Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **25** (2018). Number 1, pp. 98–117

Closure operators in modules and adjoint functors, I

A. I. Kashu

Abstract. In the present work the relations between the closure operators of two module categories are investigated in the case when the given categories are connected by two covariant *adjoint functors* $H\colon R\text{-Mod} \longrightarrow S\text{-Mod}$ and $T\colon S\text{-Mod} \longrightarrow R\text{-Mod}$. Two mappings are defined which ensure the transition between the closure operators of categories R-Mod and S-Mod. Some important properties of these mappings are proved. It is shown that the studied mappings are compatible with the order relations and with the main operations.

1. Introduction. Preliminary notions and facts

The aim of this paper is to clarify connections between the closure operators of two module categories in the adjoint situation. For that we fix an arbitrary (R, S)-bimodule ${}_RU_S$ and consider the following two covariant functors:

$$R\text{-Mod} \quad \underset{T=U\otimes_S\text{-}}{\overset{H=\mathrm{Hom}_R(U,\text{-})}{\rightleftarrows}} \quad S\text{-Mod},$$

where T is left adjoint to H. We remark that any pair of covariant adjoint functors between two module categories has such a form (up to a functorial isomorphism). This adjoint situation is characterized by two natural transformations (functorial morphisms):

$$\Phi\colon TH \to \mathbb{1}_{R-\mathrm{Mod}}, \qquad \Psi\colon \mathbb{1}_{S-\mathrm{Mod}} \to HT,$$

2010 MSC: 16D90, 16S90.

Key words and phrases: category of modules, closure operator, adjoint functors, lattice operations.

which satisfy the conditions:

$$H(\Phi_X)\cdot\Psi_{H(X)} = \mathbb{1}_{H(X)}, \tag{1.1}$$

$$\Phi_{T(Y)}\cdot T(\Psi_Y) = \mathbb{1}_{T(Y)}, \tag{1.2}$$

for every modules $X \in R$-Mod and $Y \in S$-Mod.

This situation was studied in a series of works [1–6], where the relations between *preradicals* of categories R-Mod and S-Mod are shown. The ideas and methods used in these works can partially be adopted for the investigation of connections between *closure operators* of the given categories. This question is studied by other methods in the book [9] (§ 5.13).

Now we recall some notions and facts which are necessary for the following account. A *closure operator* of R-Mod is a mapping C which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$, a submodule of M denoted by $C_M(N)$ which satisfies the conditions:

(c_1) $N \subseteq C_M(N)$ (*extension*);

(c_2) If $N_1, N_2 \in \mathbb{L}(M)$ and $N_1 \subseteq N_2$, then $C_M(N_1) \subseteq C_M(N_2)$ (*monotony*);

(c_3) For every R-morphism $f\colon M \to M'$ and $N \in \mathbb{L}(M)$ we have $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$ (*continuity*),

where $M \in R$-Mod and $\mathbb{L}(M)$ is the lattice of submodules of M ([7–13]).

We denote by $\mathbb{CO}(R)$ the class of all closure operators of R-Mod. In the class $\mathbb{CO}(R)$ the relation of *partial order* is defined as follows:

$$C \leqslant D \Leftrightarrow C_M(N) \subseteq D_M(N) \ \text{ for every } \ N \subseteq M.$$

Moreover, in $\mathbb{CO}(R)$ the operations "$\vee$" (join) and "$\wedge$" (meet) are defined by the following rules:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M(N) = \sum_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)], \tag{1.3}$$

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M(N) = \bigcap_{\alpha\in\mathfrak{A}}[(C_\alpha)_M(N)], \tag{1.4}$$

for every family $\{C_\alpha \in \mathbb{CO}(R) \mid \alpha \in \mathfrak{A}\}$ and every $N \subseteq M$. The class $\mathbb{CO}(R)$ relative to these operations is a complete "big" lattice. In particular, $\mathbb{CO}(R)$ possesses the greatest element $\mathbb{1}_R$, where $(\mathbb{1}_R)_M(N) = M$, as well as the least element $\mathbb{0}_R$, where $(\mathbb{0}_R)_M(N) = N$ for every $N \subseteq M$.

2. Mappings of closure operators in adjoint situation

Throughout of this paper we consider a pair of covariant adjoint functors $H = \mathrm{Hom}_R(U, \text{-})$ and $T = U \otimes_R \text{-}$, determined by the bimodule ${}_RU_S$

(see Section 1). Now we will define two mappings which operate between the classes of closure operators $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$ of the categories R-Mod and S-Mod. We essentially use some peculiarities of studied situation, in particular, the natural transformations Φ and Ψ with the conditions (1.1) and (1.2).

I. Mapping $C \mapsto C^*$ from $\mathbb{CO}(R)$ to $\mathbb{CO}(S)$

Let $C \in \mathbb{CO}(R)$, $Y \in S$-Mod and $n\colon N \xrightarrow{\subseteq} Y$ be an arbitrary inclusion of S-Mod. We will construct a new function C^* in S-Mod as follows. Applying T we obtain the morphism $T(n) : T(N) \to T(Y)$ of R-Mod. Using the operator C, we have the following decomposition of $T(n)$:

$$\begin{tikzcd} T(N) \arrow[r, "\overline{T(n)}"] \arrow[rrr, bend right, "T(n)"'] & \operatorname{Im} T(n) \arrow[r, "\subseteq"] & C_{T(Y)}\big(\operatorname{Im} T(n)\big) \arrow[r, "\subseteq"] & T(Y), \end{tikzcd}$$

where $\overline{T(n)}$ is the restriction of $T(n)$ to its image. We consider the natural R-morphism $\pi^n_C\colon T(Y) \to T(Y)/C_{T(Y)}\big(\operatorname{Im} T(n)\big)$. Applying H and using Ψ_Y, we obtain the composition of morphisms:

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^n_C)} H[T(Y)/C_{T(Y)}\big(\operatorname{Im} T(n)\big)].$$

Definition 1. *For every operator $C \in \mathbb{CO}(R)$ and every inclusion $n\colon N \xrightarrow{\subseteq} Y$ of S-Mod, we define the function C^* by the rule:*

$$C^*_Y(N) = \operatorname{Ker}[H(\pi^n_C) \cdot \Psi_Y]. \tag{2.1}$$

Proposition 2.1. *The function C^* defined by (2.1) is a closure operator of the category S-Mod.*

Proof. We will verify, for the function C^*, the conditions (c_1)–(c_3) of the definition of closure operator (Section 1).

(c_1) By Definition 1 $\operatorname{Im} T(n) \subseteq C_{T(Y)}\big(\operatorname{Im} T(n)\big) = \operatorname{Ker} \pi^n_C$, so $\pi^n_C \cdot T(n) = 0$, therefore $H(\pi^n_C) \cdot HT(n) = 0$. By the naturality of Ψ we have $\Psi_Y \cdot n = HT(n) \cdot \Psi_N$, therefore

$$[H(\pi^n_C) \cdot \Psi_Y \cdot n](N) = [H(\pi^n_C) \cdot HT(n) \cdot \Psi_N](N) = 0.$$

This means that $N \subseteq \operatorname{Ker}[H(\pi^n_C) \cdot \Psi_Y] = C^*_Y(N)$, so (c_1) is true.

(c_2) Let $N_1, N_2 \in \mathbb{L}(Y)$ and $N_1 \subseteq N_2$. We denote the existing inclusions as follows: $i\colon N_1 \xrightarrow{\subseteq} N_2$, $n_1\colon N_1 \xrightarrow{\subseteq} Y$, $n_2\colon N_2 \xrightarrow{\subseteq} Y$, so $n_1 = n_2 \cdot i$ and therefore $T(n_1) = T(n_2) \cdot T(i)$. Then $\operatorname{Im} T(n_1) \subseteq \operatorname{Im} T(n_2)$

and $C_{T(Y)}\big(\operatorname{Im} T(n_1)\big) \subseteq C_{T(Y)}\big(\operatorname{Im} T(n_2)\big)$. This relation implies the morphism $\pi\colon T(Y)/C_{T(Y)}\big(\operatorname{Im} T(n)\big) \to T(Y)/C_{T(Y)}\big(\operatorname{Im} T(n_2)\big)$, which defines the morphism $H(\pi)$ of the following diagram in S-Mod:

$$
\begin{tikzcd}
 & & H[T(Y)/C_{T(Y)}(\operatorname{Im} T(n_1))] \arrow[dd, dashed, "H(\pi)"] \\
Y \arrow[r, "\Psi_Y"] & HT(Y) \arrow[ur, "H(\pi_C^{n_1})"] \arrow[dr, "H(\pi_C^{n_2})"'] & \\
 & & H[T(Y)/C_{T(Y)}(\operatorname{Im} T(n_2))].
\end{tikzcd}
$$

Therefore $\operatorname{Ker}[H(\pi_C^{n_1}) \cdot \Psi_Y] \subseteq \operatorname{Ker}[H(\pi_C^{n_2}) \cdot \Psi_Y]$, which means that $C_Y^*(N_1) \subseteq C_Y^*(N_2)$, so (c_2) is true.

