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Uniform ball structures

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Abstract. A ball structure is a triple $\mathbb{B} = (X, P, B)$, where X, P are nonempty sets and, for all $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of $X, x \in B(x, \alpha)$, which is called a ball of radius α around x. We introduce the class of uniform ball structures as an asymptotic counterpart of the class of uniform topological spaces. We show that every uniform ball structure can be approximated by metrizable ball structures. We also define two types of ball structures closed to being metrizable, and describe the extremal elements in the classes of ball structures with fixed support X.

Following [2], by *ball structure* we mean a triple $\mathbb{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$, $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called *a ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. A set X is called a *support* of $\mathbb{B}, P$ is called a *set of radiuses.*

Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be ball structures, $f : X_1 \longrightarrow X_2$ We say that f is a $\succ$*-mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that

$$B_2(f(x), \beta) \subseteq f(B_1(x, \alpha))$$

for every $x \in X_1$. If there exists a surjective $\succ$-mapping $f : X_1 \longrightarrow X_2$, we write $\mathbb{B}_1 \succ \mathbb{B}_2$.

A mapping $f : X_1 \longrightarrow X_2$ is called a $\prec$*-mapping* if, for every $\alpha \in P_1$,there exists $\beta \in P_2$ such that

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

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for every $x \in X$. If there exists an injective $\prec$-mapping $f : X_1 \longrightarrow X_2$, we write $\mathbb{B}_1 \prec \mathbb{B}_2$.

A bijection $f : X_1 \longrightarrow X_2$ is called an *isomorphism* between $\mathbb{B}_1$ and $\mathbb{B}_2$ if f is a $\succ$-mapping and f is a $\prec$-mapping.

Let $\mathbb{B}_1 = (X_1, P_1, B_1)$, $\mathbb{B}_2 = (X_2, P_2, B_2)$ be ball structures with common support X. We say that $\mathbb{B}_1 \subseteq \mathbb{B}_2$ if the identity mapping id: $X \longrightarrow X$ is a $\prec$-mapping from $\mathbb{B}_1$ to $\mathbb{B}_2$. If $\mathbb{B}_1 \subseteq \mathbb{B}_2$ and $\mathbb{B}_2 \subseteq \mathbb{B}_1$, we write $\mathbb{B}_1 = \mathbb{B}_2$.

A property $\mathcal{P}$ of ball structures is called a ball property if a ball structure $\mathbb{B}$ has a property $\mathcal{P}$ provided that $\mathbb{B}$ is isomorphic to some ball structure with property $\mathcal{P}$. Now we describe four basic ball properties.

Let $\mathbb{B} = (X, P, B)$ be a ball structure. For any $x \in X$, $\alpha \in P$ put

$$B^*(x, y) = \{y \in X : x \in B(y, \alpha)\}.$$

A ball structure $\mathbb{B}^* = (X, P, B)$ is called *dual* to $\mathbb{B}$. Note that $\mathbb{B}^{**} = \mathbb{B}$.

A ball structure $\mathbb{B} = (X, P, B)$ is called *symmetric* if $\mathbb{B} = \mathbb{B}^*$.

A ball structure $\mathbb{B} = (X, P, B)$ is called *multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma(\alpha, \beta) \in P$ such that

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma(\alpha, \beta))$$

for every $x \in X$. Here

$$B(Y, \alpha) = \bigcup_{y \in Y} B(y, \alpha), \quad Y \subseteq X, \quad \alpha \in P.$$

Let $\mathbb{B} = (X, P, B)$ be a ball structure, $x, y \in X$, We say that x, y are *connected* if there exists $\alpha \in P$ such that $x \in B(y, \alpha)$, $y \in B(x, \alpha)$. A subset $Y \subseteq X$ is called connected if any two elements from Y are connected. A ball structure $\mathbb{B}$ is called *connected* if its support is connected. If $\mathbb{B}$ is symmetric and multiplicative, then connectivity is an equivalence on X, so X disintegrates into connected components.

For an arbitrary ball structure $\mathbb{B} = (X, P, B)$ we define a preodering $\leq$ on the set P by the rule $\alpha \leq \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called *cofinal* if, for every $\alpha \in P$, there exists $\beta \in P'$ such that $\alpha \leq \beta$. A *cofinality* $cf\mathbb{B}$ of $\mathbb{B}$ is the minimal cardinality of cofinal subsets of P.

