Algebra and Discrete Mathematics
Volume **28** (2019). Number 2, pp. 278–290

RESEARCH ARTICLE

On NSP constants of a matrix and their linear preservers

Marcin Skrzyński

Communicated by V. V. Kirichenko

Abstract. We collect some basic properties of NSP constants of a matrix, discuss an additive behavior of the spark, and prove two theorems characterizing linear endomorphisms which preserve the NSP constants.

1. Preliminaries and introduction

Throughout the present note, $\mathbb{F}$ stands for a field and $\mathbb{N}$ for the set of non-negative integers. The letters m and n always denote positive integers. If W is a vector space over $\mathbb{F}$, $s \in \mathbb{N} \setminus \{0\}$ and $w_1, \ldots, w_s \in W$, then we define $\operatorname{Span}_{\mathbb{F}}(w_1, \ldots, w_s)$ to be the linear subspace of W spanned by the vectors $w_1, \ldots, w_s$. We always consider $\mathbb{F}^n$ with its usual structure of a vector space over $\mathbb{F}$. The elements of $\mathbb{F}^n$ will be understood as columns. We will denote the zero vector of the space $\mathbb{F}^n$ by $\mathbf{0}$.

We define $\mathcal{M}_{m\times n}(\mathbb{F})$ as the set of all $m \times n$ matrices over $\mathbb{F}$. The set $\mathcal{M}_{m\times n}(\mathbb{F})$ will also be considered with its usual structure of a vector space over $\mathbb{F}$. We denote the $m\times n$ zero matrix by $O_{m\times n}$. For any $s \in \mathbb{N}\setminus\{0\}$ and any $t_1, \ldots, t_s \in \mathbb{F}$ we define $\operatorname{Diag}(t_1, \ldots, t_s) \in \mathcal{M}_{s\times s}(\mathbb{F})$ to be the diagonal matrix whose diagonal elements coincide with $t_1, \ldots, t_s$. The transpose of a matrix A will be denoted by A^{T}. Moreover, if $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, then we define $\operatorname{Ker}(A) = \{x \in \mathbb{F}^n : Ax = \mathbf{0}\}$. Recall that an $n \times n$ binary matrix is said to be a permutation matrix, if it has exactly one 1 in each row

2010 MSC: Primary 15A86; Secondary 15A03, 15A60.
Key words and phrases: Null Space Property, spark of a matrix, linear preserver of a function.

and in each column. The mth full linear group over $\mathbb{F}$ will be denoted by $\mathcal{GL}_m(\mathbb{F})$, i.e.,

$$\mathcal{GL}_m(\mathbb{F}) = \{U \in \mathcal{M}_{m\times m}(\mathbb{F}) :\ \det(U) \neq 0\}.$$

In the note, we will use a bit of basic algebraic geometry. Suppose therefore that W is a nonzero finite dimensional vector space over $\mathbb{F}$ and define $d = \dim W$. Let $\mathbb{F}[X_1, \ldots, X_d]$ be the polynomial ring in d variables over the field $\mathbb{F}$. A set $E \subseteq W$ is called algebraic, if there exists a linear isomorphism $\Xi : W \to \mathbb{F}^d$, a positive integer s and polynomials $f_1, \ldots, f_s \in \mathbb{F}[X_1, \ldots, X_d]$ such that

$$E = \{w \in W :\ f_1(\Xi(w)) = \ldots = f_s(\Xi(w)) = 0\}.$$

It is well known that the collection of subsets of W whose compliments are algebraic forms a topology on W. This topology is referred to as the Zariski topology. Recall also that an algebraic set $E \subseteq W$ is said to be irreducible, if there are no algebraic sets $E_1, E_2 \subseteq W$ satisfying $E_1 \neq E$, $E_2 \neq E$ and $E_1 \cup E_2 = E$. The fundamental reference for algebraic geometry is [6].

The note focuses mainly on linear preservers of certain functions. Let W be a linear space and Z a nonempty set. A linear endomorphism $\Psi : W \to W$ is said to preserve

- a map $g : W \to Z$, if $g(\Psi(w)) = g(w)$ for all $w \in W$;
- a subset E of the space W, if $\Psi(E) \subseteq E$.

Classical theory of linear preservers, dating back to F. G. Frobenius and still attracting the interest of current researchers, deals with endomorphisms of matrix spaces. In this case the linear preservers often turn out to be (U, V)-standard maps.

Definition 1. *Let $U \in \mathcal{GL}_m(\mathbb{F})$ and $V \in \mathcal{GL}_n(\mathbb{F})$. A map $\Phi : \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ is called (U, V)-standard, if either $\Phi(A) = UAV$ for any $A \in \mathcal{M}_{m\times n}(\mathbb{F})$, or $m = n$ and $\Phi(A) = UA^{\mathrm{T}}V$ for any $A \in \mathcal{M}_{m\times n}(\mathbb{F})$.*

One of the main results of the note is based on the following characterization of rank k preservers due to L. B. Beasley and T. J. Laffey [1].