(c_3) Let $f\colon Y \to Y'$ be an arbitrary S-morphism and $n\colon N \xrightarrow{\subseteq} Y$ be an inclusion. We denote $n'\colon f(N) \xrightarrow{\subseteq} Y'$. Then the R-morphism $T(f)\colon T(Y) \to T(Y')$ implies the morphism $(T(f))'\colon \operatorname{Im} T(n) \to \operatorname{Im} T(n')$, as well as the morphism $(T(f))''\colon C_{T(Y)}\big(\operatorname{Im} T(n)\big) \to C_{T(Y')}\big(\operatorname{Im} T(n')\big)$, by which we obtain the morphism $\pi\colon T(Y)/C_{T(Y)}\big(\operatorname{Im} T(n)\big) \to T(Y')/C_{T(Y')}\big(\operatorname{Im} T(n')\big)$. Then we have in S-Mod the diagram:

$$
\begin{tikzcd}
Y \arrow[r, "\Psi_Y"] \arrow[d, "f"] & HT(Y) \arrow[r, "H(\pi_C^n)"] \arrow[d, "HT(f)"] & H[T(Y)/C_{T(Y)}(\operatorname{Im} T(n))] \arrow[d, dashed, "H(\pi)"] \\
Y' \arrow[r, "\Psi_{Y'}"] & HT(Y') \arrow[r, "H(\pi_C^{n'})"] & H[T(Y')/C_{T(Y')}(\operatorname{Im} T(n'))],
\end{tikzcd}
$$

where $H(\pi) \cdot H(\pi_C^n) \cdot \Psi_Y = H(\pi_C^{n'}) \cdot \Psi_{Y'} \cdot f$. Therefore:

$$
f\big(\operatorname{Ker}[H(\pi_C^n) \cdot \Psi_Y]\big) \subseteq \operatorname{Ker}[H(\pi_C^{n'}) \cdot \Psi_{Y'}],
$$

and by definition this means that $f\big(C_Y^*(N)\big) \subseteq C_{Y'}^*\big(f(N)\big)$, so (c_3) is true, which ends the proof. $\square$

II. Mapping $D \mapsto D^*$ from $\mathbb{CO}(S)$ to $\mathbb{CO}(R)$

Now we will define in our adjoint situation (T, H) an inverse mapping from $\mathbb{CO}(S)$ to $\mathbb{CO}(R)$. Let $D \in \mathbb{CO}(S)$ and $m\colon M \xrightarrow{\subseteq} X$ be an inclusion of R-Mod. Then in S-Mod we have the morphism $H(m)\colon H(M) \to H(X)$

and by operator D we obtain the following decomposition of $H(m)$:

$$\begin{tikzcd}
H(M) \arrow[r, "\overline{H(m)}", "\cong"'] \arrow[rrr, bend left, "H(m)"] & \operatorname{Im} H(m) \arrow[r, "j_D^m", "\subseteq"'] & D_{H(X)}\left(\operatorname{Im} H(m)\right) \arrow[r, "i_D^m", "\subseteq"'] & H(X)
\end{tikzcd}$$

(we remark that $H(m)$ is a monomorphism, so its restriction $\overline{H(m)}$ is an isomorphism).

Now using T and Φ we have in R-Mod the situation:

$$\begin{tikzcd}
TH(M) \arrow[r] \arrow[rr, bend left, "TH(m)"] & T\big[D_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \arrow[r, "T(i_D^m)"] & TH(X) \arrow[r, "\Phi_X"] & X.
\end{tikzcd}$$

Definition 2. *For every closure operator $D \in \mathbb{CO}(S)$ and every inclusion $m\colon M \xrightarrow{\subseteq} X$ of R-Mod we define the function D^* by the rule:*

$$D^*_X(M) = \operatorname{Im}[\Phi_X \cdot T(i_D^m)] + M. \tag{2.2}$$

Proposition 2.2. *The function D^* defined by* (2.2) *is a closure operator of R-Mod.*

Proof. (c_1) By Definition 2 it is clear that $M \subseteq D^*_X(M)$.

(c_2) Let $M_1, M_2 \in \mathbb{L}(X)$ and $\kappa\colon M_1 \xrightarrow{\subseteq} M_2$. We denote $m_1\colon M_1 \xrightarrow{\subseteq} X$ and $m_2\colon M_2 \xrightarrow{\subseteq} X$, so $m_1 = m_2 \cdot \kappa$ and $H(m_1) = H(m_2) \cdot H(\kappa)$. Then we have in S-Mod the following situation:

$$\begin{tikzcd}
H(M_1) \arrow[r, "\overline{H(m_1)}", "\cong"'] \arrow[d, "H(k)"] \arrow[rrr, bend left, "H(m_1)"] & \operatorname{Im} H(m_1) \arrow[r, "j_D^{m_1}", "\subseteq"'] \arrow[d, dashed, "\overline{H(k)}"] & D_{H(X)}\big(\operatorname{Im} H(m_1)\big) \arrow[r, "i_D^{m_1}", "\subseteq"'] \arrow[d, dashed, "D(H(k))"] & H(X) \arrow[d, equal] \\
H(M_2) \arrow[r, "\overline{H(m_2)}"] \arrow[rrr, bend right, "H(m_2)"'] & \operatorname{Im} H(m_2) \arrow[r, "j_D^{m_2}", "\subseteq"'] & D_{H(X)}\big(\operatorname{Im} H(m_2)\big) \arrow[r, "i_D^{m_2}", "\subseteq"'] & H(X).
\end{tikzcd}$$

Here the morphism $H(\kappa)$ implies $\overline{H(\kappa)}$, as well as $D\big(H(\kappa)\big)$. Coming back in R-Mod by T, we obtain the diagram:

$$
\begin{tikzcd}
TH(M_1) \arrow[r] \arrow[dd, "TH(k)"] \arrow[rrrd, bend left, "TH(m_1)"] & T\big[D_{H(X)}\big(\operatorname{Im} H(m_1)\big)\big] \arrow[dr, "T(i_D^{m_1})"] \arrow[dd, dashed, "T[D(H(k))]"] & & \\
& & TH(X) \arrow[r, "\Phi_X"] & X. \\
TH(M_2) \arrow[r] \arrow[rrru, bend right, "TH(m_2)"'] & T\big[D_{H(X)}\big(\operatorname{Im} H(m_2)\big)\big] \arrow[ur, "T(i_D^{m_2})"] & &
\end{tikzcd}
$$

We have $T(i_D^{m_1}) = T(i_D^{m_2}) \cdot T[D(H(k))]$, therefore $\operatorname{Im} T(i_D^{m_1}) \subseteq \operatorname{Im} T(i_D^{m_2})$, which shows that $\operatorname{Im}[\Phi_X \cdot T(i_D^{m_1})] \subseteq \operatorname{Im}[\Phi_X \cdot T(i_D^{m_2})]$. Adding M to both parts, by definition we have $D_X^*(M_1) \subseteq D_X^*(M_2)$, so (c_2) is true.

(c_3) Let $f\colon X \to X'$ be a morphism of R-Mod and $m\colon M \xrightarrow{\subseteq} X$. We will verify the relation: $f\big(D_X^*(M)\big) \subseteq D_{X'}^*\big(f(M)\big)$. For that we denote: $m'\colon f(M) \xrightarrow{\subseteq} X'$ and $f'\colon M \to f(M)$ is the restriction of f, i.e. $f \cdot m = m' \cdot f'$. Applying H and using D, we obtain in S-Mod the situation:

$$
\begin{tikzcd}
H(M) \arrow[r, "\overline{H(m)}", "\cong"'] \arrow[d, "H(f')"] \arrow[rrr, bend left, "H(m)"] & \operatorname{Im} H(m) \arrow[r, "j_D^m", "\subseteq"'] \arrow[d, dashed, "\overline{H(f)}"] & D_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[r, "i_D^m", "\subseteq"'] \arrow[d, dashed, "D(H(f))"] & H(X) \arrow[d, "H(f)"] \\
H\big(f(M)\big) \arrow[r, "\overline{H(m')}", "\cong"'] \arrow[rrr, bend right, "H(m')"'] & \operatorname{Im} H(m') \arrow[r, "j_D^{m'}", "\subseteq"'] & D_{H(X')}\big(\operatorname{Im} H(m')\big) \arrow[r, "i_D^{m'}", "\subseteq"'] & H(X'),
\end{tikzcd}
$$

where $D\big(H(f)\big)$ is defined by the morphism $H(f)$.