Let (X, d) be a metric space, $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Given any $x \in X$, $r \in \mathbb{R}^+$, put

$$B_d(x, r) = \{y \in X : d(x, y) \leq r\}.$$

A ball structure $(X, \mathbb{R}^+, B_d)$ is denoted by $\mathbb{B}(X,d)$. We say that a ball structure $\mathbb{B}$ is *metrizable* if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X,d)$ for some metric space (X,d). We shall use the following metrizability criterion [2].

A ball structure $\mathbb{B}$ is metrizable if and only if $\mathbb{B}$ is symmetric, multiplicative, connected and $cf \ \ \mathbb{B} \leq \aleph_0$.

A ball structure is called *uniform* if it is symmetric and multiplicative.

In §1 we define a wide spectrum of examples of uniform ball structures related to groups and filters. In §2 we introduce some ball operations which give new uniform ball structures from a pregiven family of uniform ball structures. It is well known [1], that every uniform topological space can be approximated by pseudometrizable spaces. In §3 we prove a ball analogue of such an approximation. In §3 – 4 we introduce two types of ball structures (inductively metrizable and submetrizable) close to being metrizable. In §5 we describe extremal by inclusion elements in the classes of ball structures with fixed support.

§1 Examples

Let G be an infinite group with the identity e, γ be an infinite cardinal, $\gamma < |G|$. Denote by $\Im_e(G,\gamma)$ the family of all subsets of G of cardinality $< \gamma$ containing e. Given any $g \in G$, $F \in \Im_e(G)$, put

$$B_l(g,F) = Fg, \quad B_r(g,F) = gF.$$

The ball structures

$$(G, \Im_e(G,\gamma), B_l), \quad (G, \Im_e(G,\gamma), B_r)$$

will be denoted by $B_l(G,\gamma)$, $B_r(G,\gamma)$. Note that the mapping $g \longmapsto g^{-1}$ is an isomorphism between $B_l(G)$ and $B_r(G)$. In the case $\gamma = \aleph_0$ we write $B_l(G)$ and $B_r(G)$ instead of $B_l(G,\gamma)$ and $B_r(G,\gamma)$. It easy to see that $B_l(G)$=$B_r(G)$ if and only if the set $\{x^{-1}gx : x \in G\}$ is finite for every $g \in G$.

By metrizability criterion, $B_l(G,\gamma)$ is metrizable if and only if $\gamma = |G|$ and $cf \ \gamma = \aleph_0$. In particular, $B_l(G)$ is metrizable if and only if G is countable.

Let X be a set and let φ be a filter on X. For any $x \in X$, $F \in \varphi$, put

$$B(x,F) = \begin{cases} X \backslash F, & \text{if } x \notin F; \\ \{x\}, & \text{if } x \in F; \end{cases}$$

and denote by $\mathbb{B}(X,\varphi)$ the ball structure (X, φ, B). Note that $\mathbb{B}(X,\varphi)$ is connected if and only if either $\bigcap \varphi = \emptyset$ or $|X| = 1$. Hence, $\mathbb{B}(X,\varphi)$ is metrizable if and only if either $|X| = 1$ or $\bigcap \varphi = \emptyset$ and φ has a countable base.

Now we define a wide class of ball structures containing all ball structures of filters and almost all ball structures of groups.

Let X be a set and let $\mathcal{P}$ be a family of partitions of X. For any $x, y \in X$ and $P \in \mathcal{P}$, denote by $B(x, P)$ the set $\{y \in X : x, y$ are in the same cell of the partition $P\}$. A ball structure $(X, \mathcal{P}, B)$ is denoted by $\mathbb{B}(X, \mathcal{P})$. Clearly, $B(X, \mathcal{P})$ is symmetric. Given any $P_1, P_2 \in \mathcal{P}$, we say that P_2 is an enlargement of P_1 if $B(x, P_1) \subseteq B(x, P_2)$ for each $x \in X$. A ball structure $\mathbb{B}(X, \mathcal{P})$ is multiplicative if and only if, for any $P_1, P_2 \in \mathcal{P}$, there exists $P \in \mathcal{P}$ such that P is an enlargement of P_1 and P_2.

