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RESEARCH ARTICLE

An additive divisor problem in $\mathbb{Z}[i]$

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Abstract. Let $\tau(\alpha)$ be the number of divisors of the Gaussian integer α. An asymptotic formula for the summatory function $\sum_{N(\alpha)\le x} \tau(\alpha)\tau(\alpha+\beta)$ is obtained under the condition $N(\beta) \le x^{3/8}$. This is a generalization of the well-known additive divisor problem for the natural numbers.

1. Introduction

In 1927 A.E. Ingham [1] obtained the asymptotic formula for the number of solutions $I(x)$ the diophantic equation

$$u_1u_2 - v_1v_2 = 1$$

under conditions: $u_1, u_2, v_1, v_2 \in \mathbb{N}$, $u_1u_2 \le x$.

Obviously

$$I(x) = \sum_{n\le x} \tau(n)\tau(n+1),$$

where $\tau(n) = \sum_{n=ab} 1$ denote the number of ways n may be written as a product of two natural numbers.

Ingham proved that

$$I(x) = \frac{6}{\pi^2} x \log^2 x + O(x \log x).$$

T. Estermann [2] improved this result in form

$$I(x) = xP_2(\log x) + E(x),$$

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where $P_2(u)$ is a polynom $a_0u^2 + a_1u + a_2$ and $E(x)$ is an error term.

Estermann gave $E(x) \ll x^{\theta+\varepsilon}$, $\theta = \frac{11}{12}$. The exponent θ was subsequently improved to $\frac{5}{6}$ by D.R. Heath-Brown [3] and then to $\frac{2}{3}$ by J.-M. Deshouillers and H. Ivaniec [4]. In 1994 Y. Motohashi [5] employed powerful methods from the spectral theory of automorphic forms and obtained very precise result:

$$I(x) = \sum_{n\le x} \tau(n)\tau(n+h) = x\sum_{i=0}^{2}(\log x)^i \sum_{j=0}^{2} c_{ij} \sum_{d|h} \frac{(\log d)^j}{d} + O\left(x^{\frac{2}{3}+\varepsilon}\right)$$

holds uniformly for $1 \le h \le x^{20/27}$.

The purpose of this paper is to build the asymptotic formula for sum

$$\sum_{\substack{\alpha\in\mathbb{Z}[i]\\ 0<N(\alpha)\le x}} \tau(\alpha)\tau(\alpha+\beta)$$

where $\tau(\alpha) = \sum\limits_{\delta|\alpha} 1$ is a number of divisors of a Gaussian integer α.

Notations. Denote by $\mathbb{Z}$ the ring of Gaussian integers. We write $N(\alpha) = a^2+b^2$, $Sp(\alpha) = 2a$ for $\alpha = a+bi \in \mathbb{Z}[\alpha]$; $\varphi(\alpha) = N(\alpha)\prod\limits_{p|\alpha}(1-N(p)^{-1})$, p is prime divisor α; $e(x) = \exp(2\pi ix)$ for the real number x; the Vinogradov symbol $f \ll g$ means $f = O(g)$; ε is an arbitrary small positive number that is not necessarily the same at each occurrence; the constants implied by the O (or $\ll$) – notation depend at most on ε.

2. Statement of Result

Let β be Gaussian integer and x be real positive number. By $I(x,\beta)$ we denote the number of solutions in Gaussian integers of the equation

$$\alpha_1\alpha_2 - \alpha_3\alpha_4 = \beta, \quad N(\alpha_1\alpha_1) \le x.$$

Theorem. *For $N(\beta) \ll x^{3/8}$ and any $\varepsilon > 0$ the following formula*

$$I(x,\beta) = xP_2(\log x) + O(x^{\frac{7}{8}+\varepsilon})$$

holds.

Here $P_2(u) = A_0u^2 + A_1u + A_2$, $A_i = A_i(\beta)$, $i = 0, 1, 2$, moreover $A_i(\beta)$ are computable and $1 \ll A_i(\beta) \ll \tau(\beta)$, $A_0(\beta) > 0$.