Using T and Φ, we obtain in R-Mod the diagram:

$$
\begin{tikzcd}
TH(M) \arrow[r] \arrow[d, "TH(f')"] \arrow[rr, bend left, "TH(m)"] & T\big[D_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \arrow[r, "T(i_D^m)"] \arrow[d, dashed, "T[D(H(f))]"] & TH(X) \arrow[r, "\Phi_X"] \arrow[d, dashed, "TH(f)"] & X \arrow[d, "f"] \\
TH\big(f(M)\big) \arrow[r] \arrow[rr, bend right, "TH(m')"'] & T\big[D_{H(X')}\big(\operatorname{Im} H(m')\big)\big] \arrow[r, "T(i_D^{m'})"] & TH(X') \arrow[r, "\Phi_{X'}"] & X'.
\end{tikzcd}
$$

We have $f \cdot \Phi_X \cdot T(i_D^m) = \Phi_{X'} \cdot T(i_D^{m'}) \cdot T[D(H(f))]$, therefore $\mathrm{Im}[f \cdot \Phi_X \cdot T(i_D^m)] \subseteq \mathrm{Im}[\Phi_{X'} \cdot T(i_D^{m'})]$, which implies $f\big(\mathrm{Im}[\Phi_X \cdot T(i_D^m)] + M\big) \subseteq \mathrm{Im}[\Phi_{X'} \cdot T(i_D^{m'})] + f(M)$. By definition this means that $f\big(D_X^*(M)\big) \subseteq D_{X'}^*\big(f(M)\big)$, i.e. (c_3) is true, which ends the proof. □

3. Particular cases

As examples in continuation we verify the effect of "star" mappings defined above in some particular cases, namely for the extreme (trivial) elements of the lattices of closure operators, i.e. $C \in \{\mathbb{0}_R, \mathbb{1}_R\} \subseteq \mathbb{CO}(R)$ and $D \in \{\mathbb{0}_S, \mathbb{1}_S\} \subseteq \mathbb{CO}(S)$.

1. Let $C = \mathbb{0}_R$, where $\mathbb{0}_R$ is the least element of $\mathbb{CO}(R)$, i.e. $(\mathbb{0}_R)_X(M) = M$ for every $M \subseteq X$. By construction of C^*, in this case for every inclusion $n \colon N \xrightarrow{\subseteq} Y$ of S-Mod we have such decomposition of $T(n)$:

$$\begin{tikzcd} T(N) \arrow[r, "\overline{T(n)}"] \arrow[rr, bend right, "T(n)"'] & \mathrm{Im}\, T(n) = C_{T(Y)}\big(\mathrm{Im}\, T(n)\big) \arrow[r, "\subseteq"] & T(Y). \end{tikzcd}$$

By natural epimorphism $\pi_C^n \colon T(Y) \to T(Y)/\,\mathrm{Im}\, T(n)$ and applying H we obtain in S-Mod the composition:

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi_C^n)} H[T(Y)/\,\mathrm{Im}\, T(n)].$$

By definition of C^* we have $C_Y^*(N) = \mathrm{Ker}[H(\pi_C^n) \cdot \Psi_Y]$. We denote this operator by D°, so $D_Y^\circ(N) \overset{\text{def}}{=\!=} \mathrm{Ker}[H(\pi_C^n) \cdot \Psi_Y]$. Therefore it is verified that $\mathbb{0}_R^* = D^\circ$.

2. Let $C = \mathbb{1}_R$, where $\mathbb{1}_R$ is the greatest element of $\mathbb{CO}(R)$, i.e. $C_X(M) = X$ for every $M \subseteq X$. For the inclusion $n \colon N \xrightarrow{\subseteq} Y$ of S-Mod we have in R-Mod:

$$T(N) \xrightarrow{\overline{T(n)}} \mathrm{Im}\, T(n) \xrightarrow{\subseteq} C_{T(Y)}\big(\mathrm{Im}\, T(n)\big) = T(Y),$$

so in S-Mod we obtain the composition:

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{0} H(0) = 0$$

(since $\pi_C^n = 0$). Therefore $\mathrm{Ker}[0 \cdot \Psi_Y] = \mathrm{Ker}\, 0 = Y$ and we have $C_Y^*(N) = Y$ for every $N \subseteq Y$, which means that $\mathbb{1}_R^* = \mathbb{1}_S$.

3. Let $D = \mathbb{O}_S$, where $\mathbb{O}_S$ is the least element of $\mathbb{CO}(S)$, i.e. $D_Y(N) = N$ for every $n\colon N \xrightarrow{\subseteq} Y$ of S-Mod. Then for every inclusion $m\colon M \xrightarrow{\subseteq} X$ of R-Mod we have in S-Mod the situation:

$$\begin{tikzcd}
H(M) \arrow[r, "\overline{H(m)}", "\cong"'] \arrow[rr, bend right, "H(m)"'] & \operatorname{Im} H(m) = D_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[r, "i_D^m", "\subseteq"'] & H(X).
\end{tikzcd}$$

Now by T and Φ we obtain in S-Mod:

$$\begin{tikzcd}
TH(M) \arrow[r, "T(\overline{H(m)})", "\cong"'] \arrow[rr, bend right, "TH(m)"'] & T\big(\operatorname{Im} H(m)\big) = T\big[D_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \arrow[r, "T(i_D^m)"] & TH(X) \arrow[r, "\Phi_X"] & X.
\end{tikzcd}$$

Since $T(\overline{H(m)})$ is an isomorphism and using the naturality relation $\Phi_X \cdot TH(m) = m \cdot \Phi_M$, we have:

$$\operatorname{Im}[\Phi_X \cdot T(i_D^m)] = \operatorname{Im}[\Phi_X \cdot TH(m)] = \operatorname{Im}[m \cdot \Phi_M] = \operatorname{Im}\Phi_M \subseteq M.$$

By definition now it is clear that:

$$D_X^*(M) = \operatorname{Im}[\Phi_X \cdot T(i_D^m)] + M = M$$

for every $M \subseteq X$, i.e. $D^* = \mathbb{O}_R$ or $\mathbb{O}_S^* = \mathbb{O}_R$.

4. Let $D = \mathbb{1}_S$, where $\mathbb{1}_S$ is the greatest element of $\mathbb{CO}(S)$, i.e. $D_Y(N) = Y$ for every $N \subseteq Y$. Then for every inclusion $m\colon M \xrightarrow{\subseteq} X$ of R-Mod we have in S-Mod the situation:

$$\begin{tikzcd}
H(M) \arrow[r, "\overline{H(m)}", "\cong"'] \arrow[rrr, bend right, "H(m)"'] & \operatorname{Im} H(m) \arrow[r, "j_D^m", "\subseteq"'] & D_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[r, equal, "i_D^m"] & H(X).
\end{tikzcd}$$

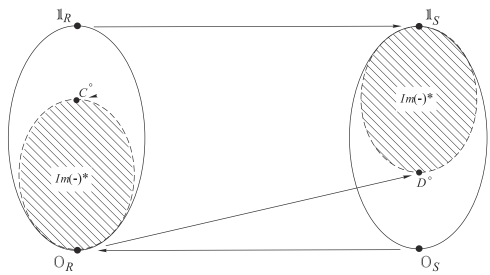
By T and Φ we obtain in R-Mod:

$$\begin{tikzcd}
TH(M) \arrow[r, "T(\overline{H(m)})", "\cong"'] \arrow[rrr, bend right, "TH(m)"'] \arrow[d, "\Phi_M"] & T\big(\operatorname{Im} H(m)\big) \arrow[r, "T(j_D^m)"] & T\big[D_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \arrow[r, equal, "T(i_D^m)"] \arrow[dr, dashed] & TH(X) \arrow[d, "\Phi_X"] \\
M \arrow[rrr, "m", "\subseteq"'] & & & X.
\end{tikzcd}$$

Therefore in this case $\operatorname{Im}[\Phi_X \cdot T(i_D^m)] = \operatorname{Im}\Phi_X$ and $D_X^*(M) = \operatorname{Im}\Phi_X + M$. We denote this operator by C°, i.e. $C_X^\circ(M) \overset{\text{def}}{=\!=} \operatorname{Im}\Phi_X + M$, so it is proved that $\mathbb{1}_S^* = C^\circ$.