A ball structure $\mathbb{B}$ is called *cellular* if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X, \mathcal{P})$ for some set X and some family $\mathcal{P}$ of partitions of X. Given any ball structure $\mathbb{B} = (X, P, B)$, $x, y \in X$ and $\alpha \in P$, we say that x, y are *α-path connected* if there exists a sequence $x_0, x_1, ..., x_n$, $x_0 = x$, $x_n = y$ such that

$$x_{i+1} \in B(x_i, \alpha), \quad x_i \in B(x_{i+1}, \alpha)$$

for every $i \in \{0, 1, ..., n-1\}$. For any $x \in X$, $\alpha \in P$, put

$$B^{\Box}(x, \alpha) = \{y \in X : x, y \textit{ are } \alpha - \textit{path connected}\}.$$

A ball structure $\mathbb{B}^{\Box}(X, P, B^{\Box})$ is called a *cellularization* of $\mathbb{B}$. By [2], a ball structure $\mathbb{B}$ is cellular if and only if $\mathbb{B} = \mathbb{B}^{\Box}$. A metrizable ball structure $\mathbb{B}$ is cellular if and only if $\mathbb{B}$ is isomorphic to $\mathbb{B}(X, d)$ for some non-Archimedian metric space.

Every ball structure $\mathbb{B}(X, \varphi)$ of a filter φ on X is cellular. A ball structure $\mathbb{B}(G, \gamma)$ of a group G is cellular if and only if either $\gamma > \aleph_0$ or $\gamma = \aleph_0$ and every finite subsets of G generates a finite subgroup.

§2 Constructions

Let $\{\mathbb{B}_\lambda = (X_\lambda, P, B_\lambda) : \lambda \in I\}$ be a family of ball structures with pairwise disjoint supports and common set of radiuses, $X = \bigcup_{\lambda \in I} X_\lambda$. For every $x \in X$, $x \in X_\lambda$ and every $\alpha \in P$, put $B(x, \alpha) = B_\lambda(x, \alpha)$. A ball structure $\mathbb{B} = (X, P, B)$ is called a *disjoint union* of the family $\{\mathbb{B}_\lambda : \lambda \in I\}$. Every uniform ball structure is a disjoint union of its connected components.

Let $\{\mathbb{B}_\lambda = (X, P_\lambda, B_\lambda) : \lambda \in I\}$ be a family of ball structures with common support. Suppose that, for any $\lambda_1, \lambda_2 \in I$, there exists $\lambda \in I$ such that $\mathbb{B}_{\lambda_1} \subseteq \mathbb{B}_\lambda$, $\mathbb{B}_{\lambda_2} \subseteq \mathbb{B}_\lambda$. For every $\lambda \in I$, choose a copy $P'_\lambda = f_\lambda(P_\lambda)$ of P_λ such that the family $\{P'_\lambda : \lambda \in I\}$ is disjoint. Put $P = \bigcup_{\lambda \in I} P'_\lambda$. For all $x \in X$, $\beta \in P$, $\beta \in P'_\lambda$, put $B(x, \beta) = B_\lambda(x, f_\lambda^{-1}(\beta))$.

A ball structure $\mathbb{B} = (X, P, B)$ is called an *inductive limit* of the family $\{\mathbb{B}_\lambda : \lambda \in I\}$. Clearly, $\mathbb{B}_\lambda \subseteq \mathbb{B}$ for every $\lambda \in I$. If every $\mathbb{B}_\lambda$ is uniform, $\mathbb{B}$ is uniform.

Let $\mathbb{B} = (X, P, B)$ be a ball structure, $Y \subseteq X$. For any $y \in Y$, $\alpha \in P$, put $B_Y(y, \alpha) = B(y, \alpha) \bigcap Y$. A ball structure $\mathbb{B}_Y = (Y, P, B_Y)$ is called a substructure of $\mathbb{B}$. If $\mathbb{B}$ is uniform, then $\mathbb{B}_Y$ is uniform.