3. Auxiliary Results

Let δ_0, δ be the Gaussian rationals ($\delta_0, \delta \in \mathbb{Q}(i)$) not necessarily integers. Let for $Re(s) > 1$

$$\zeta(s, \delta, \delta_0) = \sum_{\substack{\omega \in \mathbb{Z}[i] \\ \omega \neq -\delta}} e\left(\frac{1}{2} Sp(\delta_0 \omega)\right) N(\delta + \omega)^{-s}.$$

Lemma 1 (see [5],lemmas 1 and 3). *The function $\zeta(s, \delta, \delta_0)$ is entire function if $\delta_0 \notin \mathbb{Z}[i]$. For $\delta_0 \in \mathbb{Z}[i]$, $\zeta(s, \delta, \delta_0)$ is holomorphic except at $s = 1$, where it has a simple pole and*

$$\zeta(s, \delta, 0) = \frac{\pi}{s-1} + a_0(\delta) + a_1(\delta)(s-1) + \dots$$

where

$$a_0(\delta) = \begin{cases} \pi E + 4L'(1, \chi_4) & \text{if } \delta \in \mathbb{Z}[i], \\ \pi E + 4L'(1, \chi_4) + \sum\limits_{\beta \in B} N(\delta + \beta) + b_0(\delta) & \text{if } 0 < N(\delta) < 1; \end{cases}$$

E is the Euler constant, $L'(s, \chi_4) = \frac{d}{ds} L(s, \chi_4)$, $L(s, \chi_4)$ is L-Dirichlet function with non-principal character mod 4; $b_0(\delta) = -4 + O\left(N^{1/2}(\delta)\right)$, B denotes the set $\{0, \pm 1, \pm i\}$. Moreover, the functional equation

$$\pi^{-s} \Gamma(s) \zeta(s, \delta, \delta_0) = \pi^{-(1-s)} \Gamma(1-s) \zeta(1-s, -\delta_0, \delta) e\left(-\frac{1}{2} Sp(\delta_0 \delta)\right) \quad (1)$$

holds.

Let $\alpha, \beta, \gamma \in \mathbb{Z}[i]$. We define the Kloosterman sum for the ring of Gaussian integer

$$K(\alpha, \beta; \gamma) = \sum_{\substack{\xi, \xi' \pmod{\gamma} \\ \xi \cdot \xi' \equiv 1 (\gamma)}} e\left(\frac{1}{2} Sp\left(\frac{\alpha \xi + \beta \xi'}{\gamma}\right)\right).$$

Lemma 2. *Let α, β, γ be Gaussian integers, $\gamma \neq 0$. Then the estimate*

$$|K(\alpha, \beta; \gamma)| \ll (N(\gamma) N((\alpha, \beta; \gamma)))^{1/2} \tau(\gamma) \quad (2)$$

holds, (where $(\alpha, \beta; \gamma)$ is the greate common divisor of α, β, γ). Moreover,

$$K(\alpha, \beta; \gamma) = \sum_{\delta | (\alpha, \beta, \gamma)} N(\delta) K\left(1, \frac{\alpha\beta}{\delta^2}; \frac{\gamma}{\delta}\right). \quad (3)$$

This lemma follow from a multiplicative property of $K(\alpha,\beta;\gamma)$ on γ and the Bombieri estimate of an exponential sum on the algebraic curve over the finite field. The formula (3) is a generalized Kuznetsov's identity for Kloosterman sums.

Lemma 3. *Let* $\alpha_0,\gamma\in\mathbb{Z}[i]$, $(\alpha_0,\gamma)=\beta$, $N(\beta)<N(\gamma)$. *Then for* $N(\gamma)\ll x^{2/3+\varepsilon}$ *we have*

$$\sum_{\substack{\alpha\equiv\alpha_0(\gamma)\\ N(\alpha)\le x}}\tau(\alpha)=c_0(\alpha_0,\gamma)\frac{x}{N(\gamma)}\log\frac{x}{N(\beta)}+ \\ +c_1(\alpha_0,\gamma)\frac{x}{N(\gamma)}+O\left(x^{1/2+\varepsilon}N(\gamma)^{-1/4}\right),$$

where $c_0(\alpha_0,\gamma)=\pi^2N(\beta)\varphi\left(\frac{\gamma}{\beta}\right)N^{-1}(\gamma)\tau(\beta)$,

$$c_1(\alpha_0,\gamma)=\pi^2\sum_{\delta|\beta}\left[2E-1+2\frac{L'(1,\chi_4)}{L(1,\chi_4)}+\sum_{p|\gamma/\delta}{}^{*}\log\frac{N(p)}{N(p)-1}\right] \\ \prod_{\gamma|\gamma/\delta}{}^{*}(1-N^{-1}(p)).$$

Proof. Without loss of generality we will consider only a case $(\alpha_0,\gamma)=1$.