Totalizing the mentioned above facts, we can present the general situation on images of extreme elements:

$$\mathbb{CO}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{CO}(S)$$



Proposition 3.1. *The "star" mappings act on the extreme closure operators as follows:*

$$\mathbb{0}_R^* = D^\circ, \quad \mathbb{1}_R^* = \mathbb{1}_S; \qquad \mathbb{0}_S^* = \mathbb{0}_R, \quad \mathbb{1}_S^* = C^\circ.$$

4. Partial order and "star" mappings

In this section we will study the behaviour of the mappings $C \mapsto C^*$ and $D \mapsto D^*$ relative to the partial order in the classes $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$.

Proposition 4.1. *The "star" mappings are monotone, i.e. they preserve the relations of partial order:*

a) $C_1 \leqslant C_2 \Rightarrow C_1^* \leqslant C_2^*$;

b) $D_1 \leqslant D_2 \Rightarrow D_1^* \leqslant D_2^*$.

Proof. a) We verify the monotony of the mapping $C \mapsto C^*$ from $\mathbb{CO}(R)$ to $\mathbb{CO}(S)$. Let $C_1, C_2 \in \mathbb{CO}(R)$ and $C_1 \leqslant C_2$. For every inclusion $n\colon N \xrightarrow{\subseteq} Y$ of S-Mod by the construction of Definition 1 and using the relation $C_1 \leqslant C_2$ we have: $(C_1)_{T(Y)}\big(\operatorname{Im} T(n)\big) \subseteq (C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big)$. This implies in R-Mod the morphism π from the diagram:

$$\begin{tikzcd}
 & T(Y)/(C_1)_{T(Y)}\big(\operatorname{Im} T(n)\big) \arrow[dd, dashed, "\pi"] \\
T(Y) \arrow[ur, "\pi_{C_1}^n"] \arrow[dr, "\pi_{C_2}^n"'] & \\
 & T(Y)/(C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big),
\end{tikzcd}$$

where $\pi^n_{C_1}$ and $\pi^n_{C_2}$ are the natural morphisms. By H and Ψ we obtain in S-Mod the situation:

$$\begin{tikzcd}
& & H\big[T(Y)/(C_1)_{T(Y)}\big(\operatorname{Im} T(n)\big)\big] \arrow[dd, dashed, "H(\pi)"] \\
Y \arrow[r, "\Psi_Y"] & HT(Y) \arrow[ur, "H(\pi^n_{C_1})"] \arrow[dr, "H(\pi^n_{C_2})"'] & \\
& & H\big[T(Y)/(C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big)\big],
\end{tikzcd}$$

where $H(\pi) \cdot H(\pi^n_{C_1}) \cdot \Psi_Y = H(\pi^n_{C_1}) \cdot \Psi_Y$. Therefore

$$\operatorname{Ker}[H(\pi^n_{C_1}) \cdot \Psi_Y] \subseteq \operatorname{Ker}[H(\pi^n_{C_2}) \cdot \Psi_Y],$$

which by definition means that $(C_1^*)_Y(N) \subseteq (C_2^*)_Y(N)$ for every $N \subseteq Y$, i.e. $C_1^* \leqslant C_2^*$.

b) Now we will verify the monotony of the mapping $D \mapsto D^*$ from $\mathbb{CO}(S)$ to $\mathbb{CO}(R)$. Let $D_1, D_2 \in \mathbb{CO}(S)$ and $D_1 \leqslant D_2$. For an arbitrary inclusion $m\colon M \xrightarrow{\subseteq} X$ of R-Mod we follow the construction of operators D_1^* and D_2^*. Since $D_1 \leqslant D_2$, we have the inclusion i of the diagram:

$$\begin{tikzcd}
& & (D_1)_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[dd, dashed, "\cap\!\shortmid\, i"] \arrow[dr, "i^m_{D_1}", "\subseteq"'] & \\
H(M) \arrow[r, "\overline{H(m)}", "\cong"'] & \operatorname{Im} H(m) \arrow[ur, "j^m_{D_1}", "\subseteq"'] \arrow[dr, "j^m_{D_2}", "\subseteq"'] & & H(X). \\
& & (D_2)_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[ur, "i^m_{D_2}", "\subseteq"'] &
\end{tikzcd}$$

Therefore in R-Mod we obtain the situation:

$$\begin{tikzcd}
& & T[(D_1)_{H(X)}(\operatorname{Im} H(m))] \arrow[dd, dashed, "T(i)"] \arrow[dr, "T(i^m_{D_1})"] & & \\
TH(M) \arrow[r, "T(\overline{H(m)})", "\cong"'] & T(\operatorname{Im} H(m)) \arrow[ur, "T(j^m_{D_1})"] \arrow[dr, "T(j^m_{D_2})"'] & & TH(X) \arrow[r, "\Phi_X"] & X. \\
& & T[(D_2)_{H(X)}(\operatorname{Im} H(m))] \arrow[ur, "T(i^m_{D_2})"'] & &
\end{tikzcd}$$

By commutativity of diagram we have $\operatorname{Im}[\Phi_X \cdot T(i^m_{D_1})] \subseteq \operatorname{Im}[\Phi_X \cdot T(i^m_{D_2})]$ and adding M to both parts by definition we obtain that $(D_1)^*_X(M) \subseteq (D_2)^*_X(M)$ for every $M \subseteq X$, i.e. $D_1^* \leqslant D_2^*$. $\square$

We remark that from the particular cases of Section 3 and by monotony of "star" mappings follows

Corollary 4.2. a) *For every operator $C \in \mathbb{CO}(R)$ we have $C^* \geqslant D^\circ$.*

b) *For every operator $D \in \mathbb{CO}(S)$ we have $D^* \leqslant C^\circ$.* □

In continuation we will prove some more properties of "star" mappings, related to the partial order in $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$.

Proposition 4.3. a) *For every operator $C \in \mathbb{CO}(R)$, the relation $C \geqslant C^{**}$ is true.*

b) *For every operator $D \in \mathbb{CO}(S)$, the relation $D \leqslant D^{**}$ is true.*

Proof. a) Let $C \in \mathbb{CO}(R)$ and $m\colon M \xrightarrow{\subseteq} X$ be an arbitrary inclusion of R-Mod. Then in S-Mod we have the morphism $H(m)\colon H(M) \to H(X)$. We follow the construction of C^* for the inclusion $n\colon \operatorname{Im} H(m) \xrightarrow{\subseteq} H(X)$. In S-Mod we have:

$$\begin{tikzcd}
H(M) \arrow[r, "\overline{H(m)}", "\cong"'] \arrow[rr, bend right=15, "H(m)"] & \operatorname{Im} H(m) \arrow[r, "n", "\subseteq"'] & H(X).
\end{tikzcd}$$

Using T and C we obtain in R-Mod:

$$\begin{tikzcd}
TH(M) \arrow[r, "T(\overline{H(m)})"] \arrow[d, "\overline{TH(m)}"] \arrow[rr, bend right=10, "TH(m)"] & T\big(\operatorname{Im} H(m)\big) \arrow[r, "T(n)"] \arrow[d, "\overline{T(n)}"] & TH(X) \\
\operatorname{Im} TH(m) \arrow[r, equal] & \operatorname{Im} T(n) \arrow[r, "\subseteq"'] & C_{TH(X)}\big(\operatorname{Im} T(n)\big). \arrow[u, "\cup\!\mid"']
\end{tikzcd}$$

Now we consider the natural morphism

$$\pi_C^n\colon TH(X) \to TH(X)/C_{TH(X)}\big(\operatorname{Im} T(n)\big).$$

Applying H and adding $\Psi_{H(X)}$, we have in S-Mod:

$$H(X) \xrightarrow{\Psi_{H(X)}} HTH(X) \xrightarrow{H(\pi_C^n)} H[TH(X)/C_{TH(X)}\big(\operatorname{Im} T(n)\big)].$$

By Definition 1 we have:

$$C^*_{H(X)}\big(\operatorname{Im} H(m)\big) = \operatorname{Ker}[H(\pi_C^n) \cdot \Psi_{H(X)}]. \tag{4.1}$$