Theorem 1. *Let $\mathbb{F}$ be a field with at least four elements, let $\Phi : \mathcal{M}_{m\times n}(\mathbb{F}) \to \mathcal{M}_{m\times n}(\mathbb{F})$ be a linear automorphism, and let k be a positive integer satisfying $k \leqslant \min\{m, n\}$. Suppose that*

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{F}) :\ \mathrm{rank}(A) = k \implies \mathrm{rank}(\Phi(A)) = k$$

(in other words, Φ preserves the set $\{A \in \mathcal{M}_{m\times n}(\mathbb{F}) :\ \mathrm{rank}(A) = k\}$). Then there exist matrices $U \in \mathcal{GL}_m(\mathbb{F})$ and $V \in \mathcal{GL}_n(\mathbb{F})$ such that Φ is the (U, V)-standard map.

Further information on linear preservers can be found in [2], for example.

To formulate the definition of NSP constants we need some additional notation. We denote by $\mathbb{R}_+$ the set of all positive real numbers. We define $N = \{1, \ldots, n\}$. Let $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{F}^n$. For an arbitrary set $S \subseteq N$ we define $x_S = [y_1, \ldots, y_n]^{\mathrm{T}} \in \mathbb{F}^n$ by

$$y_j = \begin{cases} x_j, & \text{if } j \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, in the case when $\mathbb{F} = \mathbb{C}$, the field of complex numbers, $\|x\|_1 = |x_1| + \ldots + |x_n|$ will denote the ℓ^1 norm of the vector x.

Definition 2. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $k \in \mathbb{N}$. The NSP constant $\gamma_k(A)$, also referred to as the NSP constant of order k, is defined to be the infimum of the set of all $\gamma \in \mathbb{R}_+$ such that*

$$\forall\, S \subseteq N\, \forall\, x \in \operatorname{Ker}(A) : \#S \leqslant k \implies \|x_S\|_1 \leqslant \gamma \|x_{N\setminus S}\|_1,$$

where $\#S$ stands for the cardinality of S.

The notion of NSP (Null Space Property) constants was introduced in [3]. It is one of the basic notions in the mathematical theory of compressed sensing. Let us now turn to another basic concept of this theory.

Definition 3. *Denote the columns of a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{F})$ by $K_1, \ldots, K_n$. The spark of A is defined to be the infimum of the set of all $s \in \mathbb{N} \setminus \{0\}$ such that*

$$\exists\, j_1, \ldots, j_s \in N : \begin{cases} j_1 < \ldots < j_s, \\ K_{j_1}, \ldots, K_{j_s} \text{ are linearly dependent over } \mathbb{F}. \end{cases}$$

We list below a few quite obvious but useful properties of the spark of a matrix.

Proposition 1. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. Then the following hold true:*

(i) $\operatorname{spark}(UA) = \operatorname{spark}(A)$ *for any* $U \in \mathcal{GL}_m(\mathbb{F})$;
(ii) $\operatorname{spark}(AD) = \operatorname{spark}(A)$ *for any diagonal matrix* $D \in \mathcal{GL}_n(\mathbb{F})$;
(iii) $\operatorname{spark}(AP) = \operatorname{spark}(A)$ *for any* $n \times n$ *permutation matrix* P;
(iv) *either* $\operatorname{spark}(A) = +\infty$, *or* $\operatorname{spark}(A) \leqslant \operatorname{rank}(A) + 1$;
(v) $\operatorname{spark}(A) = +\infty$ *if and only if* $\operatorname{rank}(A) = n$;
(vi) $\operatorname{spark}(A) = 1$ *if and only if* A *has a zero column.*

Let us finally state a result proved in [7].

Theorem 2. *The following conditions are equivalent for a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and a non-negative integer k:*
(1) $\gamma_k(A) \neq +\infty$;
(2) $k < \mathrm{spark}(A)$.

We will make use of the above theorem in the next section. For more information about NSP constants, the spark of a matrix, and the mathematical theory of compressed sensing consult, e.g., [4] and [5].

Null Space Property and the NSP constants, as well as the spark of a matrix, are studied mainly from a computational point of view. However, numerous classical examples always encourage to ask the question about linear preservers of a given quantitative characteristic of matrices. In [8] some theorems on linear preservers of the spark of a matrix are proved. The main goal of the present note is to give analogous results for NSP constants. We will also discuss an additive property of the spark. The note is organized as follows. In Section 1 we collect certain more or less known elementary facts about NSP constants. Section 2 is devoted to the spark of a matrix. The characterizations of linear preservers of NSP constants are proved in Section 3.

2. Some basic properties of NSP constants

Let us begin with NSP constants of products of matrices. We will denote by S_n the symmetric group of the set N.

Proposition 2. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and $k \in \mathbb{N}$. Then*
(i) $\gamma_k(UA) = \gamma_k(A)$ *for any* $U \in \mathcal{GL}_m(\mathbb{C})$,
(ii) $\gamma_k(zA) = \gamma_k(A)$ *for any* $z \in \mathbb{C} \setminus \{0\}$,
(iii) $\gamma_k(AB) \geqslant \gamma_k(B)$ *for any* $B \in \mathcal{M}_{n\times n}(\mathbb{C})$,
(iv) $\gamma_k(AP) = \gamma_k(A)$ *for any $n \times n$ permutation matrix P.*