Let $\{\mathbb{B}_\lambda = (X_\lambda, P_\lambda, B_\lambda) : \lambda \in I\}$ be an arbitrary family of ball structures. By *box product* of this family we mean a ball structure

$$\prod_{\lambda\in I} \mathbb{B}_\lambda = (\prod_{\lambda\in I} X_\lambda, \prod_{\lambda\in I} P_\lambda, B),$$

where

$$B(x, p) = \{y \in \prod_{\lambda\in I} X_\lambda : pr_\lambda(y) \in B_\lambda(pr_\lambda(x), pr_\lambda(p)), \lambda \in I\}$$

for all

$$x \in \prod_{\lambda\in I} X_\lambda, p \in \prod_{\lambda\in I} P_\lambda.$$

If every ball structure $\mathbb{B}_\lambda$ is uniform, then $\prod_{\lambda\in I} \mathbb{B}_\lambda$ is uniform. Note also that

$$\mathbb{B}_\gamma \prec \prod_{\lambda\in I} \mathbb{B}_\lambda, \quad \prod_{\lambda\in I} \mathbb{B}_\lambda \succ \mathbb{B}_\gamma$$

for every $\gamma \in I$.

Let $\mathbb{B} = (X, P, B)$ be a ball structure. A subset $Y \subseteq X$ is called *bounded* if there exist $x \in X$, $\alpha \in P$ such that $Y \subseteq B(x, \alpha)$. We say that $\mathbb{B}$ is *bounded* if its support is bounded. Let $\mathbb{B}$ be a connected uniform ball structure, $x_0 \in X, Y \subseteq X$. Then Y is bounded if and only if there exists $\alpha \in P$ such that $Y \subseteq \mathbb{B}(x_0, \alpha)$. A box product of an arbitrary family of bounded ball structures is bounded. It is metrizable if and only if every $\mathbb{B}_\lambda, \lambda \in I$ is metrizable and all but finitely many of them are bounded.

We define also two modifications of box products. Let $\mathcal{F}$ be a family of all finite subsets of I. The first modification is

$$\overset{\vee}{\prod_{\lambda\in I}} \mathbb{B}_\lambda = (\prod_{\lambda\in I} X_\lambda, \Im \times \prod_{\lambda\in I} P_\lambda, \check{B}),$$

where

$$\check{B}(x, (F, p)) = \{y \in \prod_{\lambda\in I} X_\lambda : pr_\lambda(y) \in B_\lambda(pr_\lambda(x), pr_\lambda(p)) \quad for \ \ every \ \lambda \notin F\}.$$

The second modification is

$$\prod_{\lambda\in I}^{\wedge}\mathbb{B}_\lambda=(\prod_{\lambda\in I}X_\lambda,\Im\times\prod_{\lambda\in I}P_\lambda,\hat{B}),$$

where

$$\hat{B}(x,(F,p))=\{y\in\prod_{\lambda\in I}X_\lambda : pr_\lambda(y)\in B_\lambda(pr_\lambda(x),pr_\lambda(p)),$$
$$\lambda\in F\ \ and\ \ pr_\lambda(x)=pr_\lambda(y),\quad \lambda\notin F\}.$$

Clearly,

$$\prod_{\lambda\in I}^{\wedge}\mathbb{B}_\lambda\subseteq\prod_{\lambda\in I}\mathbb{B}_\lambda\subseteq\prod_{\lambda\in I}^{\vee}\mathbb{B}_\lambda.$$

§3 Approximations

A ball structure $\mathbb{B}$ is called *pseudometrizable* if $\mathbb{B}$ is a disjoint union of metrizable ball structures.

Theorem 3.1. *Every uniform ball structure $\mathbb{B}=(X,P,B)$ is an inductive limit of some family of pseudometrizable ball structures.*