We denote $i^n_{C^*}\colon C^*_{H(X)}\big(\operatorname{Im} H(m)\big) \xrightarrow{\subseteq} H(X)$ and consider the following commutative diagram in R-Mod:

$$\begin{tikzcd}
TH(X) \arrow[r, "T(\Psi_{H(X)})"] \arrow[rr, bend left, "1_{TH(X)}"] & THTH(X) \arrow[r, "\Phi_{TH(X)}"] \arrow[d, "TH(\pi^n_C)"] & TH(X) \arrow[r, "\Phi_X"] \arrow[d, "\pi^n_C"] & X \arrow[d, "\pi^m_C"] \\
T\big[C^*_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \arrow[u, "T(i^n_{C^*})"] \arrow[r, dashed, "0"] & TH[TH(X)/A] \arrow[r, "\Phi_{\overline{TH(X)}}"] & TH(X)/A \arrow[r, "(1/C)\Phi_X"] & X/C_X(M),
\end{tikzcd}$$

where $A = C_{TH(X)}\big(\operatorname{Im} T(n)\big)$ and $(1/C)\Phi_X$ is defined by Φ_X. From the definition of $C^*_{H(X)}\big(\operatorname{Im} H(m)\big)$ (see (4.1)) we have $H(\pi^n_C)\cdot\Psi_{H(X)}\cdot i^n_{C^*} = 0$, therefore $TH(\pi^n_C)\cdot T(\Psi_{H(X)})\cdot T(i^n_{C^*}) = 0$. From commutativity of diagram we obtain $\pi^m_C\cdot\Phi_X\cdot 1_{TH(X)}\cdot T(i^n_{C^*}) = 0$, so $\operatorname{Im}[\Phi_X\cdot T(i^n_{C^*})] \subseteq \operatorname{Ker}\pi^n_C = C_X(M)$. Since $M \subseteq C_X(M)$, now we have $\operatorname{Im}[\Phi_X\cdot T(i^n_{C^*})] + M \subseteq C_X(M)$. The left part of this relation by definition represents the module $C^{**}_X(M)$, therefore we obtain $C^{**}_X(M) \subseteq C_X(M)$, for every $M \subseteq X$, i.e. $C^{**} \leqslant C$ proving a).

b) To verify the part b) we consider an operator $D \in \mathbb{CO}(S)$ and an inclusion $n\colon N \xrightarrow{\subseteq} Y$ of S-Mod. Using the operator $D^* \in \mathbb{CO}(R)$ we obtain the decomposition of $T(n)$:

$$\begin{tikzcd}
T(N) \arrow[r, "\overline{T(n)}"] \arrow[rrr, bend right, "T(n)"'] & \operatorname{Im} T(n) \arrow[r, "j^n_{D^*}", "\subseteq"'] \arrow[rr, bend left, "m", "\subseteq"'] & D^*_{T(Y)}\big(\operatorname{Im} T(n)\big) \arrow[r, "i^n_{D^*}", "\subseteq"'] & T(Y).
\end{tikzcd}$$

We denote by m the inclusion $m\colon \operatorname{Im} T(n) \xrightarrow{\subseteq} T(Y)$ and by $\pi^m_{D^*}$ the natural morphism $\pi^m_{D^*}\colon T(Y) \to T(Y)/D^*_{T(Y)}\big(\operatorname{Im} T(n)\big)$. Applying H and using Ψ, we obtain in S-Mod the composition:

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^m_{D^*})} H\big[T(Y)/D^*_{T(Y)}\big(\operatorname{Im} T(n)\big)\big]$$

and by definition we have:

$$D^{**}_Y(N) = \operatorname{Ker}[H(\pi^m_{D^*})\cdot\Psi_Y]. \tag{4.2}$$

Now we apply the transition $D \mapsto D^*$ to the inclusion $m\colon \operatorname{Im} T(n) \xrightarrow{\subseteq} T(Y)$ of R-Mod. With the help of H we have in S-Mod the situation:

$$\begin{tikzcd}
HT(N) \arrow[r, "H(\overline{T(n)})"] \arrow[rrrr, bend right, "HT(n)"'] & H\big(\operatorname{Im} T(n)\big) \arrow[r, "\overline{H(m)}", "\cong"'] \arrow[rrr, bend left, "H(m)"] & \operatorname{Im} H(m) \arrow[r, "j^m_D", "\subseteq"'] & D_{HT(Y)}\big(\operatorname{Im} H(m)\big) \arrow[r, "i^m_D", "\subseteq"'] & HT(Y).
\end{tikzcd}$$

Returning in R-Mod and using Φ, we obtain the diagram:

$$\begin{tikzcd}[column sep=small]
THT(N) \arrow[r, "TH(\overline{T(n)})"] \arrow[d, "\Phi_{T(N)}"] \arrow[rrrr, bend right=12, "THT(n)"'] & TH(\operatorname{Im} T(m)) \arrow[r, "T(\overline{H(m)})"] \arrow[d, "\Phi_{\operatorname{Im} T(n)}"] \arrow[rrr, bend left=20, "TH(m)"] & T(\operatorname{Im} H(m)) \arrow[r, "T(j^m_D)"] & T[D_{HT(Y)}(\operatorname{Im} H(m))] \arrow[r, "T(i^m_D)"] \arrow[d, dashed, "\overline{\Phi^m_{T(Y)}}"] \arrow[dr, dashed, "\Phi^m_{T(Y)}"] & THT(Y) \arrow[d, "\Phi_{T(Y)}"] \\
T(N) \arrow[r] & \operatorname{Im} T(n) \arrow[rr, "j^n_{D^*}", "\subseteq"'] & & D^*_{T(Y)}(\operatorname{Im} T(n)) \arrow[r, "i^n_{D^*}", "\subseteq"'] & T(Y),
\end{tikzcd}$$

where $\Phi^m_{T(Y)} = \Phi_{T(Y)} \cdot T(i^m_D)$. By Definition 2 we have:

$$D^*_{T(Y)}\big(\operatorname{Im} T(n)\big) = \operatorname{Im}[\Phi_{T(Y)} \cdot T(i^m_D)] + \operatorname{Im} T(n). \tag{4.3}$$

We denote by $i^m_{D^*}$ the inclusion $i^m_{D^*} \colon D^*_{T(Y)}\big(\operatorname{Im} T(n)\big) \xrightarrow{\subseteq} T(Y)$ and by $\pi^m_{D^*}$ the natural morphism $\pi^m_{D^*} \colon T(Y) \to T(Y)/D^*_{T(Y)}\big(\operatorname{Im} T(n)\big)$, so $\pi^m_{D^*} \cdot i^m_{D^*} = 0$.

Using D and Ψ, we obtain in S-Mod the diagram:

$$\begin{tikzcd}
HT(N) \arrow[d, "H(\overline{T(n)})"] & N \arrow[l, "\Psi_N"'] \arrow[r, "\subseteq"'] \arrow[d, dashed, "\Psi'_N"] & D_Y(N) \arrow[r, "k", "\subseteq"'] \arrow[d, dashed, "\Psi''_N"] & Y \arrow[d, "\Psi_Y"] \\
H\big(\operatorname{Im} T(n)\big) \arrow[r, "\overline{H(m)}"] & \operatorname{Im} H(m) \arrow[r, "\subseteq"'] & D_{HT(Y)}\big(\operatorname{Im} H(m)\big) \arrow[r, "\subseteq"'] & HT(Y),
\end{tikzcd}$$

where $\Psi'_N = \overline{H(m)} \cdot H\big(\overline{T(n)}\big) \cdot \Psi_N$ and $\kappa \colon D_Y(N) \xrightarrow{\subseteq} Y$ is the inclusion. The morphism Ψ_Y implies the morphism Ψ''_N, by which (using the last but one diagram) we obtain in S-Mod:

$$\begin{tikzcd}[column sep=small]
D_Y(N) \arrow[r, dashed, "\Psi''_N"] \arrow[d, "k", "\cap|"'] & D_{HT(Y)}(\operatorname{Im} H(m)) \arrow[r, "\Psi_{D_{HT(Y)}(\operatorname{Im} H(m))}"] \arrow[d, "i^m_D", "\cap|"'] & HT[D_{HT(Y)}(\operatorname{Im} H(m))] \arrow[r, "H(\overline{\Phi^m_{T(Y)}})"] & H[D^*_{T(Y)}(\operatorname{Im} T(n))] \arrow[dl, "H(i^m_{D^*})"'] \arrow[d, "0"] \\
Y \arrow[r, "\Psi_Y"] & HT(Y) \arrow[r, "1_{HT(Y)}"] & HT(Y) \arrow[r, "H(\pi^m_{D^*})"] & H[T(Y)/D^*_{T(Y)}(\operatorname{Im} T(n))],
\end{tikzcd}$$

where $1_{HT(Y)} = H(\Phi_{T(Y)}) \cdot \Psi_{HT(Y)}$. As we mentioned above, by construction $\pi^m_{D^*} \cdot\ i^m_{D^*} = 0$, therefore $H(\pi^m_{D^*}) \cdot H\ (i^m_{D^*}) = 0$. Therefore:

$$\begin{aligned}
&H(\pi^m_{D^*}) \cdot\ 1_{TH(Y)} \cdot \Psi_Y \cdot k \\
&\qquad = H(\pi^m_{D^*}) \cdot H\ (i^m_{D^*}) \cdot H\big(\overline{\Phi^m_{T(Y)}}\big) \cdot \Psi_{D_{HT(Y)}(\operatorname{Im} H(m))} \cdot \Psi'' = 0.
\end{aligned}$$