Proof. Property (ii) is an immediate consequence of (i). Property (i) follows in turn from the fact that $\mathrm{Ker}(UA) = \mathrm{Ker}(A)$. Property (iii) is a consequence of the inclusion $\mathrm{Ker}(B) \subseteq \mathrm{Ker}(AB)$. Let us proceed to (iv). Denote by E the totality of $\gamma \in \mathbb{R}_+$ such that

$$\forall\, S \subseteq N\, \forall\, x \in \mathrm{Ker}(A) : \#S \leqslant k \implies \|x_S\|_1 \leqslant \gamma \|x_{N\setminus S}\|_1$$

and by F the totality of $\gamma \in \mathbb{R}_+$ such that

$$\forall\, S \subseteq N\, \forall\, x \in \mathrm{Ker}(AP) : \#S \leqslant k \implies \|x_S\|_1 \leqslant \gamma \|x_{N\setminus S}\|_1.$$

Moreover, for any vector $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{C}^n$ and any permutation $\sigma \in \mathrm{S}_n$ define $x_\sigma = [x_{\sigma(1)}, \ldots, x_{\sigma(n)}]^{\mathrm{T}}$. It is easy to see that if $x \in \mathbb{C}^n$,

$\sigma \in \mathrm{S}_n$ and $S \subseteq N$, then the vectors $(x_\sigma)_S$ and $x_{\sigma(S)}$ differ only in the arrangement of coordinates. Notice next that

$$\exists\, \tau \in \mathrm{S}_n : \operatorname{Ker}(A) = \{x_\tau : x \in \operatorname{Ker}(AP)\}.$$

Finally, pick an arbitrary $\gamma \in F$, an arbitrary $x \in \operatorname{Ker}(AP)$, and an arbitrary $S \subseteq N$ satisfying $\#S \leqslant k$. Then $\#\tau(S) = \#S$ and $N \setminus \tau(S) = \tau(N \setminus S)$. Since

$$\|(x_\tau)_S\|_1 = \|x_{\tau(S)}\|_1 \leqslant \gamma \|x_{N\setminus\tau(S)}\|_1 = \gamma \|x_{\tau(N\setminus S)}\|_1 = \gamma \|(x_\tau)_{N\setminus S}\|_1,$$

we obtain $\gamma \in E$. Therefore, $F \subseteq E$. The converse inclusion can be shown analogously. Hence $\gamma_k(AP) = \inf F = \inf E = \gamma_k(A)$. $\square$

Observe that if $\gamma_k(A) \neq +\infty$, then

$$\forall\, S \subseteq N\, \forall\, x \in \operatorname{Ker}(A) : \#S \leqslant k \implies \|x_S\|_1 \leqslant \gamma_k(A)\|x_{N\setminus S}\|_1$$

and the set E defined in the above proof contains the interval $(\gamma_k(A), +\infty)$. Notice also that

$$\gamma_k(O_{m\times n}) = \begin{cases} 0, & \text{if } k = 0, \\ +\infty, & \text{if } k \in \mathbb{N} \setminus \{0\}. \end{cases}$$

We will now focus on the sequence of all NSP constants of a given matrix.

Proposition 3. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Then*

(i) $\gamma_0(A) = 0$,
(ii) *the sequence $\{\gamma_k(A)\}_{k=0}^{\infty}$ is weakly increasing,*
(iii) $\forall\, k \in \mathbb{N} : \gamma_k(A) = +\infty \iff k \geqslant \operatorname{spark}(A)$,
(iv) $\gamma_k(A) = +\infty$ *for each $k \in \mathbb{N} \setminus \{0\}$ (in other words, the sequence of all NSP constants of A coincides with that of the zero matrix) if and only if A has a zero column.*

Proof. Properties (i) and (ii) follow immediately from the definition of NSP constants. Property (iii) says exactly the same thing as Theorem 2. Finally, we can deduce from (iii) that $\gamma_k(A) = +\infty$ for each $k \in \mathbb{N} \setminus \{0\}$ if and only if $\operatorname{spark}(A) = 1$. But $\operatorname{spark}(A) = 1$ if and only if A has a zero column. The proof is complete. $\square$

Proposition 4. *The following conditions are equivalent for a matrix $A \in \mathcal{M}_{m\times n}(\mathbb{C})$:*

(1) $\gamma_k(A) \neq +\infty$ *for any $k \in \mathbb{N}$;*
(2) $\operatorname{rank}(A) = n$;
(3) $\{\gamma_k(A)\}_{k=0}^{\infty}$ *is the zero sequence;*
(4) *there exists a positive integer ℓ such that $\gamma_\ell(A) = 0$.*

Proof. We infer from property (iii) in the previous proposition that all NSP constants of the matrix A are finite if and only if $\operatorname{spark}(A) = +\infty$. On the other hand, $\operatorname{spark}(A) = +\infty$ if and only if $\operatorname{rank}(A) = n$. Hence the first two conditions are equivalent. Suppose now that $\operatorname{rank}(A) = n$. Then $\operatorname{Ker}(A) = \{\mathbf{0}\}$. Consequently, it follows from the definition of NSP constants that $\gamma_k(A) = 0$ for any $k \in \mathbb{N}$. We have thus proved the implication (2) $\Longrightarrow$ (3). The implication (3) $\Longrightarrow$ (4) is obvious. Finally, suppose that condition (4) is satisfied. Since $1 \leqslant \ell$, we get

$$\forall\, S \subseteq N\, \forall\, x \in \operatorname{Ker}(A) : \#S = 1 \implies \|x_S\|_1 \leqslant \gamma_\ell(A)\|x_{N\setminus S}\|_1$$

(see the observations below the proof of Proposition 2). Hence all coordinates of any vector belonging to $\operatorname{Ker}(A)$ are equal to 0. In other words, $\operatorname{Ker}(A) = \{\mathbf{0}\}$ or equivalently $\operatorname{rank}(A) = n$. This completes the proof. $\square$

Unlike with the spark, right multiplication by a nonsingular diagonal matrix can change NSP constants. We will now discuss a few remarks concerning this fact.

