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On groups of finite normal rank

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Abstract. In this article the investigation of groups of finite normal rank is continued. The finiteness of normal rank of nonabelian p-group G is proved where G has a normal elementary abelian p-subgroup A for which quotient group G/A is isomorphic to the direct product of finite number of quasicyclic p-groups.

A number of authors studied the groups in which finiteness conditions were laid on some systems of their subgroups [1]. Earlier the author investigated the groups of finite F-rank [2], where F was some system of nonabelian finitely generated subgroups of a group and some classes of groups of finite normal rank.

In this article the investigation of groups of finite normal rank is continued.

Definition. *We shall say that a group G has finite normal rank r, if r is a minimal number with the property that for any finite set of elements $g_1, g_2, ..., g_n$ of a group G there are the elements $h_1, h_2, ..., h_m$ of G such that $m \leq r$ and*

$$< h_1, h_2, ..., h_m >^G = < g_1, g_2, ..., g_n >^G .$$

In the case when there is not such number r, the normal rank of group G is considered to be infinite.

We shall use the notation $r_n(G)$ for the normal rank of group G. The special rank of group G is denoted by the generally accepted symbol $r(G)$.

The principal result of this article is the theorem.

Theorem. *Let G be a nonabelian p-group, where p is a prime number. Let A be a normal subgroup of G, which is an elementary abelian p-group. Quotient group G/A is isomorphic to the direct product of l quasicyclic p-groups. If subgroup A can be generated as a G-subgroup by n elements, i.e.*

$$A =< a_1, a_2, ...a_n >^G,$$

and n, l are the finite numbers, then the normal rank of group G is finite and $r_n(G) \leq n + l$.

This result was announced in [3] earlier.

We shall need the following lemma in proof of the theorem.

Lemma. *The normal rank of wreath product of group of prime order p and direct product of l quasicyclic p-groups is equal to $l + 1$.*

Proof. Let A be the basis of wreath product W,$W =< a > wr(X_{j=1}^l P_j)$, where P_j is a quasicyclic p-group. We shall prove at first that for any $b_1, b_2, ..., b_n$ from A there is such element $b \in A$, for which

$$< b_1, b_2, ..., b_n >^G =< b >^G .$$

Since the group $W = \cup_{i=1}^{\infty}(< a > wr(X_{j=1}^l < g_{ji} >)$, $|g_{ji}| = p^i$, then the elements $b_1, b_2, ..., b_n$ are contained in subgroup $V =< a > wr(X_{j=1}^l < g_{ij} >)$ for some number i. The upper central series of subgroup V is

$$E = Z_0 < Z_1 < Z_2 < ... < Z_{lp^i-1} < Z_{lp^i} < V,$$

where Z_{lp^i} is the basis of wreath product V, factors Z_{k+1}/Z_k, $k = 0, 1, ..., lp^i$ have orders p, factors V/Z_{lp^i} is isomorphic to the direct product of l cyclic groups of orders p^i [4] .

The subgroups $B_k =< b_k >^{X_{j=1}^l < g_{ij} >}$, $k = 1, 2, ..., n$ are normal in group V, therefore intersections $B_k \cap Zj$, $j = 1, 2, ..., lp^i$ are nontrivial. Since the factors Z_{k+1}/Z_k, $k = 0, 1, ..., lp^i$ are cyclic of prime order, then the equalities $B_k \cap Z_q = Z_q$, $q = 0, 1, ..., t_k$, are valid, where $t_k \leq lp^i$.

From here it follows that $B_k = Z_{t_k}$, therefore for any $m_1, m_2 \leq n$ the one from subgroups B_{m_1}, B_{m_2} embeds in another. Consequently the subgroups $B_k, k = 1, 2, ..., n$ form a series of embeded subgroups

$$B_{k_1} < B_{k_2} < ... < B_{k_n} = B,$$

where $B =< b_1, b_2, ..., b_n >^{X_{j=1}^l < g_{ji} >}$. Thererore $B =< b >^{X_{j=1}^l < g_{ji} >}$, where $b = b_{k_n}$. From here follows the equality

$$< b_1, b_2, ..., b_n >^G =< b >^G .$$

Now we shall prove that for any $c_1, c_2, ..., c_r \in W$ the subgroup $C =< c_1, c_2, ..., c_r >^G$ can be generated as G-subgroup by no more than $l+1$ elements. It is sufficient to consider the case $C_1 \not\leq A$, where $C_1 =< c_1, c_2, ..., c_r >$. Since $C_1A/A \simeq C_1/(C_1 \cap A)$, the subgroup C_1 is finite and $r(C_1A/A) \leq l$, then we can choose the elements $d_1, d_2, ..., d_{s+u}$ such that

$$C =< d_1, d_2, ..., d_s, d_{s+1}, ..., d_{s+u} >^G$$

and $d_i \in A, i = 1, 2, ..., s, d_{s+1}, d_{s+2}, ..., d_{s+u} \notin A, u \leq l$. As we proved, there is the element $d \in A$ for which

$$< d_1, d_2, ..., d_s >^G =< d >^G,$$

therefore $C =< d, d_{s+1}, ..., d_{s+u} >^G$. Consequently the normal rank of wreath product W is no more than $l+1$.