This shows that $D_Y(N) \subseteq \operatorname{Ker}[H(\pi^m_{D^*}) \cdot \Psi_Y] \overset{\text{def}}{=\!=} D^{**}_Y(N)$ for every $N \subseteq Y$, which means that $D \leqslant D^{**}$. $\square$

Remark. In this case we mention that the proved above facts are perfectly concordant with the results for preradicals in adjoint situation, where $r \geqslant r^{**}$ and $s \leqslant s^{**}$ for every preradicals r of R-Mod and s of S-Mod ([5, 6]).

5. Lattice operations and "star" mappings

Now we will study the behaviour of "star" mappings in the adjoint situation (T, H) with respect to lattice operations "$\wedge$" (meet) and "$\vee$" (join) in the classes $\mathbb{CO}(R)$ and $\mathbb{CO}(S)$.

Proposition 5.1. *The mapping $C \mapsto C^*$ from $\mathbb{CO}(R)$ to $\mathbb{CO}(S)$ preserves the meet of closure operators, i.e.*

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)^* = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha)^*$$

for every family of operators $\{C_\alpha \in \mathbb{CO}(R) \mid \alpha \in \mathfrak{A}\}$.

Proof. Let $\{C_\alpha \in \mathbb{CO}(R) \mid \alpha \in \mathfrak{A}\}$ be an arbitrary family of closure operators of R-Mod and $n\colon N \xrightarrow{\subseteq} Y$ be an inclusion of S-Mod. By definition of mapping $C \mapsto C^*$, for any $\alpha \in \mathfrak{A}$ we have in R-Mod the morphisms:

$$\begin{tikzcd} T(N) \arrow[r, "\overline{T(n)}"] \arrow[rrr, bend left, "T(n)"] & \operatorname{Im} T(n) \arrow[r, "j^n_{C_\alpha}", "\subseteq"'] & (C_\alpha)_{T(Y)}(\operatorname{Im} T(n)) \arrow[r, "i^n_{C_\alpha}", "\subseteq"'] & T(Y) \arrow[r, "\pi^n_{C_\alpha}"] & T(Y)/(C_\alpha)_{T(Y)}(\operatorname{Im} T(n)). \end{tikzcd}$$

Using H and Ψ, we obtain in S-Mod the composition:

$$Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi^n_{C_\alpha})} H\big[T(Y)/(C_\alpha)_{T(Y)}\big(\operatorname{Im} T(n)\big)\big]$$

and by Definition 1 we have: $(C_\alpha)^*_Y(N) = \operatorname{Ker}[H(\pi^n_{C_\alpha}) \cdot \Psi_Y]$.

Similarly, for $C = \bigwedge_{\alpha\in\mathfrak{A}} C_\alpha$ from the definition of C^* we have:

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)^*_Y(N) = \operatorname{Ker}[H(\pi^n_{\wedge C_\alpha}) \cdot \Psi_Y],$$

where $\pi^n_{\wedge C_\alpha} : T(Y) \to T(Y)/\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)_{T(Y)}\big(\operatorname{Im} T(n)\big)$ is the natural morphism.

Now we observe that is true the equality:

$$\operatorname{Ker}[H(\pi^n_{\wedge C_\alpha})] = \bigcap_{\alpha\in\mathfrak{A}} [\operatorname{Ker}[H(\pi^n_{C_\alpha})], \tag{5.1}$$

since $\mathrm{Ker}(\pi^n_{\wedge C_\alpha}) = \big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)_{T(Y)}\big(\mathrm{Im}\, T(n)\big) = \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_{T(Y)}\big(\mathrm{Im}\, T(n)\big)\big] = \bigcap_{\alpha\in\mathfrak{A}} [\mathrm{Ker}(\pi^n_{C_\alpha})]$. Using this relation we obtain:

$$\begin{aligned}\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)^*_Y(N) &\overset{\text{def}}{=\!=} \mathrm{Ker}[H(\pi^n_{\wedge C_\alpha})\cdot\Psi_Y] = \Psi_Y^{-1}\big[\mathrm{Ker}\, H(\pi^n_{\wedge C_\alpha})\big] \\ &\overset{(5.1)}{=\!=} \Psi_Y^{-1}\Big[\bigcap_{\alpha\in\mathfrak{A}} \mathrm{Ker}\, H(\pi^n_{C_\alpha})\Big] = \bigcap_{\alpha\in\mathfrak{A}} \Big[\Psi_Y^{-1}\big(\mathrm{Ker}\, H(\pi^n_{C_\alpha})\big)\Big] \\ &= \bigcap_{\alpha\in\mathfrak{A}} \mathrm{Ker}[H(\pi^n_{C_\alpha})\cdot\Psi_Y] = \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)^*_Y(N)\big] = \Big(\bigwedge_{\alpha\in\mathfrak{A}} C^*_\alpha\Big)_Y(N)\end{aligned}$$

for every $N \subseteq Y$. This shows that $\big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\big)^* = \bigwedge_{\alpha\in\mathfrak{A}} (C_\alpha)^*$. □

Proposition 5.2. *The mapping $D \mapsto D^*$ from $\mathbb{CO}(S)$ to $\mathbb{CO}(R)$ preserves the join of closure operators, i.e.*

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\Big)^* = \bigvee_{\alpha\in\mathfrak{A}} (D_\alpha)^*$$

for every family of operators $\{D_\alpha \in \mathbb{CO}(S) \mid \alpha \in \mathfrak{A}\}$.

Proof. Let $\{D_\alpha \in \mathbb{CO}(S) \mid \alpha \in \mathfrak{A}\}$ be an arbitrary family of closure operators of S-Mod and $m\colon M \xrightarrow{\subseteq} X$ be an inclusion of R-Mod. By definition $(D_\alpha)^*_X(M) = \mathrm{Im}[\Phi_X \cdot T(i^m_{D_\alpha})] + M$, where $i^m_{D_\alpha}\colon (D_\alpha)_{H(X)}\big(\mathrm{Im}\, H(m)\big) \xrightarrow{\subseteq} H(X)$. The same rule applied for the operator $\bigvee_{\alpha\in\mathfrak{A}} D_\alpha$ and inclusion m leads to equality:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\Big)^*_X(M) = \mathrm{Im}[\Phi_X \cdot\ T(i^m_{\vee D_\alpha})] + M,$$

where $i^m_{\vee D_\alpha}\colon \big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big)_{H(X)}\big(\mathrm{Im}\, H(m)\big) \xrightarrow{\subseteq} H(X)$. Since

$$\mathrm{Im}\,\big(i^m_{\vee D_\alpha}\big) = \sum_{\alpha\in\mathfrak{A}} \mathrm{Im}\,\big(i^m_{D_\alpha}\big),$$

we obtain:

$$\mathrm{Im}\, T(i^m_{\vee D_\alpha}) = \sum_{\alpha\in\mathfrak{A}} \mathrm{Im}\, T\big(i^m_{D_\alpha}\big). \tag{5.2}$$