We have for $c=1+\varepsilon$:

$$\sum_{\substack{\alpha\equiv\alpha_0(\gamma)\\ N(\alpha)\le x}}\tau(\alpha)-\sum_{\substack{\alpha=\alpha_0+\beta\gamma\\ \beta\in B}}\tau(\alpha)=$$

$$=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(F(s)-\sum_{\beta\in B}\frac{\tau(\alpha_0+\beta\gamma)}{N(\alpha_0+\beta\gamma)^s}\right)\frac{x^s}{s}ds+O\left(\frac{x^c}{TN(\gamma)}\right),\qquad(4)$$

where

$$F(s)=N(\gamma)^{-2s}\sum_{\substack{\alpha_1,\alpha_2(\mathrm{mod}\ \gamma)\\ \alpha_1\alpha_2\equiv\alpha_0(\gamma)}}\zeta\left(s,\frac{\alpha_1}{\gamma},0\right)\zeta\left(s,\frac{\alpha_2}{\gamma},0\right)=\sum_{\substack{\alpha\equiv\alpha_0(\gamma)\\ \alpha\in\mathbb{Z}[i]}}\frac{\tau(\alpha)}{N(\alpha)^s}.$$

From lemma 1 we have the functional equation

$$F(s)=\frac{\pi^{2(2s-1)}}{N^{2s}(\gamma)}\frac{\Gamma^2(1-s)}{\Gamma^2(s)}\Psi(1-s),$$

where

$$\Psi(s)=\sum_{\omega}\frac{1}{N(\omega)^s}\sum_{\alpha\beta=\omega}\Phi(\alpha,\beta;\gamma),$$

$$\Phi(\alpha,\beta;\gamma)\sum_{\substack{\alpha_1,\alpha_2(\mathrm{mod}\ \gamma)\\ \alpha_1\alpha_2\equiv\alpha_0(\gamma)}} e\left(\frac{1}{2}Sp\left(\frac{\alpha\alpha_1+\beta\alpha_2}{\gamma}\right)\right).$$

Moreover, $F(0)=0$ if $N(\gamma)>1$ and $\alpha\not\equiv 0(\mathrm{mod}\ \gamma)$.

By lemma 1 we obtain

$$G(s)=F(s)-\sum_{\beta\in B}\frac{\tau(\alpha_0+\beta\gamma)}{N(\alpha_0+\beta\gamma)^s}\ll\begin{cases} N(\gamma)^{-1+\varepsilon} & \text{if } Re(s)=1+\varepsilon,\\ N(\gamma)^{1/2+\varepsilon}T^3 & \text{if } Re(s)=-\frac{1}{4}.\end{cases} \tag{5}$$

Applying Phragmen-Lindelöf principle we infer

$$G(-\varepsilon+it)\ll N(\gamma)^{1/5+\varepsilon}T^{12/5+\varepsilon}\quad \text{for } |t|\le T.$$

To deal with integral in (4) we move the segment of integration to $Re(s)=-\varepsilon$.

By the theorem of residues we obtain

$$\sum_{\substack{\alpha\equiv\alpha_0(\gamma)\\ N(\alpha)\le x}}\tau(\alpha)=\mathrm{res}_{s=0}\left(G(s)\frac{x^s}{s}\right)+\mathrm{res}_{s=1}\left(G(s)\frac{x^s}{s}\right)+$$

$$+\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT}G(s)\frac{x^s}{s}ds++O\left(x^{\varepsilon}\right)+O\left(N(\gamma)^{1/5+\varepsilon}T^{12/5+\varepsilon}\right)+$$

$$+O\left(\frac{x^{1+\varepsilon}}{TN(\gamma)}\right). \tag{6}$$

Further,

$$\mathrm{res}_{s=0}\left(G(s)\frac{x^s}{s}\right)=\frac{\pi^2 x\log x}{N(\gamma)}\prod_{\gamma|\alpha}(1-N(\gamma)^{-1})+$$

$$+\frac{\pi^2 x}{N(\gamma)}\prod_{p|\alpha}(1-N(p)^{-1})\left[-1+2\left(E+\frac{L'(1,\chi_4)}{L(1,\chi_4)}+\sum_{p|\delta}\frac{\log N(p)}{N(p)-1}\right)\right], \tag{7}$$

$$\mathrm{res}_{s=0}\left(G(s)\frac{x^s}{s}\right)=\mathrm{res}_{s=0}\left(-\sum_{\beta\in B}\frac{\tau(\alpha_0+\beta\gamma)}{N(\alpha_0+\beta\gamma)^s}\frac{x^s}{s}\right)\ll N(\gamma)^{\varepsilon}.$$

