g_{v_{n+p_i,1}} g_{v_{n1}}^{-1})|_{T_\chi} \in C_{\mathrm{Aut}\, T_\chi}(\langle b_i|_{T_\chi}\rangle).$$

   Taking $p = gcd(p_1, ..., p_k)$ we have

$$b_i^{g_{v_{n+p,1}}}|_{T_\chi} = b_i^{g_{v_{n1}}}|_{T_\chi},$$

$$(g_{v_{n+p,1}} g_{v_{n1}}^{-1})|_{T_\chi} \in C_{\mathrm{Aut}\, T_\chi}(\langle b_i|_{T_\chi}\rangle)$$

   for $i = 1, ..., k$ and $n \geq n_0$. Therefore in virtue of (1) and that $H_1 = \langle b_1|_{T_\chi}, ..., b_k|_{T_\chi}\rangle$ is weakly branch subgroup of the group $\mathrm{Aut}\, T_\chi$ we have

$$(g_{v_{n+p,1}} g_{v_{n1}}^{-1})|_{T_\chi} \in \bigcap_{i=1}^{k} C_{\mathrm{Aut}\, T_\chi}(\langle b_i|_{T_\chi}\rangle) = \\ = C_{\mathrm{Aut}\, T_\chi}(\langle b_1|_{T_\chi}, ..., b_k|_{T_\chi}\rangle) = C_{\mathrm{Aut}\, T_\chi}(H_1) = 1$$

   for $n \geq n_0$.

   Thus $\{g_{v_{n1}}|_{T_\chi} \mid n \in \mathbb{N}\}$ is ultimately periodic, and we have by (4) that $\{g_{v_{n1}} \mid n \in \mathbb{N}\}$ is ultimately periodic too.

□

Proof of theorem 4. It follows from the corollary 2 that there is $b_1 \in A$ such that $gb_1 : u_0 \longrightarrow u_0$. The sequence $\{(gb_1)_{v_{n1}} \mid n \in \mathbb{N}\}$ is ultimately periodic by the theorem 3. Therefore there is $k \in \mathbb{N}$ such that $\{(gb_1)_{v_{n1}} \mid n \geq k\}$ is periodic.

Let us denote by $h = (gb_1)_{v_{n1}}$. There is $b_2 \in A$ such that

$$gb_1b_2 = (h, ..., h)^{(k)}$$

by the corollary 3. For h we have $h : u \longrightarrow u$, and the sequence $\{h_{v_{n1}} \mid n \in \mathbb{N}\}$ is periodic. Let this period be m.

There is $a_1 \in A$ such that

$$ha_1 = (h, ..., h)^{(m)}$$

by the corollary 3. Let us denote by $a = a_1^{-1}$, $b = (b_1b_2)^{-1}$. We have

$$h = (h, ..., h)^{(m)}a,$$

$$g = (h, ..., h)^{(k)}b,$$

and statement is proved.

□

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