Using this equality, by definitions we have:

$$\begin{aligned}
\big(\bigvee_{\alpha\in\mathfrak{A}} D_\alpha\big)^*_X(M) &= \mathrm{Im}[\Phi_X \cdot\ T(i^m_{\vee D_\alpha})] + M = \Phi_X[\mathrm{Im}\, T(i^m_{\vee D_\alpha})] + M \\
&\overset{(5.2)}{=\!=} \Phi_X\big[\sum_{\alpha\in\mathfrak{A}} \mathrm{Im}\, T\big(i^m_{D_\alpha}\big)\big] + M = \big[\sum_{\alpha\in\mathfrak{A}} \Phi_X\big(\mathrm{Im}\, T(i^m_{D_\alpha})\big)\big] + M \\
&= \big(\sum_{\alpha\in\mathfrak{A}} \mathrm{Im}[\Phi_X \cdot\ T(i^m_{D_\alpha})]\big) + M = \sum_{\alpha\in\mathfrak{A}} \big(\mathrm{Im}[\Phi_X \cdot\ T(i^m_{D_\alpha})] + M\big) \\
&= \sum_{\alpha\in\mathfrak{A}} \big[\big(D_\alpha\big)^*_X(M)\big] = \big(\bigvee_{\alpha\in\mathfrak{A}} D^*_\alpha\big)_X(M)
\end{aligned}$$

for every $M \subseteq X$. Therefore we obtain $\big(\bigvee\limits_{\alpha\in\mathfrak{A}} D_\alpha\big)^* = \bigvee\limits_{\alpha\in\mathfrak{A}} \big(D_\alpha\big)^*$. □

6. Product of closure operators and "star" mappings

We remember that besides lattice operations, in the class of closure operators $\mathbb{CO}(R)$ also the operation of multiplication is defined by the rule:

$$(C_1 \cdot C_2)_X(M) = (C_1)_X[(C_2)_X(M)]$$

for every operators $C_1, C_2 \in \mathbb{CO}(R)$ and $M \subseteq X$. In continuation we will show how the "star" mappings act to the product of closure operators.

Proposition 6.1. *For every closure operators $C_1, C_2 \in \mathbb{CO}(R)$ the relation $(C_1 \cdot C_2)^* \geqslant C_1^* \cdot C_2^*$ is true.*

Proof. Let $C_1, C_2 \in \mathbb{CO}(R)$ and $n\colon N \xrightarrow{\subseteq} Y$ be an arbitrary inclusion of S-Mod. By definitions we have:

$$(C_1 \cdot C_2)^*_Y(N) = \mathrm{Ker}[H(\pi^n_{C_1 \cdot C_2}) \cdot \Psi_Y],$$

where $\pi^n_{C_1 \cdot C_2}\colon T(Y) \to T(Y)/(C_1 \cdot C_2)_{T(Y)}\big(\mathrm{Im}\, T(n)\big)$ is the natural morphism.

On the other hand, to define $[C_1^* \cdot C_2^*]_Y(N)$ we consider in S-Mod the inclusions:

$$\begin{tikzcd} N \arrow[r, "l"] \arrow[rr, bend right, "n"'] & (C_2)^*_Y(N) \arrow[r, "\kappa"] & Y, \end{tikzcd}$$

i.e. $n = \kappa \cdot l$. Therefore $T(n) = T(\kappa) \cdot T(l)$ and $\mathrm{Im}\, T(n) \subseteq \mathrm{Im}\, T(\kappa)$.

Now we apply the transition $C_1 \mapsto C_1^*$ for the inclusion κ:

$$\mathrm{Im}\, T(\kappa) \overset{\subseteq}{\to} (C_1)_{T(Y)}\big(\mathrm{Im}\, T(\kappa)\big) \overset{\subseteq}{\to} T(Y) \xrightarrow{\pi^\kappa_{C_1}} T(Y)/(C_1)_{T(Y)}\big(\mathrm{Im}\, T(\kappa)\big).$$

By Definition 1 we have:

$$(C_1^* \cdot C_2^*)_Y(N) = (C_1)_Y^*[(C_2)_Y^*(N)] = \operatorname{Ker}[H(\pi_{C_1}^{\kappa}) \cdot \Psi_Y].$$

Similarly

$$(C_2)_Y^*(N) = \operatorname{Ker}[H(\pi_{C_2}^{n}) \cdot \Psi_Y],$$

where $\pi_{C_2}^n \colon T(Y) \to T(Y)/(C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big)$ is the natural morphism. So in S-Mod we obtain the situation:

$$(C_2)_Y^*(N) \xrightarrow[\subseteq]{k} Y \xrightarrow{\Psi_Y} HT(Y) \xrightarrow{H(\pi_{C_2}^n)} H[T(Y)/(C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big)],$$

where by construction $H(\pi_{C_2}^n) \cdot \Psi_Y \cdot \kappa = 0$.

Applying T and completing the diagram we have in R-Mod:

$$\begin{tikzcd}
T[(C_2)_Y^*(N)] \arrow[r, "T(\kappa)"] & T(Y) \arrow[r, shift left, "T(\Psi_Y)"] \arrow[dr, "\pi_{C_2}^n"'] & THT(Y) \arrow[l, shift left, dashed, "\Phi_{TY}"] \arrow[r, "TH(\pi_{C_2}^n)"] & TH[T(Y)/(C_2)_{T(Y)}(\operatorname{Im} T(n))] \arrow[dl, dashed, "\Phi_{\overline{T(Y)}}"] \\
 & & T(Y)/(C_2)_{T(Y)}(\operatorname{Im} T(n)). &
\end{tikzcd}$$

By naturality of Φ the equality $\pi_{C_2}^n \cdot \Phi_{T(Y)} = \Phi_{\overline{T(Y)}} \cdot TH(\pi_{C_2}^n)$ is true, therefore

$$\Phi_{\overline{T(Y)}} \cdot TH(\pi_{C_2}^n) \cdot T(\Psi_Y) = \pi_{C_2}^n \cdot \Phi_{T(Y)} \cdot T(\Psi_Y) = \pi_{C_2}^n \cdot 1_{T(Y)} = \pi_{C_2}^n.$$

From the remark that $H(\pi_{C_2}^n) \cdot \Psi_Y \cdot \kappa = 0$ it follows that $TH(\pi_{C_2}^n) \cdot T(\Psi_Y) \cdot T(\kappa) = 0$. Therefore

$$\begin{aligned}
\operatorname{Im} T(\kappa) &\subseteq \operatorname{Ker}[TH(\pi_{C_2}^n) \times T(\Psi_Y)] \\
&\subseteq \operatorname{Ker}[\Phi_{\overline{T(Y)}} \cdot TH(\pi_{C_2}^n) \cdot T(\Psi_Y)] = \operatorname{Ker} \pi_{C_2}^n = (C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big),
\end{aligned}$$

i.e. $\operatorname{Im} T(\kappa) \subseteq (C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big)$. This relation implies the inclusion:

$$\begin{aligned}
(C_1)_{T(Y)}\big(\operatorname{Im} T(\kappa)\big) &\subseteq (C_1)_{T(Y)}\big[(C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big)\big] \\
&\overset{\text{def}}{=\!=} (C_1 \cdot C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big),
\end{aligned}$$

which in its turn defines the epimorphism:

$$\pi \colon T(Y)/(C_1)_{T(Y)}\big(\operatorname{Im} T(\kappa)\big) \to T(Y)/(C_1 \cdot C_2)_{T(Y)}\big(\operatorname{Im} T(n)\big).$$

Applying H we obtain in S-Mod the situation:

$$\begin{tikzcd}
 & & H[T(Y)/(C_1)_{T(Y)}(\operatorname{Im} T(\kappa))] \arrow[d, dashed, "H(\pi)"] \\
Y \arrow[r, "\Psi_Y"] & HT(Y) \arrow[ur, "H(\pi^{\kappa}_{C_1})"] \arrow[r, "H(\pi^{n}_{C_1 \cdot C_2})"'] & H[T(Y)/(C_1 \cdot C_2)_{T(Y)}(\operatorname{Im} T(n))].
\end{tikzcd}$$

Now it is obvious that $\operatorname{Ker}[H(\pi^{\kappa}_{C_1}) \cdot \Psi_Y] \subseteq \operatorname{Ker}[H(\pi^{n}_{C_1 \cdot C_2}) \cdot \Psi_Y]$, which by definition means that $(C_1^*)_Y[(C_2^*)_Y(N)] \subseteq (C_1 \cdot C_2)^*_Y(N)$ for every $N \subseteq Y$. Therefore $C_1^* \cdot C_2^* \leqslant (C_1 \cdot C_2)^*$. □

Similar statement takes place for the mapping $D \mapsto D^*$.