Observe that by lemma 2

$$\sum_{\alpha\beta=\omega}|\Phi(\alpha,\beta;\gamma)|=\sum_{\alpha\beta=\omega}|K(\alpha,\beta\alpha_0;\gamma)|\ll N(\gamma)^{1/2}N((\omega,\gamma))^{1/2}\tau(\gamma)\tau(\omega).$$

Now by termwise integration and applying the Stirling formula for the gamma function and the method of stationary phase we get

$$\frac{1}{2\pi i}\int_{-\varepsilon-iT}^{-\varepsilon+iT} G(s)\frac{x^s}{s}ds =$$

$$= \sum_{\substack{\omega \\ 0<N(\omega)\le Y}} \frac{\pi^2}{N(\omega)} \sum_{\alpha\beta=\omega} \Phi(\alpha,\beta;\gamma)\frac{y^{3/8}}{4\sqrt{2/\pi}} e\left(-\frac{1}{8}-\frac{1}{2\pi}y^{1/4}\right)\cdot$$

$$\cdot\left(1+O\left(y^{-1/8}\right)\right)+O\left(\frac{x^{1+\varepsilon}}{TN(\gamma)}\right)+O\left(x^{\varepsilon}\right)+$$

$$+O\left(\sum_{N(\omega)>Y} y^{-\varepsilon}T^{1+4\varepsilon}N(\gamma)^{1/2+\varepsilon}N((\omega,\gamma))^{1/2}\tau(\omega)N(\omega)^{-1}\right), \quad (8)$$

where $Y \le X = \left(\frac{4}{\pi}\right)^4 \frac{T^4N^2(\gamma)}{x}$, $y = \frac{\pi^4 xN(\omega)}{N^2(\gamma)}$.

The assertion of the lemma follow from (4),(6)–(8) if we put

$$T = x^{1/2}N(\gamma)^{-3/4},\ Y = x^{1/3}.$$

□

4. Proof of the theorem

We start the proof of our theorem by observing that

$$\tau(\alpha) = 2\#\{\gamma|\alpha; N(\gamma)\le x^{1/2}\} - \#\{\gamma|\alpha; N(\alpha)x^{-1/2}\le N(\gamma)\le x^{1/2}\}$$

whenever α, $N(\alpha)\le x$.

Hence

$$\sum_{N(\alpha)\le x}\tau(\alpha)\tau(\alpha+\beta) =$$

$$= \sum_{N(\gamma)\le x^{1/2}}\left(2\left\{\sum_{\substack{\alpha\equiv\beta(\gamma)\\ N(\alpha-\beta)\le x}}\tau(\alpha)-1\right\}-\left\{\sum_{\substack{\alpha\equiv\beta(\gamma)\\ N(\alpha-\beta)\le N(\gamma)x^{1/2}}}\tau(\alpha)-1\right\}\right) =$$

$$= 2\sum_{N(\gamma)\le x^{1/2}}\sum_{\substack{\alpha\equiv\beta(\gamma)\\ N(\alpha)\le x}}\tau(\alpha) - \sum_{N(\gamma)\le x^{1/2}}\sum_{\substack{\alpha\equiv\beta(\gamma)\\ N(\alpha)\le N(\gamma)x^{1/2}}}\tau(\alpha)+O(x^{7/8+\varepsilon})$$

Indeed, we have

$$N(\alpha-\beta) = |\alpha-\beta|^2 \ge \left||\alpha|^2-|\beta|^2\right| = N(\alpha)-N(\beta) \quad \text{for } N(\alpha)\ge N(\beta),$$

and

$$N(\alpha-\beta)\le|\alpha|^2+|\beta|^2=N(\alpha)+N(\beta).$$

Therefore we carry an error in the asymptotic formula $\ll N(\beta)x^{1/2}\ll x^{7/8}$ if we replace the condition $N(\alpha-\beta)\le x$ on the condition $N(\alpha)\le x$ (we take into account that $N(\beta)\le x^{3/8}$).

