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On p-nilpotency of finite group with normally embedded maximal subgroups of some Sylow subgroups

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Abstract. Let G be a finite group and P be a p-subgroup of G. If P is a Sylow subgroup of some normal subgroup of G, then we say that P is normally embedded in G. Groups with normally embedded maximal subgroups of Sylow p-subgroup, where $(|G|, p-1) = 1$, are studied. In particular, the p-nilpotency of such groups is proved.

Introduction

All groups considered in this paper will be finite. Our notation is standard and taken mainly from [1], [2].

Let $\mathcal{M}(G)$ be the set of all maximal subgroups of Sylow subgroups of a group G. One of the first results related to the study of the structure of a group with given restrictions on $\mathcal{M}(G)$ belongs to Srinivasan, see [3]. In particular, in [3] it is proved that a group G is supersolvable, if every subgroup of $\mathcal{M}(G)$ is normal in G. Subsequently, groups with restrictions on subgroups of $\mathcal{M}(G)$ have been studied in the works of many authors, see the literature in [4].

A subgroup H of G is said to be S-embedded in G, see [5], if G has a normal subgroup N such that HN is S-permutable in G and

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$H \cap N \leqslant H_{sG}$, where H_{sG} is the largest S-permutable subgroup of G contained in H. In the paper [5] the structure of the groups depending on S-embedded subgroups is studied. In particular, the p-nilpotency of a group G for which every subgroup of $\mathcal{M}(P)$ is S-embedded in G, where P is a Sylow p-subgroup of G and $p \in \pi(G)$ such that $(|G|, p-1) = 1$ follows from [5, Theorem 2.3].

In the present paper, we study another generalization of normality.

Definition. A subgroup H of a group G is said normally embedded in G, if for every Sylow subgroup P of H, there is a normal subgroup K of G such that P is Sylow subgroup of K, see [6, I.7.1].

A series of results related to the structure of a group with normally embedded subgroups is presented in [6].

The following examples show that S-embedded and normally embedded are different concepts.

In the symmetric group S_5 of degree 5 some maximal subgroup H of a Sylow 2-subgroup is a Sylow 2-subgroup in the normal alternating subgroup A_5 of degree 5, i.e. H is normally embedded in S_5. But, H is not S-embedded. In the alternating group A_4 of degree 4 some maximal subgroup M of a Sylow 2-subgroup is not normally embedded in A_4. But, M is S-embedded.

In this paper, the structure of a group G under the condition that every subgroup of $\mathcal{M}(P)$ is normally embedded in G is studied, where P is a Sylow p-subgroup of G and $p \in \pi(G)$ such that $(|G|, p-1) = 1$.

The following theorem is proved.

Theorem. *Let G be a group, H be a normal subgroup of G such that G/H is p-nilpotent and P be a Sylow p-subgroup of H, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If every subgroup of $\mathcal{M}(P)$ is normally embedded in G, then G is p-nilpotent.*

1. Preliminaries

In this section we collect lemmas used in the proof of the main theorem presented in Section 2.

The Fitting subgroup and the Frattini subgroup of G are denoted by $F(G)$ and $\Phi(G)$, respectively; we write Z_m for a cyclic group of orders m; $\mathrm{O}_p(G)$ and $\mathrm{O}_{p'}(G)$ denote the greatest normal p-subgroup of G and the greatest normal p'-subgroup of G, respectively. By $\pi(G)$ denote the set of all prime divisors of the order of G; by H^G denote the normal closure of a subgroup H in a group G, i.e. the smallest normal subgroup of G containing H. We write $H \,\mathrm{ne}\, G$ for normally embedded subgroup H of G and $G = [A]B$ for the semidirect product of some subgroups A and B with the normal subgroup A.

If the orders of chief factors of G are either equal to p or not divisible on p then G is called p-supersolvable. We denote by $p\mathfrak{U}$ the class of all p-supersolvable groups. A group that has a normal Sylow p-subgroup is called p-closed and a group that has a normal p'-Hall subgroup is called p-nilpotent.

Let G be a group of order $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, where $p_1 > p_2 > \ldots > p_k$. We say that G has an ordered Sylow tower of supersolvable type if there exists a series

$$1 = G_0 < G_1 < G_2 < \ldots < G_{k-1} < G_k = G$$

of normal subgroups of G such that G_i/G_{i-1} is isomorphic to a Sylow p_i-subgroup of G for each $i = 1, 2, \ldots, k$.