Proposition 6.2. *For every closure operators $D_1, D_2 \in \mathbb{CO}(S)$ the relation $(D_1 \cdot D_2)^* \leqslant D_1^* \cdot D_2^*$ is true.*

Proof. Let $D_1, D_2 \in \mathbb{CO}(S)$ and $m\colon M \xrightarrow{\subseteq} X$ be an arbitrary inclusion of R-Mod. We apply the mapping $\mathbb{CO}(S) \xrightarrow{(-)^*} \mathbb{CO}(R)$ of Definition 2 in the following three cases.

1) For the product $D_1 \cdot D_2$ and inclusion m:

$$(D_1 \cdot D_2)^*_X(M) = \operatorname{Im}[\Phi_X \cdot T(i^m_{D_1 \cdot D_2})] + M,$$

where $i^m_{D_1 \cdot D_2}\colon (D_1 \cdot D_2)_{H(X)}\big(\operatorname{Im} H(m)\big) \xrightarrow{\subseteq} H(X)$.

2) For the operator D_2 and inclusion m:

$$(D_2)^*_X(M) = \operatorname{Im}[\Phi_X \cdot T(i^m_{D_2})] + M,$$

where $i^m_{D_2}\colon (D_2)_{H(X)}\big(\operatorname{Im} H(m)\big) \xrightarrow{\subseteq} H(X)$.

3) For the operator D_1 and inclusion $\kappa\colon (D_2)^*_X(M) \xrightarrow{\subseteq} H(X)$:

$$(D_1)^*_X[(D_2)^*_X(M)] = \operatorname{Im}[\Phi_X \cdot T(i^{\kappa}_{D_1})] + M,$$

where $i^{\kappa}_{D_1}\colon (D_1)_{H(X)}\big(\operatorname{Im} H(\kappa)\big) \xrightarrow{\subseteq} H(X)$.

From the definition of $(D_2)^*_X(M)$ we have in R-Mod the situation:

$$\begin{tikzcd}
T\big[(D_2)_{H(X)}(\operatorname{Im} H(m))\big] \arrow[r, "T(i^m_{D_2})"] \arrow[drr, dashed, "f"] & TH(X) \arrow[r, "\Phi_X"] & X \\
 & & (D_2)^*_X(M) = \operatorname{Im}[\Phi_X \cdot T(i^m_{D_2})] + M, \arrow[u, "\kappa"', "\cup|"]
\end{tikzcd}$$

where f is the restriction of $\Phi_X \cdot T(i^m_{D_2})$ to $(D_2)^*_X(M)$, i.e. $\kappa \cdot f = \Phi_X \cdot T(i^m_{D_2})$.

Applying T we obtain in R-Mod the diagram:

$$\begin{tikzcd}
HT\big[\big(D_2\big)_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \arrow[r, "HT(i^m_{D_2})"] \arrow[drr, dashed, "H(f)"] & HTH(X) \arrow[r, "H(\Phi_X)"] & H(X) \\
\big(D_2\big)_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[u, "\Psi_{(D_2)_{H(X)}(\operatorname{Im} H(m))}"] \arrow[r, "i^m_{D_2}", "\subseteq"'] & H(X) \arrow[u, "\Psi_{H(X)}"] \arrow[ur, dashed, "1_{H(X)}"] & H[(D_2)^*_X(M)]. \arrow[l, "H(\kappa)"] \arrow[u, "H(\kappa)"']
\end{tikzcd}$$

From its commutativity it follows that:

$$H(\Phi_X) \cdot HT(i^m_{D_2}) \cdot \Psi_{(D_2)_{H(X)}(\operatorname{Im} H(m))} = H(\Phi_X) \cdot \Psi_{H(X)} \cdot i^m_{D_2} = 1_{H(X)} \cdot i^m_{D_2} = i^m_{D_2}\,.$$

Therefore $\operatorname{Im} i^m_{D_2} \subseteq \operatorname{Im}[H(\Phi_X) \cdot HT(i^m_{D_2})] = \operatorname{Im}[H(\kappa) \cdot H(f)] \subseteq \operatorname{Im} H(\kappa)$, i.e. $\big(D_2\big)_{H(X)}\big(\operatorname{Im} H(m)\big) \subseteq \operatorname{Im} H(\kappa)$. This relation implies the inclusion:

$$\begin{aligned}
&(D_1 \cdot D_2)_{H(X)}\,(\operatorname{Im} H(m)) \\
&\qquad \overset{\text{def}}{=\!=} \big(D_1\big)_{H(X)}\big[\big(D_2\big)_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \overset{i}{\subseteq} \big(D_1\big)_{H(X)}\big(\operatorname{Im} H(\kappa)\big),
\end{aligned}$$

so in S-Mod we have the situation:

$$\begin{tikzcd}
\big(D_1 \cdot D_2\big)_{H(X)}\big(\operatorname{Im} H(m)\big) \arrow[dr, "i^m_{D_1 \cdot D_2}", "\subseteq"'] \arrow[dd, "i"', "\cap\!\!\mid"] & \\
& H(X), \\
\big(D_1\big)_{H(X)}\big(\operatorname{Im} H(\kappa)\big) \arrow[ur, "i^\kappa_{D_1}", "\subseteq"'] &
\end{tikzcd}$$

which implies in R-Mod the diagram:

$$\begin{tikzcd}
T\big[\big(D_1 \cdot D_2\big)_{H(X)}\big(\operatorname{Im} H(m)\big)\big] \arrow[dr, "T(i^m_{D_1 \cdot D_2})"] \arrow[dd, "T(i)"] & & \\
& TH(X) \arrow[r, "\Phi_X"] & X. \\
T\big[\big(D_1\big)_{H(X)}\big(\operatorname{Im} H(\kappa)\big)\big] \arrow[ur, "T(i^\kappa_{D_1})"] & &
\end{tikzcd}$$

Now it is clear that $\mathrm{Im}[\Phi_X \cdot T(i^m_{D_1 \cdot D_2})] \subseteq \mathrm{Im}[\Phi_X \cdot T(i^\kappa_{D_1})]$. Adding M to both parts, by definition we have $(D_1 \cdot D_2)^*_X(M) \subseteq (D_1^* \cdot D_2^*)_X(M)$ for every $X \subseteq M$, therefore $(D_1 \cdot D_2)^* \leqslant D_1^* \cdot D_2^*$. □

References

[1] L. Bican, P. Jambor, T. Kepka , P. Nemec, *Preradicals and change of rings,* Comment. Math. Carolinae, **16**, No.2, 1975, pp. 201–217.

[2] A. I. Kashu, *Preradicals in adjoint situation,* Mat. Issled., vyp. **48**, 1978, pp. 48–64 (in Russian).

[3] A. I. Kashu, *On correspondence of preradicals and torsions in adjoint situation,* Mat. Issled., vyp. **56**, 1980, pp. 62–84 (in Russian).

[4] A. I. Kashu, *Radicals and torsions in modules*, Kishinev, Ştiinţa, 1983 (in Russian).

[5] A. I. Kashu, *Radicals of modules and adjoint functors* (preprint), Academy of Sciences of MSSR, Institute of Mathematics. Kishinev, 1984 (in Russian).

[6] A. I. Kashu, *Functors and torsions in categories of modules*, Academy of Sciences of RM, Institute of Mathematics. Kishinev, 1997 (in Russian).

[7] D. Dikranjan, E. Giuli, *Factorizations, injectivity and compactness in categories of modules*, Commun. in Algebra, v. **19**, No.1, 1991, pp. 45–83.

[8] D. Dikranjan, E. Giuli, *Closure operators* I, Topology and its applications, v. **27**, 1987, pp. 129–143.

[9] D. Dikranjan, W. Tholen, *Categorical structure of closure operators*, Kluwer Academic Publishers, 1995.

[10] A. I. Kashu, *Closure operators in the categories of modules, Part* I, Algebra and Discrete Math., v. **15** (2013), No.2, pp. 213–228.

[11] A. I. Kashu, *Closure operators in the categories of modules, Part* II, Algebra and Discrete Math., v. **16** (2013), No.1, pp. 81–95.

[12] A. I. Kashu, *Closure operators in the categories of modules,Part* III, Bulet. Acad. Şt. RM, Matematica, No.1(74), 2014, pp. 90–100.

[13] A. I. Kashu, *Closure operators in the categories of modules, Part* IV, Bulet. Acad. Şt. RM, Matematica, No.3(76), 2014, pp. 13–22.

Contact information

A. I. Kashu

Institute of Mathematics and Computer Science, Academy of Sciences of Moldova,
5 Academiei str., Chişinău,
MD – 2028 MOLDOVA
E-Mail(s): alexei.kashu@math.md

Received by the editors: 07.07.2017.