For proving the equality $r_n(W) = l+1$ we numerate the elements of subgroup $X_{j=1}^{\infty} P_j$ as $h_1, h_2, ...$ and assume $a^{h_i} = a_i$. According to the structure of subgroup W the subgroup $A_0 =< a_i a_j^{-1}, i, j = 1, 2, ... >$ is normal in W and quotient group W/A_0 is isomorphic to the direct product of a group of prime order p and l quasicyclic p-groups. Since the normal rank of quotient group W/A_0 is equal to $l+1$ and $r_n(W/A_0) \leq r_n(W)$, where $r_n(W) \leq l+1$, then we have the equality $r_n(W) = l+1$. Lemma is proved. □

Proof of the theorem. At first we shall prove that for any finite set of elements $b_1, b_2, ..., b_k$ of A there are the elements $c_1, c_2, ..., c_t$ of A such that $t \leq n$ and $< b_1, b_2, ..., b_k >^G =< c_1, c_2, ..., c_t >^G$. We shall prove at first this statement by the induction on number v of elements $a_1, a_2, ..., a_v$, where $A =< a_1, a_2, ..., a_v >^G$. If $v = 1$ then $A =< a_1 >^G$, therefore group G is isomorphic to some quotient group of wreath product of a group of prime order p and direct product of l quasicyclic p-groups. From the proof of the lemma it follows that there is an element $b \in A$ for which

$$< b_1, b_2, ..., b_k >^G =< b >^G .$$

Let our statement be valid for $u = n-1$. Let $u = n$ and

$$B =< b_1, b_2, ..., b_k >^G, A_1 =< a_1, a_2, ..., a_{n-1} >^G .$$

If subgroup B is contained in A_1 then according to the inductive assumption there are such elements $c_1, c_2, ..., c_t, t \leq n$ that $B =< c_1, c_2, ..., c_t >^G$. Let now $B \not\leq A_1$. Quotient group G/A_1 is isomorphic to some quotient group of wreath product of a group of prime order p and direct product of l quasicyclic p-groups. From this and isomorphism $BA_1/A_1 \simeq B/B \cap A_1$

it follows by the lemma that there is an element $b \in B$ for which $B/B \cap A_1 =< b(B \cap A_1) >^G$. Consequently for every $b_i, i = 1, 2, ..., k$, there are such integers $n_1, n_2, ..., n_{r_i}$ and the elements $g_1, g_2, ..., g_{r_i}$ of G that the equalities

$$b_i = (b^{n_1})^{g_1}(b^{n_2})^{g_2}...(b^{n_{r_i}})^{g_{r_i}}h_i$$

are valid, where $h_i \in (B \cap A_1)$. Since the element b belongs to the subgroup B then $B =< b, h_1, h_2, ..., h_k >^G$, therefore

$$B =< b >^G< h_1, h_2, ..., h_k >^G . \quad (1)$$

According to the inductive assumption and inclusion $< h_1, h_2, ..., h_k >^G \leq A_1$ there are such elements $d_1, d_2, ..., d_m$ of A that $m \leq n - 1$ and

$$< h_1, h_2, ..., h_k >^G =< d_1, d_2, ..., d_m >^G .$$

From this equality and (1) it follows that $B =< b, d_1, ..., d_m >^G, m \leq n - 1$. Our statement is proved.

Let now $B =< b_1, b_2, ..., b_k >^G$, where even if one from the elements $b_i, i = 1, 2, ..., k$ does not belong to the subgroup A. Since the subgroup D generated by the elements $b_1, b_2, ..., b_k$ is finite, then the intersection $D \cap A$ is finite too. Therefore there are the elements $c_1, c_2, ..., c_j, j \leq n$, for which $< D \cap A >^G =< c_1, c_2, ..., c_j >^G$. Since quotient group G/A is a direct product of l locally cyclic groups and $DA/A \simeq D/D \cap A$, then there are such elements $c_{j+1}, ..., c_{i+y}$ of D that

$$< D >^G =< c_1, c_2, ..., c_{j+y} >^G,$$

$y \leq l$. Consequently the equality $B =< c_1, c_2, ..., c_{j+y} >^G$ is valid, where $j + y \leq n + l$. The theorem is proved. □

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