Proof. We may suppose that $B(X,\alpha)=B^*(X,\alpha)$ for all $x\in X$, $\alpha\in P$. Denote by I the family of all subsets of P of the form $\{\alpha_n\in P : n\in\omega\}$ such that $\alpha_n\leq\alpha_{n+1}$, $n\in\omega$ and, for any $n,m\in\omega$, there exists $k(n,m)\in\omega$ such that

$$B(B(x,n),m)\subseteq B(x,k(n,m))$$

for every $x\in X$. For every $\lambda\in I$, $\lambda=\{\alpha_n : n\in\omega\}$, put $P_\lambda=\{\alpha_n : n\in\omega\}$. By metrizability criterion, every connected component of $\mathbb{B}_\lambda=(X,P_\lambda,B_\lambda)$, $B_\lambda(x,\alpha_n)=B(x,\alpha_n)$ is metrizable, so $\mathbb{B}_\lambda$ is pseudometrizable . It is easy to check that $\mathbb{B}$ is an inductive limit of the family $\{\mathbb{B}_\lambda : \lambda\in I\}$. $\square$

A ball structure $\mathbb{B}$ is called *inductively metrizable* if $\mathbb{B}$ is an inductive limit of metrizable ball structures.

Theorem 3.2. *For every uniform ball structure $\mathbb{B}=(X,P,B)$ the following statements are equivalent*

(i) $\mathbb{B}$ is inductively metrizable;

(ii) there exists a metric space (X,d) such that $\mathbb{B}(X,d)\subseteq\mathbb{B}$;

(iii) there exists a subset $P'\subseteq P,|P'|\leq\aleph_0$ and $x_0\in X$ such that $\bigcup_{\alpha\in P'}B(x_0,\alpha)=X$.

Proof. The implications $(i)\Longrightarrow(ii)\Longrightarrow(iii)$ are trivial.

$(iii) \Longrightarrow (i)$. We may suppose that $P' = \{\alpha_n : n \in \omega\}$, $\alpha_n \leq \alpha_{n+1}$, $n \in \omega$ and, for any $n, m \in \omega$ there exists $k(n,m) \in \omega$ such that

$$B(B(x,n),m) \subseteq B(x,k(n,m))$$

for every $x \in X$. Then $\mathbb{B}' = (X, P', B')$, $B'(x,\alpha) = B(x,\alpha)$, $\alpha \in P'$ is a metrizable ball structure. Consider the family $\Im$ of all metrizable ball structures on X such that $\mathbb{B}' \subseteq \mathbb{B}''$ for every $\mathbb{B}'' \in \Im$. Clearly, $\mathbb{B}$ is an inductive limit of $\Im$. $\Box$

By Theorem 3.2, a ball structure $\mathbb{B}(X,\varphi)$ of a filter φ on X, $|X| > 1$ is inductively metrizable if and only if there exists a countable subset ψ of φ such that $\bigcap \psi = \emptyset$. A ball structure $\mathbb{B}(G,\gamma)$ is inductively metrizable if and only if it is metrizable.

§4 Submetrizability

A uniform ball structure $\mathbb{B} = (X, P, B)$ is called *submetrizable* if there exists an unbounded metrizable ball structure $\mathbb{B}' = (X, P', B')$ such that $\mathbb{B} \subseteq \mathbb{B}'$.

Let $\mathbb{B} = (X, P, B)$ be a ball structure, $f : X \longrightarrow \mathbb{R}$. We say that f is a function of *bounded oscilation* (with respect to $\mathbb{B}$) if, for every $\alpha \in P$, there exists a natural number $n(\alpha)$ such that

$$diam \;\; f(B(x,\alpha)) \leq n(\alpha)$$

for every $x \in X$, where $diam \;\; A = sup\{|a - b| : a, b \in A\}$. Clearly, every bounded function is of bounded oscilation.

Theorem 4.1. *For every uniform ball structure $\mathbb{B} = (X, P, B)$ the following statements are equivalent*

(i) $\mathbb{B}$ is submetrizable;

(ii) there exists an unbounded function $f : X \longrightarrow \mathbb{R}$ of bounded oscilation.

Proof. $(i) \Longrightarrow (ii)$. Let (X,d) be an unbounded metric space such that $\mathbb{B} \subseteq \mathbb{B}(X,d)$. Fix an arbitrary point $x_0 \in X$ and put $f(x) = d(x, x_0)$. Since (X,d) is unbounded, f is unbounded. If $x, y \in X$, then

$$|f(x) - f(y)| = |d(x_0,x) - d(x_0,y)| \leq d(x,y).$$

Hence, f is of bounded oscilation on (X,d). Since $\mathbb{B} \subseteq \mathbb{B}(X,d)$, f is of bounded oscilation with respect to $\mathbb{B}$.