Proposition 5. *Let $t_1, \ldots, t_n \in \mathbb{C} \setminus \{0\}$. Define $D = \operatorname{Diag}(t_1, \ldots, t_n)$ and*

$$c = \frac{\max\{|t_1|, \ldots, |t_n|\}}{\min\{|t_1|, \ldots, |t_n|\}}.$$

Then for an arbitrary $A \in \mathcal{M}_{m\times n}(\mathbb{C})$ and an arbitrary $k \in \mathbb{N}$, the following hold true:

(i) *$c^{-1}\gamma_k(A) \leqslant \gamma_k(AD) \leqslant c\gamma_k(A)$;*

(ii) *if $|t_1| = \ldots = |t_n|$, then $\gamma_k(AD) = \gamma_k(A)$.*

Moreover, if $n - 1 \leqslant m$ and $|t_q| \neq |t_r|$ for some $q, r \in N$, then there exists a matrix $B \in \mathcal{M}_{m\times n}(\mathbb{C})$ with $\gamma_1(B) \neq \gamma_1(BD)$.

Proof. Notice first that

$$\operatorname{Ker}(AD) = \{[t_1^{-1}x_1, \ldots, t_n^{-1}x_n]^{\mathrm{T}} : [x_1, \ldots, x_n]^{\mathrm{T}} \in \operatorname{Ker}(A)\}.$$

Let $E \subseteq \mathbb{R}_+$ be defined as in the proof of Proposition 2. If $\gamma \in E$, $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \operatorname{Ker}(A)$ and $\tilde{x} = [t_1^{-1}x_1, \ldots, t_n^{-1}x_n]^{\mathrm{T}}$, then for any set $S \subseteq N$ satisfying $\#S \leqslant k$ we have

$$\begin{aligned} \min\{|t_1|, \ldots, |t_n|\}\|\tilde{x}_S\|_1 &\leqslant \sum_{j\in S} |t_j||t_j^{-1}x_j| = \|x_S\|_1 \leqslant \gamma\|x_{N\setminus S}\|_1 \\ &\leqslant \gamma \max\{|t_1|, \ldots, |t_n|\} \sum_{j\in N\setminus S} |t_j^{-1}x_j| = \gamma \max\{|t_1|, \ldots, |t_n|\}\|\tilde{x}_{N\setminus S}\|_1 \end{aligned}$$

and hence $\|\tilde{x}_S\|_1 \leqslant c\gamma \|\tilde{x}_{N\setminus S}\|_1$. It follows that

$$\forall\, \gamma \in E : \gamma_k(AD) \leqslant c\gamma.$$

Consequently, $\gamma_k(AD) \leqslant c \cdot \inf E = c\gamma_k(A)$.

Let us turn to the lower bound in property (i). Notice that $D^{-1} = \mathrm{Diag}(t_1^{-1}, \ldots, t_n^{-1})$. Applying the upper bound to ADD^{-1} therefore yields

$$\gamma_k(A) = \gamma_k(ADD^{-1}) \leqslant \frac{\max\{|t_1^{-1}|, \ldots, |t_n^{-1}|\}}{\min\{|t_1^{-1}|, \ldots, |t_n^{-1}|\}} \gamma_k(AD) = c\gamma_k(AD)$$

and we are done.

Property (ii) is an immediate consequence of (i).

Suppose finally that $|t_q| \neq |t_r|$ for some $q, r \in N$ and that $n - 1 \leqslant m$. Then $n \geqslant 2$. Consider the set

$$V = \{[x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{C}^n : x_q = x_r,\ x_j = 0 \text{ for all } j \in N \setminus \{q, r\}\}.$$

Since V is a one-dimensional linear subspace of $\mathbb{C}^n$ and $n - \dim V = n - 1 \leqslant m$, there exists a matrix $B \in \mathcal{M}_{m\times n}(\mathbb{C})$ such that $\mathrm{Ker}(B) = V$. An easy direct calculation shows that $\gamma_1(B) = 1$. On the other hand,

$$\begin{aligned} \mathrm{Ker}(BD) &= \{[t_1^{-1}x_1, \ldots, t_n^{-1}x_n]^{\mathrm{T}} : [x_1, \ldots, x_n]^{\mathrm{T}} \in V\} \\ &= \{[y_1, \ldots, y_n]^{\mathrm{T}} \in \mathbb{C}^n : y_r = t_q t_r^{-1} y_q,\ y_j = 0 \text{ for all } j \in N \setminus \{q, r\}\} \end{aligned}$$

and hence $\gamma_1(BD) = \max\{|t_q t_r^{-1}|, |t_q^{-1} t_r|\} \neq 1$. The "moreover" part follows. □

We conclude with an example related to the above proposition.