Now, by lemma 3 we obtain

$$\sum_{N(\alpha)\le x^{1/2}}\ \sum_{\substack{\alpha\equiv\beta(\gamma)\\ N(\alpha)\le x}} = \sum_{\delta|\beta}\ \sum_{\substack{N(\gamma)\le x^{1/2}N(\delta)^{-1}\\ (\gamma,\beta/\delta)=1}}\ \sum_{\substack{\alpha\equiv\beta(\mathrm{mod}\ \gamma\delta)\\ N(\alpha)\le x}}\tau(\alpha)=$$

$$=\sum_{\delta|\beta}\ \sum_{\substack{N(\gamma)\le\frac{x^{1/2}}{N(\delta)}\\ (\gamma,\alpha_0/\delta)=1}}\left\{\frac{\pi^2x}{N^2(\gamma\delta)}N(\delta)\varphi(\gamma)\tau(\delta)\left(\log\frac{x}{N(\delta)}-1\right)+\right.$$

$$\left.+\frac{2\pi^2x}{N(\gamma)}\sum_{t|\delta}\left(E+\frac{L'(1,\chi_4)}{L(1,\chi_4)}+\sum_{p|\gamma\delta/t}\log\frac{N(p)}{N(p)-1}\right)\prod_{p|\gamma\delta/t}(1-N(p)^{-1}\right\}+$$

$$+O\left(\sum_{\delta|\beta}\ \sum_{N(\gamma)\le\frac{x^{1/2}}{N(\delta)}}x^{1/2}N(\gamma\delta)^{-1/4}\right).$$

Using the equality

$$\varphi(\alpha)=N(\alpha)\prod_{p|\alpha}(1-N(p)^{-1})=N(\alpha)\sum_{\delta|\alpha}\frac{\mu(\delta)}{N(\delta)}$$

we infer

$$\sum_{\substack{N(\alpha)\le x\\ (\alpha,\beta)=1}}\frac{\varphi(\delta)}{N(\delta)}=\sum_{\substack{N(\alpha)\le x\\ (\alpha,\beta)=1}}\sum_{\delta|\alpha}\frac{\mu(\delta)}{N(\delta)}=\sum_{\substack{N(\delta)\le x\\ (\alpha\delta,\beta)=1}}\frac{\mu(\delta)}{N(\delta)}\sum_{N(\alpha)\le\frac{x}{N(\delta)}}1=$$

$$=\prod_{p|\beta}(1-N(p)^{-1})\left(\pi x\sum_{\substack{N(\alpha)\le x\\ (\delta,\beta)=1}}\frac{\mu(\delta)}{N^2(\delta)}+O(x^{1/3})\right)$$

$$=c_0(\beta)x+O(x^{1/3}). \tag{11}$$

where $c_0(\beta)=c_0\frac{\varphi(\beta)}{N(\beta)}\prod_{p|\beta}(1-N(\rho)^{-2}),\quad c_0=const.$

Therefore

$$\sum_{\substack{N(\gamma)\le\frac{x^{1/2}}{N(\delta)}\\ (\gamma,\beta/\delta)=1}}c_0(\beta/\delta)\left(\log x+1+O\left(x^{1/3}\right)\right). \tag{12}$$

Hence, from (10),(12), we get

$$\sum_{N(\gamma)\le x^{1/2}} \sum_{\substack{\alpha\equiv\beta(\gamma)\\ N(\alpha)\le x}} \tau(\alpha) = x(a_0(\beta)\log^2 x + + a_1(\beta)\log x + a_2(\beta)) + O(x^{7/8+\varepsilon}). \quad (13)$$

Similarly

$$\sum_{N(\gamma)\le x^{1/2}} \sum_{\substack{\alpha\equiv\beta(\gamma)\\ N(\alpha)\le N(\gamma)x^{1/2}}} \tau(\alpha) = x(b_1(\beta)\log x + b_2(\beta)) + O(x^{7/8+\varepsilon}). \quad (14)$$

From (9),(13),(14) we obtain the assertion of theorem. □

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