$(i) \Longrightarrow (ii)$. For all $x \in X$, $n \in \omega$, put

$$B'(x,n) = \{y \in X : |f(x) - f(y)| \leq n\}.$$

Clearly, the ball structure $\mathbb{B}' = (X, \omega, B')$ is symmetric, multiplicative, connected and $cf\mathbb{B}' = \aleph_0$. Hence, $\mathbb{B}$ is metrizable. To show that $\mathbb{B} \subseteq \mathbb{B}'$, fix an arbitrary $\alpha \in P$, choose $n(\alpha)$ such that diam $f(B(x,\alpha)) \leq n(\alpha)$ for every $x \in X$. Then $B(x,\alpha) \subseteq B'(x, n(\alpha))$ for every $x \in X$. $\square$

A connected uniform ball structure $\mathbb{B} = (X, P, B)$ is called *ordinal* if there exists a cofinal well ordered by $\leq$ subset of P. Clearly, every metrizable ball structure is ordinal.

Theorem 4.2. *For every ordinal ball structure $\mathbb{B} = (X, P, B)$, the following statements are equivalent*

(i) $\mathbb{B}$ is metrizable;

(ii) $\mathbb{B}$ is submetrizable;

Proof. The implication $(i) \Longrightarrow (ii)$ is trivial.

$(ii) \Longrightarrow (i)$. We may suppose that P is well ordered. Assume that $\mathbb{B}$ is not metrizable so cf $P > \aleph_0$. By Theorem 4.1, there exists an unbounded function $f : X \longrightarrow \mathbb{R}$ of bounded oscilation. Choose a countable subset $X' \subseteq X$ such that $f(X')$ is unbounded in $\mathbb{R}$. Since cf $P > \aleph_0$, there exists $X_0 \in X$, $\alpha \in P$ such that $X' \subseteq B(x_0, \alpha)$. We get a contradiction to the definition of function of bounded oscilation. $\square$

The next results give us examples of nonmetrizable submetrizable ball structures.

Theorem 4.3. *If a group G has a normal subgroup H of countable index, then $\mathbb{B}_l(G)$ is submetrizable.*

Proof. Let $\Im$ be a family of all finite subsets of G containing the identity e of G. Given any $g \in G$, $F \in \Im$ put

$$B'_l(g, FN) = FNg.$$

Denote by $\Im'$ the family of all subsets $Y \subseteq G$ of the form $Y = FN$, $F \in \Im$. Then $\mathbb{B}' = (G, \Im', B')$ is metrizable ball structure and $\mathbb{B} \subseteq \mathbb{B}'$. $\square$

Theorem 4.4. *Let φ be a filter on an infinite set X, $Y = \bigcap \varphi$. Then $\mathbb{B}(X, \varphi)$ is submetrizable if and only if one of the following statements holds*

(i) Y is infinite;

(ii) Y is finite and there exists a filter ψ on X with a countable base such that $\varphi \subseteq \psi$ and $Y \notin \psi$.

Proof. If (i) holds, choose a countable subset $\{y_n : n \in \omega\}$ of Y and put $f(y_n) = n$, $n \in \omega$ and $f(x) = 0$ for every $x \in X \setminus \{y_n : n \in \omega\}$. Then $f : X \longrightarrow \mathbb{R}$ is an unbounded function of bounded oscilation. Apply Theorem 4.1.

If (ii) holds, choose a base $\{F_n : n \in \omega\}$ of ψ such that $F_{n+1} \subset F_n$ for every $n \in \omega$. For every $x \in X$, put

$$f(x) = \begin{cases} n, & \text{if } x \in F_n \setminus F_{n+1}; \\ 0, & \text{otherwise.} \end{cases}$$

Since f is an unbounded function of bounded oscilation, we can apply Theorem 4.1.