Lemma 1 ([6, I.7.3]). *Let U be a normally embedded p-subgroup of a group G, K a normal subgroup of G. Then:*

(1) *if $U \leqslant H \leqslant G$, then $U \,\mathrm{ne}\, H$;*
(2) *$UK/K \,\mathrm{ne}\, G/K$;*
(3) *$U \cap K \,\mathrm{ne}\, G$;*
(4) *if K is a p-group, then $UK \,\mathrm{ne}\, G$ and $U \cap K$ is normal in G;*
(5) *$U^g \,\mathrm{ne}\, H$ for all $g \in G$.*

Lemma 2. *Let H be a normal subgroup of G and every maximal subgroup of Sylow p-subgroup of H is normally embedded in G. If N is normal in G, then every maximal subgroup of every Sylow p-subgroup of HN/N is normally embedded in G/N. In particular, if N is normal in G and every maximal subgroup of Sylow p-subgroup of G is normally embedded in G, then every maximal subgroup of every Sylow p-subgroup of G/N is normally embedded in G/N.*

Proof. By Lemma 1 (5), it follows that X_1 is normally embedded in G for any Sylow p-subgroup X of H and any maximal subgroup X_1 of X. Let $\overline{P_1} = X/N$ is a maximal subgroup of Sylow p-subgroup $\overline{P}$ of HN/N. Then $N \leqslant X \leqslant HN$ and there exists a Sylow p-subgroup P in HN such that $\overline{P} = PN/N$. By [1, VI.4.6], there exist the Sylow p-subgroups H_p in H and N_p in N such that $P = H_pN_p$, hence $\overline{P} = H_pN/N$. Further,

$N \leqslant X < PN \leqslant H_pN$ and $X = (X \cap H_p)N$ by Dedekind's identity. Since $H_p \cap N = X \cap H_p \cap N$, we have

$$p = |\overline{P} : \overline{P_1}| = |H_pN/N : X/N| = |H_pN : X|$$
$$= |H_pN : (X \cap H_p)N| = \frac{|H_p||N||X \cap H_p \cap N|}{|H_p \cap N||X \cap H_p||N|} = |H_p : X \cap H_p|.$$

So, $X \cap H_p$ is a maximal subgroup in H_p. By hypothesis, $X \cap H_p$ is normally embedded in G. By Lemma 1 (2), $(X \cap H_p)N/N = X/N$ is normally embedded in G/N.

For $H = G$ we obtain the second part of the lemma. □

Lemma 3 ([7, Lemma 5]). *Let G be a p-solvable group. Assume that G does not belong to $p\mathfrak{U}$, but $G/K \in p\mathfrak{U}$ for all non-trivial normal subgroups K of G. Then:*

(1) $Z(G) = \mathrm{O}_{p'}(G) = \Phi(G) = 1$;
(2) *G contains a unique minimal normal subgroup N, $N = F(G) = \mathrm{O}_p(G) = C_G(N)$;*
(3) *G is primitive; $G = [N]M$, where M is maximal in G with trivial core;*
(4) *N is an elementary Abelian subgroup of order p^n, $n > 1$;*
(5) *if M is Abelian, then M is cyclic of order dividing $p^n - 1$, and n is the smallest natural number such that $p^n \equiv 1 \pmod{|M|}$.*

A non-nilpotent group whose proper subgroups are all nilpotent is called a Schmidt group.

Lemma 4 ([8]). *Let S be a Schmidt group. Then:*

(1) *$S = [P]Q$, where P is a normal Sylow p-subgroup, Q is a non-normal Sylow q-subgroup, p and q are distinct primes;*
(2) *$Q = \langle y \rangle$ is cyclic and $y^q \in Z(S)$;*
(3) *$|P/P'| = p^m$, where m is the order of p modulo q;*
(4) *the chief series of S has the following system of indexes: $p, p, \dots, p, p^m, q, \dots, q$; number of indexes equal to p coincides with n, where $p^n = |P'|$; number of indexes equal to q coincides with b, where $q^b = |Q|$.*

Lemma 5. *Let $p \in \pi(G)$ and $(|G|, p-1) = 1$. Then G is p-supersolvable if and only if G is p-nilpotent. In particular, if a Sylow p-subgroup is cyclic, then G is p-nilpotent.*

Proof. It is clear that every p-nilpotent group is p-supersolvable. Conversely. Let G be a group of the smallest order such that G is p-supersolvable, but is not p-nilpotent. Let H be an arbitrary proper subgroup of G. Then H is p-supersolvable and $(|H|, p-1) = 1$. Therefore in view of the choice G, the subgroup H is p-nilpotent and G is a minimal non-p-nilpotent group. By [9, Theorem 10.3.3], G is a Schmidt group. By Lemma 4 (1), $G = [P]Q$, where P is a Sylow p-subgroup and Q is a cyclic Sylow q-subgroup. Since G is p-supersolvable, then by Lemma 4 (4), the order of p modulo q is equal 1, i.e. $m = 1$. Hence q divides $p - 1$. This is a contradiction.

In particular, if a Sylow p-subgroup is cyclic, then G is p-supersolvable. Then G is p-nilpotent by what has been proved above. The lemma is proved. □

Corollary 1. *Let p be the smallest prime of $\pi(G)$. Then G is p-supersolvable if and only if G is p-nilpotent.*

Example 1. The symmetric group $G = S_3$ of degree 3 is 3-supersolvable, but is not 3-nilpotent. Hence, the condition $(|G|, p-1) = 1$ in Lemma 5 can not be removed.

Example 2. A group $G = Z_5 \times ([Z_7]Z_3)$ is 5-supersolvable and is 5-nilpotent. In addition, $(|G|, 5-1) = 1$, and the prime divisor 5 of $|G|$ is not the smallest.

Evidently, if a p-subgroup P of G is normally embedded in G, then P is a Sylow subgroup of P^G.