Example 1. Let us focus for a while on the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{C}).$$

Observe that $\mathrm{spark}(A) = 3$ and $\mathrm{Ker}(A) = \mathrm{Span}_{\mathbb{C}}([1, 1, 1]^{\mathrm{T}})$. Consequently,

$$\gamma_k(A) = \begin{cases} 0, & \text{if } k = 0, \\ 1/2, & \text{if } k = 1, \\ 2, & \text{if } k = 2, \\ +\infty, & \text{if } k \geqslant 3. \end{cases}$$

For an arbitrary $\varepsilon \in [2, 3]$ define now $D_\varepsilon = \mathrm{Diag}(2, \varepsilon, 3)$. Since $\mathrm{Ker}(AD_\varepsilon) = \mathrm{Span}_{\mathbb{C}}([2^{-1}, \varepsilon^{-1}, 3^{-1}]^{\mathrm{T}})$, we get

$$\gamma_1(AD_\varepsilon) = \max\left\{ \frac{3\varepsilon}{2(\varepsilon+3)}, \frac{6}{5\varepsilon}, \frac{2\varepsilon}{3(\varepsilon+2)} \right\} = \frac{3\varepsilon}{2(\varepsilon+3)}.$$

Therefore $\{\gamma_1(AD_\varepsilon) : \varepsilon \in [2,3]\} = \left[\frac{3}{5}, \frac{3}{4}\right]$. Notice that the quotient c defined in Proposition 5 is equal to $3/2$ for all matrices D_ε.

3. An additive property of the spark of a matrix

We start with a simple observation based on a little bit of algebraic geometry.

Lemma 1. *Let V be a nonzero finite dimensional vector space over an infinite field and let $p \in \mathbb{N} \setminus \{0\}$. Consider proper linear subspaces $V_1, \dots, V_p \subset V$ and a finite (possibly empty) family $\mathcal{W}$ of proper linear subspaces of V. Let $x \in V_1 \cap \ldots \cap V_p$. Suppose that $x \notin \bigcup \mathcal{W}$. Then there is a vector $y \in V$ such that $x + y \notin V_1 \cup \ldots \cup V_p \cup \bigcup \mathcal{W}$.*

Proof. The subspaces $V_1, \dots, V_p$ are proper algebraic subsets of V. Since the underlying field is infinite, the space V is irreducible and hence $V_1 \cup \ldots \cup V_p \neq V$. Consequently, $Y := V \setminus (V_1 \cup \ldots \cup V_p)$ is a nonempty Zariski open subset of V. The translation $\Theta : V \ni v \longmapsto x + v \in V$ is a homeomorphism with respect to the Zariski topology on V. But $\bigcup \mathcal{W}$ is a proper algebraic subset of V. This yields that $\bigcup \mathcal{W}$ is nowhere dense in V equipped with the Zariski topology. We therefore get that $\Theta(Y)$ is not contained in $\bigcup \mathcal{W}$. Pick a vector $y \in Y$ such that $\Theta(y) = x + y \notin \bigcup \mathcal{W}$. Since $y \notin V_1 \cup \ldots \cup V_p$ and $x \in V_1 \cap \ldots \cap V_p$, we have $x + y \notin V_1 \cup \ldots \cup V_p$. The proof is complete. □

We will use the above lemma to prove the key result of the section.

Theorem 3. *Let $\mathbb{F}$ be an infinite field and let $A \in \mathcal{M}_{m\times n}(\mathbb{F})$. Suppose that $2 \leqslant \operatorname{spark}(A) \leqslant m$. Then there exists a matrix $B \in \mathcal{M}_{m\times n}(\mathbb{F})$ such that $\operatorname{spark}(A + B) > \operatorname{spark}(A)$ and all elements of the first column of B are equal to 0.*

Proof. Denote the columns of A by $K_1, \dots, K_n$. Observe that $s := \operatorname{spark}(A) \leqslant n$. We will now find $L_1, \dots, L_n \in \mathbb{F}^m$ such that $L_1 = \mathbf{0}$ and any s vectors among $K_1 + L_1, \dots, K_n + L_n$ are linearly independent over $\mathbb{F}$.

Let $L_1 = \ldots = L_{s-1} = \mathbf{0}$. Notice that $\operatorname{Span}_{\mathbb{F}}(K_1, \dots, K_{s-1})$ is a proper linear subspace of $\mathbb{F}^m$. Pick some vector $v \in \mathbb{F}^m \setminus \operatorname{Span}_{\mathbb{F}}(K_1, \dots, K_{s-1})$ and let

$$L_s = \begin{cases} \mathbf{0}, & \text{if } K_s \notin \operatorname{Span}_{\mathbb{F}}(K_1, \dots, K_{s-1}), \\ v, & \text{otherwise.} \end{cases}$$

The definition of the spark of a matrix yields that $K_1, \dots, K_{s-1}$ are linearly independent over $\mathbb{F}$. It is clear that $K_s + L_s \notin \operatorname{Span}_{\mathbb{F}}(K_1, \dots, K_{s-1})$. Hence the vectors $K_1 + L_1, \ldots, K_s + L_s$ are also linearly independent over $\mathbb{F}$. If $s = n$, then we are done.