Assume that Y is finite and $\mathbb{B}(X, \varphi)$ is submetrizable. By Theorem 4.1, there exists an unbounded function $f : X \longrightarrow \mathbb{R}$ of bounded oscilation. For every $n \in \omega$, put $X_n = \{x \in X : |f(x)| > n\}$. Let ψ be a filter on X with the base $X_n : n \in \omega$. Take an arbitrary $F \in \varphi$. Since f is of bounded oscilation, f is bounded on $X \setminus F$. If follows that $X_m \subseteq F$ for some $m \in \omega$, so $\varphi \subseteq \psi$. $\square$

Theorem 4.5. *Let a ball structure $\mathbb{B}$ is a disjoint union of the family $\{\mathbb{B}_\lambda = (X_\lambda, P, B_\lambda) : \lambda \in I\}$ of uniform ball structures. Then $\mathbb{B}$ is submetrizable if and only if either I is infinite of there exists $\lambda \in I$ such that $\mathbb{B}_\lambda$ is submetrizable.*

Proof. If I is infinite, choose a countable subset $\{\lambda_n : n \in \omega\}$ of I, put $f|X_{\lambda_n} \equiv n$, $n \in \omega$ and $f|X_\lambda \equiv 0$, $\lambda \notin \{\lambda_n : n \in \omega\}$. Apply Theorem 4.1.

If $\mathbb{B}_\lambda$ is submetrizable, we fix an unbounded function $f : X_\gamma \longrightarrow \mathbb{R}$ of bounded oscilation and put $f|X_\gamma \equiv 0$ for every $\gamma \neq \lambda$. Apply Theorem 4.1.

On the other hand, assume that I is finite and $\mathbb{B}_\lambda$ is not submetrizable for every $\lambda \in I$. By Theorem 4.1, $\mathbb{B}$ is not submetrizable. $\square$

§5 Extremalities

Fix a set X and denote by $\mathcal{B}(X)$ the class of all ball structures with the support X. Clearly, every bounded ball structure is a maximal by inclusion $\subseteq$ element of $\mathcal{B}(X)$, and any two bounded ball structures on X coincide. On the other hand, the discrete ball structure $(X, \{p\}, B)$, $B(x, p) = \{x\}$, $x \in X$ is a minimal by inclusion element of $\mathcal{B}(X)$. We can easily avoid these trivialities considering some natural subclasses of $\mathcal{B}(X)$.

An unbounded uniform ball structure $\mathbb{B}$ is called *prebounded* if $\mathbb{B} \subseteq \mathbb{B}'$, $\mathbb{B}'$ is unbounded and uniform, implies $\mathbb{B} \subseteq \mathbb{B}'$.

Theorem 5.1. *For every unbounded uniform ball structure $\mathbb{B} = (X, P, B)$, there exists a prebounded ball structure $\mathbb{B}'$ such that $\mathbb{B} \subseteq \mathbb{B}'$. Every prebounded ball structure is not submetrizable.*

Proof. The first statement follows directly from Zorn lemma and the construction of inductive limit. To prove the second statement, we take an unbounded metric space (X, d) and define a new metric d' on X

such that (X, d') is unbounded and $\mathbb{B}(X, d) \subset \mathbb{B}(X, d')$. Fix an arbitrary element $x_0 \in X$. For every $x \in X$, choose $n \in \omega$ such that

$$n^2 \le d(x) < (n+1)^2$$

and put $f(x) = n$. For every $x \in X$, $m \in \omega$, denote

$$B'(x, m) = \{y \in X : |f(x) - f(y)| \le m\}.$$

Consider the ball structure $B' = (X, \omega, B')$. By metrizability criterion, $\mathbb{B}'$ is metrizable. Clearly, $\mathbb{B} \subset \mathbb{B}'$. $\square$

Theorem 5.2. *Let X be an infinite set and let $\varphi = \{F \subseteq X : X \backslash F \ \ is \ \ finite\}$. Then $\mathbb{B}(X, \varphi) \subseteq \mathbb{B}$ for every connected uniform ball structure $\mathbb{B} = (X, P, B)$.*

Proof. Take an arbitrary $F \in \varphi$. Since $\mathbb{B}$ is connected and uniform, the finite subset $X \backslash F$ is bounded. Choose $\alpha \in P$ such that $X \backslash F \subseteq B(x, \alpha)$ for every $x \in X \backslash F$. Then *id* $B(x, F) \subseteq B(x, \alpha)$ for every $x \in X$. Hence, $\mathbb{B}(X, \varphi) \subseteq \mathbb{B}$. $\square$

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