Lemma 6. *Let G be a group, $\Phi(G) = 1$, P be a Sylow subgroup of G with unprimary order and N be a unique minimal normal subgroup of G. If every subgroup of $\mathcal{M}(P)$ is normally embedded in G and N is Abelian, then N is not contained in P.*

Proof. Suppose that $N \leqslant P$. If $N = P$, then by hypothesis, every maximal subgroup S of P is normally embedded in G. Then by Lemma 1 (4), S is normal in G. Since the order of P is not equal to a prime, we have a contradiction with the fact that N is a minimal normal subgroup in G.

In the following we assume that $N < P$. Since $\Phi(G) = 1$, it follows that there exists a maximal subgroup M of G such that N is not contained in M. Hence $G = NM$. By [2, Lemma 2.36], $N \cap M = 1$ and $G = [N]M$. Then by Dedekind's identity, $P = P \cap [N]M = [N](P \cap M)$, where

$P \cap M \neq 1$. Let T be a maximal subgroup of P such that $P \cap M \leqslant T$. Since N is a unique minimal normal subgroup of G, it follows that $N \leqslant T^G$. Now, $P = NT \leqslant T^G$, but by hypothesis, T is a Sylow subgroup of T^G, a contradiction. □

Lemma 7. *Let P be a Sylow p-subgroup of G. If every subgroup of $\mathcal{M}(P)$ is normally embedded in G and $(|G|, p-1) = 1$, then G is p-nilpotent.*

Proof. We use induction on the order of G. Since $(|G/N|, p-1) = 1$ and by Lemma 2, every maximal subgroup of every Sylow p-subgroup of G/N is normally embedded in G/N for any normal subgroup $N \neq 1$ of G, then all quotients of G satisfy the hypotheses of the lemma.

By the inductive hypothesis, $O_{p'}(G) = 1$. Since the class of all p-nilpotent groups is a saturated formation, then $\Phi(G) = 1$ and $N = F(G) = O_p(G)$ is a unique minimal normal subgroup G. Hence there is a Sylow p-subgroup R of G such that $N \subseteq R$. Since R and P are conjugate in G, then by Lemma 1 (5), it follows that every maximal subgroup of R is normally embedded in G. If $|R| = p$, then G is p-nilpotent by Lemma 5. Therefore, we further assume that $|R| > p$. By Lemma 6, N is not contained in R. This is a contradiction. The lemma is proved. □

2. Proof of the theorem

In view of Lemma 5, we prove that G is p-supersolvable.

By Lemma 1 (1), every maximal subgroup of Sylow p-subgroup P of H is normally embedded in H and $(|H|, p-1) = 1$. By Lemma 7, H is p-nilpotent. Since by hypothesis, G/H is p-nilpotent, then G is p-solvable.

We use induction on the order of G. Let N be an arbitrary non-trivial normal subgroup of G. Clearly, HN/N is normal in G/N and

$$(G/N)/(HN/N) \cong G/(HN) \cong (G/H)/(HN/H)$$

is p-nilpotent. Besides, by Lemma 2, every maximal subgroup of every Sylow p-subgroup of HN/N is normally embedded in G/N and $(|G/N|, p-1) = 1$. Hence the quotients G/N satisfy the hypotheses of the theorem.

By the inductive hypothesis, G/N is p-supersolvable. By Lemma 3, $Z(G) = O_{p'}(G) = \Phi(G) = 1$, G contains a unique minimal normal subgroup

$$N = F(G) = O_p(G) = C_G(N), \ G = [N]M,$$

N is an elementary Abelian subgroup of order p^n, $n > 1$, M is a maximal subgroup of G.

Since $N \leqslant H$, then N is contained in every Sylow p-subgroup P of H. By Lemma 6, we have a contradiction. The theorem is proved.

Corollary 2. *Let G be a group, H be a normal subgroup of group G such that G/H is p-nilpotent and P be a Sylow p-subgroup of H, where p is the smallest in $\pi(G)$. If every subgroup of $\mathcal{M}(P)$ is normally embedded in G, then G is p-nilpotent.*

Corollary 3. *Let G be a group and P be a Sylow p-subgroup of G, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If every subgroup of $\mathcal{M}(P)$ is normally embedded in G, then G is p-nilpotent.*

Corollary 4. *Let G be a group and P be a Sylow p-subgroup of G, where p is the smallest in $\pi(G)$. If every subgroup of $\mathcal{M}(P)$ is normally embedded in G, then G is p-nilpotent.*

Corollary 5. *Let G be a group. If every subgroup of $\mathcal{M}(G)$ is normally embedded in G, then G possesses an ordered Sylow tower of supersolvable type.*

Proof. Let p be the smallest prime of $\pi(G)$ and P be a Sylow p-subgroup of G. Then by hypothesis, every subgroup of $\mathcal{M}(P)$ is normally embedded in G. By Corollary 4, G is p-nilpotent. By Lemma 1 (1) and the inductive hypothesis, a Hall p'-subgroup of G has an ordered Sylow tower of supersolvable type. Consequently, G has an ordered Sylow tower of supersolvable type. □

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