Suppose that $s < n$ and proceed by recursion on $j \in \{s+1, \ldots, n\}$. Let $L_1, \ldots, L_{j-1}$ be already defined. If K_j is not a linear combination, over $\mathbb{F}$, of any $s-1$ vectors among $K_1 + L_1, \ldots, K_{j-1} + L_{j-1}$, then define $L_j = \mathbf{0}$. Otherwise, observe that the families

- $\{V_1, \ldots, V_p\}$ of all linear subspaces of $\mathbb{F}^m$ which are spanned by $s-1$ vectors among $K_1 + L_1, \ldots, K_{j-1} + L_{j-1}$ and contain K_j,
- $\mathcal{W}$ of all linear subspaces of $\mathbb{F}^m$ which are spanned by $s-1$ vectors among $K_1 + L_1, \ldots, K_{j-1} + L_{j-1}$ and do not contain K_j

together with $x = K_j$ satisfy the assumptions of Lemma 1, and define L_j to be the vector y yielded by this lemma. Recall now that any s (and hence any $s-1$) vectors among $K_1 + L_1, \ldots, K_{j-1} + L_{j-1}$ are linearly independent over $\mathbb{F}$. Next, Lemma 1 implies that $K_j + L_j$ is not a linear combination, over $\mathbb{F}$, of any $s-1$ vectors among $K_1 + L_1, \ldots, K_{j-1} + L_{j-1}$. It follows that any s vectors among $K_1 + L_1, \ldots, K_j + L_j$ are linearly independent over $\mathbb{F}$. The recursion is done.

Define therefore $B \in \mathcal{M}_{m\times n}(\mathbb{F})$ as the matrix whose columns coincide with $L_1, \ldots, L_n$. Then

$$\operatorname{spark}(A+B) \geqslant s+1 > \operatorname{spark}(A)$$

and, obviously, all elements of the first column of B are equal to 0. The proof is complete. $\square$

Corollary 1. *Let $\mathbb{F}$ be an infinite field and let $A \in \mathcal{M}_{m\times n}(\mathbb{F}) \setminus \{O_{m\times n}\}$. Suppose that* $\operatorname{spark}(A) \leqslant m$. *Then*

$$\exists\, \tilde{B} \in \mathcal{M}_{m\times n}(\mathbb{F}) : \begin{cases} \operatorname{spark}(\tilde{B}) = 1, \\ \operatorname{spark}(A+\tilde{B}) > m. \end{cases}$$

Proof. First, suppose additionally that $\operatorname{spark}(A) \geqslant 2$. Theorem 3 then yields the existence of a positive integer ℓ and matrices $B_1, \ldots, B_\ell \in \mathcal{M}_{m\times n}(\mathbb{F})$ with the following properties: $\operatorname{spark}(A + B_1 + \ldots + B_\ell) > m$, the first column of any of $B_1, \ldots, B_\ell$ is the zero column. Notice that $\operatorname{spark}(B_1 + \ldots + B_\ell) = 1$.

Now assume that $\operatorname{spark}(A) = 1$. In other words, A has a zero column. Let us denote the columns of A by $K_1, \ldots, K_n$. Since $A \neq O_{m\times n}$, at least one of them, say K_r, is not the zero column. Define $C_1 \in \mathcal{M}_{m\times n}(\mathbb{F})$ to be the matrix whose jth column, for an arbitrary $j \in N$, coincides with

$$\begin{cases} \mathbf{0}, & \text{if } K_j \neq \mathbf{0}, \\ K_r, & \text{otherwise.} \end{cases}$$

It is evident that $\mathrm{spark}(C_1) = 1$ and $\mathrm{spark}(A + C_1) = 2$. If $m = 1$, then we are done. Hence let us continue under the additional assumption that $m \geqslant 2$. Pick an $n \times n$ permutation matrix P such that the first column of the product C_1P is the zero column. We have $\mathrm{spark}((A + C_1)P) = \mathrm{spark}(A + C_1) = 2$. In view of the first part of the proof, there exists a matrix $C_2 \in \mathcal{M}_{m\times n}(\mathbb{F})$ such that $\mathrm{spark}((A + C_1)P + C_2) > m$ and the first column of C_2 is the zero column. Define finally $C_3 = (C_1P + C_2)P^{-1}$. Since the first column of $C_1P + C_2$ is the zero column, we get $\mathrm{spark}(C_3) = \mathrm{spark}(C_1P + C_2) = 1$. Moreover,

$$\begin{aligned}\mathrm{spark}(A + C_3) &= \mathrm{spark}(APP^{-1} + (C_1P + C_2)P^{-1}) \\ &= \mathrm{spark}(((A + C_1)P + C_2)P^{-1}) = \mathrm{spark}((A + C_1)P + C_2) > m.\end{aligned}$$

Thus we complete the proof by letting

$$\tilde{B} = \begin{cases} B_1 + \ldots + B_\ell & \text{if } \mathrm{spark}(A) \geqslant 2, \\ C_1 & \text{if } \mathrm{spark}(A) = 1 = m, \\ C_3 & \text{if } \mathrm{spark}(A) = 1 \text{ and } m \geqslant 2. \end{cases} \quad \square$$

A minor comment seems noteworthy. Recall that if $\mathbb{F}$ is a finite field and $n \geqslant (\#\mathbb{F})^m$, then by the pigeonhole principle $\max\{\mathrm{spark}(A) : A \in \mathcal{M}_{m\times n}(\mathbb{F})\} = 2$ (see [9, Example 2]). Hence the infiniteness assumption cannot be removed from Theorem 3 and Corollary 1.

4. Two theorems on preserving NSP constants

We are already in a position to state and prove our first main result.

Theorem 4. *Let $\Psi : \mathcal{M}_{m\times n}(\mathbb{C}) \to \mathcal{M}_{m\times n}(\mathbb{C})$ be a linear endomorphism. Suppose that*

$$\exists\, \ell \in \{1, \ldots, m\}\, \forall\, A \in \mathcal{M}_{m\times n}(\mathbb{C}) : \gamma_\ell(\Psi(A)) = \gamma_\ell(A)$$

(in other words, Ψ preserves the NSP constant of a certain order $\ell \in \{1, \ldots, m\}$). Then Ψ is bijective.

Proof. Assume, to the contrary, that Ψ is not bijective. Hence there exists a matrix $A_0 \in \mathcal{M}_{m\times n}(\mathbb{C}) \setminus \{O_{m\times n}\}$ such that $\Psi(A_0) = O_{m\times n}$. Since

$$\gamma_\ell(A_0) = \gamma_\ell(\Psi(A_0)) = \gamma_\ell(O_{m\times n}) = +\infty,$$

Proposition 3 (iii) yields $m \geqslant \ell \geqslant \mathrm{spark}(A_0)$. It now follows from Corollary 1 that

$$\exists\, \tilde{B} \in \mathcal{M}_{m\times n}(\mathbb{C}) : \begin{cases} \mathrm{spark}(\tilde{B}) = 1, \\ \mathrm{spark}(A_0 + \tilde{B}) > m \geqslant \ell. \end{cases}$$

Therefore we have

$$+\infty = \gamma_\ell(\tilde{B}) = \gamma_\ell(\Psi(\tilde{B})) = \gamma_\ell(\Psi(A_0 + \tilde{B})) = \gamma_\ell(A_0 + \tilde{B}) \neq +\infty,$$

a contradiction. □

Notice that the above proof is analogous to that of [8, Proposition 2.2].

We turn to a comment on the assumptions of Theorem 4. Suppose that $m < n$. Let $A \in \mathcal{M}_{m\times n}(\mathbb{C})$. Then $\operatorname{spark}(A) \leqslant m+1$. Combining this fact with Proposition 3 (iii) implies that $\gamma_k(A) = +\infty$ for any integer $k \geqslant m+1$. Recall also that $\gamma_0(A) = 0$. Hence if $k \in \mathbb{N} \setminus \{1, \ldots, m\}$, then every linear endomorphism (not necessarily bijective) of $\mathcal{M}_{m\times n}(\mathbb{C})$ preserves the NSP constant of order k.

Consider now a matrix $U \in \mathcal{GL}_m(\mathbb{C})$, an $n \times n$ permutation matrix P and nonzero complex numbers $t_1, \ldots, t_n$ such that $|t_1| = \ldots = |t_n|$. Define $D = \operatorname{Diag}(t_1, \ldots, t_n)$. It follows from Proposition 2 and Proposition 5 (ii) that the linear automorphism

$$\Phi : \mathcal{M}_{m\times n}(\mathbb{C}) \ni A \longmapsto UAPD \in \mathcal{M}_{m\times n}(\mathbb{C})$$

preserves all NSP constants. In other words,

$$\forall\, k \in \mathbb{N}\, \forall\, A \in \mathcal{M}_{m\times n}(\mathbb{C}) : \gamma_k(\Phi(A)) = \gamma_k(A).$$

Our second main result is that under some additional assumption the converse of the above statement holds true.

Theorem 5. *Suppose that $m \geqslant n$. Let $\Phi : \mathcal{M}_{m\times n}(\mathbb{C}) \to \mathcal{M}_{m\times n}(\mathbb{C})$ be a linear endomorphism. Then the following conditions are equivalent:*

(1) *Φ preserves all NSP constants;*
(2) *Φ preserves the NSP constant of order* 1*;*
(3) *there exist nonzero complex numbers $t_1, \ldots, t_n$ satisfying $|t_1| = \ldots = |t_n|$, an $n \times n$ permutation matrix P and a matrix $U \in \mathcal{GL}_m(\mathbb{C})$ such that $\Phi(A) = UAPD$ for any $A \in \mathcal{M}_{m\times n}(\mathbb{C})$, where $D = \operatorname{Diag}(t_1, \ldots, t_n)$.*

Proof. The implications (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (1) are obvious. Let us therefore assume that Φ preserves the NSP constant of order 1. We need to show that condition (3) is satisfied.

Propositions 3 and 4 yield

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{C}) : \begin{cases} \operatorname{rank}(A) = n & \Longleftrightarrow \quad \gamma_1(A) = 0, \\ \operatorname{spark}(A) = 1 & \Longleftrightarrow \quad \gamma_1(A) = +\infty. \end{cases}$$

Hence we obtain two important properties:

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{C}) :\ \mathrm{rank}(A) = n \implies \mathrm{rank}(\Phi(A)) = n, \qquad (*)$$

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{C}) :\ \mathrm{spark}(A) = 1 \implies \mathrm{spark}(\Phi(A)) = 1. \qquad (**)$$

It follows from Theorem 4 that Φ is bijective. Combining property $(*)$ with Theorem 1 therefore implies that Φ is a (U,V)-standard map for some $U \in \mathcal{GL}_m(\mathbb{C})$ and some $V \in \mathcal{GL}_n(\mathbb{C})$.

If $m = n = 1$, then obviously we are done. Suppose for a moment that $m = n \geqslant 2$ and

$$\forall\, A \in \mathcal{M}_{m\times n}(\mathbb{C}) :\ \Phi(A) = UA^{\mathrm{T}}V.$$

A quick inspection of the proof of [8, Lemma 2.6] shows that there is a matrix $B_1 \in \mathcal{M}_{n\times n}(\mathbb{C})$ with $\mathrm{spark}(B_1) = 1$ and $\mathrm{spark}(B_1^{\mathrm{T}}V) = 2$. Since

$$\mathrm{spark}(\Phi(B_1)) = \mathrm{spark}(UB_1^{\mathrm{T}}V) = \mathrm{spark}(B_1^{\mathrm{T}}V),$$

we get a contradiction to property $(**)$. We have thus proved that $\Phi(A) = UAV$ for all $A \in \mathcal{M}_{m\times n}(\mathbb{C})$, even if $m = n \geqslant 2$.

Suppose now that V^{-1} does not coincide with any product of the form $\tilde{D}\tilde{P}$, where $\tilde{D} \in \mathcal{GL}_n(\mathbb{C})$ is a diagonal matrix and $\tilde{P}$ is an $n\times n$ permutation matrix. A simple inspection of the proof of [8, Lemma 2.3] then yields the existence of a matrix $B_2 \in \mathcal{M}_{m\times n}(\mathbb{C})$ such that $\mathrm{spark}(B_2) = 2$ and $\mathrm{spark}(B_2V^{-1}) = 1$. But

$$\mathrm{spark}(\Phi(B_2V^{-1})) = \mathrm{spark}(UB_2V^{-1}V) = \mathrm{spark}(B_2),$$

and we again arrive at a contradiction. Thus $V = PD$ for some $n \times n$ permutation matrix P and some diagonal matrix $D \in \mathcal{GL}_n(\mathbb{C})$.

Finally, let $t_1, \ldots, t_n \in \mathbb{C} \setminus \{0\}$ be such that $D = \mathrm{Diag}(t_1, \ldots, t_n)$. Suppose that

$$\exists\, q, r \in N :\ |t_q| \neq |t_r|.$$

Then by Proposition 5 there exists a matrix $B_3 \in \mathcal{M}_{m\times n}(\mathbb{C})$ with $\gamma_1(B_3) \neq \gamma_1(B_3D)$. Since

$$\begin{cases} \gamma_1(B_3P^{-1}) = \gamma_1(B_3), \\ \gamma_1(\Phi(B_3P^{-1})) = \gamma_1(UB_3P^{-1}PD) = \gamma_1(B_3D), \end{cases}$$

we have a contradiction to the assumption that Φ preserves the NSP constant of order 1. Hence $|t_1| = \ldots = |t_n|$. Condition (3) follows. $\square$

For the sake of completeness, let us notice that the proof of Theorem 5 is analogous to that of [8, Theorem 2.1].

References

[1] L. B. Beasley, T. J. Laffey, *Linear operators on matrices: the invariance of rank-k matrices*, Linear Algebra Appl., **133**, 1990, pp. 175–184.

[2] L. B. Beasley, C.-K. Li, M.-H. Lim, R. Loewy, B. R. McDonald, S. Pierce (eds.), *A survey of linear preserver problems*, Linear Multilinear Algebra, **33**, No. 1-2, 1992.

[3] A. Cohen, W. Dahmen, R. DeVore, *Compressed sensing and best k-term approximation*, J. Am. Math. Soc., **22**, No. 1, 2009, pp. 211–231.

[4] D. L. Donoho, M. Elad, *Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ^1 minimization*, Proc. Natl. Acad. Sci. USA, **100**, No. 5, 2003, pp. 2197–2202.

[5] M. Fornasier, H. Rauhut, *Compressive Sensing*, in: O. Scherzer (ed.), *Handbook of Mathematical Methods in Imaging*, Springer, 2011, pp. 187–228.

[6] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, 1977.

[7] M. Skrzyński, *Remarks on the spark of a matrix and the Null Space Property*, Publ. Inst. Math., Nouv. Sér., accepted.

[8] M. Skrzyński, *A note on preserving the spark of a matrix*, Ann. Univ. Paedagog. Crac., Stud. Math., **14**, 2015, pp. 63–67.

[9] M. Skrzyński, *Remarks on algebraic and geometric properties of the spark of a matrix*, in: A. Gomolińska et al. (eds.), *Recent Results in Pure and Applied Mathematics Podlasie 2014*, Bialystok University of Technology Publishing Office, 2014, pp. 43–48.

Contact information

M. Skrzyński

Institute of Mathematics, Cracow University of Technology, ul. Warszawska 24, 31-155 Kraków, Poland

E-Mail(s): pfskrzyn@cyf-kr.edu.pl

Received by the editors: 09.05.2018.