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Some remarks on Φ-sharp modules

Ahmad Yousefian Darani and Mahdi Rahmatinia

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Abstract. The purpose of this paper is to introduce some new classes of modules which is closely related to the classes of sharp modules, pseudo-Dedekind modules and TV-modules. In this paper we introduce the concepts of Φ-sharp modules, Φ-pseudo-Dedekind modules and Φ-TV-modules. Let R be a commutative ring with identity and set $\mathbb{H} = \{M \mid M$ is an R-module and $\mathrm{Nil}(M)$ is a divided prime submodule of $M\}$. For an R-module $M \in \mathbb{H}$, set $T = (R \setminus Z(M)) \cap (R \setminus Z(R))$, $\mathfrak{T}(M) = T^{-1}(M)$ and $P := (\mathrm{Nil}(M) :_R M)$. In this case the mapping $\Phi : \mathfrak{T}(M) \longrightarrow M_P$ given by $\Phi(x/s) = x/s$ is an R-module homomorphism. The restriction of Φ to M is also an R-module homomorphism from M in to M_P given by $\Phi(m/1) = m/1$ for every $m \in M$. An R-module $M \in \mathbb{H}$ is called a Φ-sharp module if for every nonnil submodules N, L of M and every nonnil ideal I of R with $N \supseteq IL$, there exist a nonnil ideal $I' \supseteq I$ of R and a submodule $L' \supseteq L$ of M such that $N = I'L'$. We prove that Many of the properties and characterizations of sharp modules may be extended to Φ-sharp modules, but some can not.

1. Introduction

We assume throughout this paper all rings are commutative with $1 \neq 0$ and all modules are unitary. An element x of an integral domain R is called primal if whenever $x \mid y_1y_2$, with $x, y_1, y_2 \in R$, then $x = z_1z_2$ where $z_1 \mid y_1$ and $z_2 \mid y_2$. Cohn in [18] introduced the concept of Schreier domains.

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An integral domain R is called a pre-Schreier domain if every nonzero element of R is primal. If in addition R is integrally closed, then R is called a Schreier domain. In [27], Z. Ahmad, T. Dumitrescu and M. Epure introduced the notion of sharp domains. A domain R is said to be a sharp domain if whenever $I \supseteq AB$ with I, A, B nonzero ideals of R, then there exist ideals $A' \supseteq A$ and $B' \supseteq B$ such that $I = A'B'$. Let R be a ring with identity and $\mathrm{Nil}(R)$ be the set of nilpotent elements of R. Recall from [19] and [12], that a prime ideal P of R is called a divided prime ideal if $P \subset (x)$ for every $x \in R \backslash P$; thus a divided prime ideal is comparable to every ideal of R. Badawi in [9], [10], [12], [13], [14] and [15], the scond-named author investigated the class of rings $\mathcal{H} = \{R \mid R$ is a commutative ring with $1 \neq 0$ and $\mathrm{Nil}(R)$ is a divided prime ideal of $R\}$. Anderson and Badawi in [6] and [7] generalized the concept of Prüfer, Dedekind, Krull and Bezout domain to context of rings that are in the class $\mathcal{H}$. Lucas and Badawi in [11] generalized the concept of Mori domains to the context of rings that are in the class $\mathcal{H}$. Also, authors this paper in [25] generalized the concept of sharp domains to the context of rings that are in the class $\mathcal{H}$. Let R be a ring, $Z(R)$ the set of zero divisors of R and $S = R \setminus Z(R)$. Then $T(R) := S^{-1}R$ denoted the total quotient ring of R. We start by recalling some background material. A nonzero divisor of a ring R is called a regular element and an ideal of R is said to be regular if it contains a regular element. An ideal I of a ring R is said to be a nonnil ideal if $I \nsubseteq \mathrm{Nil}(R)$. If I is a nonnil ideal of $R \in \mathcal{H}$, then $\mathrm{Nil}(R) \subset I$. In particular, it holds if I is a regular ideal of a ring $R \in \mathcal{H}$. Recall from [6] that for a ring $R \in \mathcal{H}$, the map $\phi : T(R) \longrightarrow R_{\mathrm{Nil}(R)}$ given by $\phi(a/b) = a/b$, for $a \in R$ and $b \in R \setminus Z(R)$, is a ring homomorphism from $T(R)$ into $R_{\mathrm{Nil}(R)}$ and ϕ resticted to R is also a ring homomorphism from R into $R_{\mathrm{Nil}(R)}$ given by $\phi(x) = x/1$ for every $x \in R$. Let $R \in \mathcal{H}$. Then R is called a ϕ-sharp ring if whenever for nonnil ideals I, A, B of R with $I \supseteq AB$, then $I = A'B'$ for nonnil ideals A', B' of R where $A' \supseteq A$ and $B' \supseteq B$ [25].

For a nonzero ideal I of R let $I^{-1} = \{x \in T(R) : xI \subseteq R\}$. It is obvious that $II^{-1} \subseteq R$. An ideal I of R is called invertible, if $II^{-1} = R$. The ν-closure of I is the ideal $I_\nu = (I^{-1})^{-1}$ and I is called divisorial ideal (or $\nu-$ ideal) if $I_\nu = I$. A nonzero ideal I of R is called t-ideal if $I = I_t$ in which

$$I_t = \bigcup \{J_\nu \mid J \subseteq I \text{ is a nonzero finitely generated ideal of } R\}.$$

Let $R \in \mathcal{H}$. Then a nonnil ideal I of R is called ϕ-invertible if $\phi(I)$ is an invertible ideal of $\phi(R)$. A nonnil ideal I is ϕ-ν-ideal if $\phi(I)$ is a ν-ideal of $\phi(R)$ [11]. A nonnil ideal I of R is a ϕ-t-ideal if $\phi(I)$ is a t-ideal of

$\phi(R)$ [25]. Let $R \in \mathcal{H}$. Then R is called a ϕ-pseudo-Dedekind ring if the ν-closure of each nonnil ideal of R is ϕ-invertible. Also, R is said to be a ϕ-TV ring in which every ϕ-t-ideal is a ϕ-ν-ideal [25].

Let R be a ring and M be an R-module. Then M is a multiplication R-module if every submodule N of M has the form IM for some ideal I of R. If M be a multiplication R-module and N a submodule of M, then $N = IM$ for some ideal I of R. Hence $I \subseteq (N :_R M)$ and so $N = IM \subseteq (N :_R M)M \subseteq N$. Therefore $N = (N :_R M)M$ [16]. Let M be a multiplication R-module, $N = IM$ and $L = JM$ be submodules of M for fome ideals I and J of R. Then, the product of N and L is denoted by $N.L$ or NL and is defined by IJM [5]. An R-module M is called a cancellation module if $IM = JM$ for two ideals I and J of R implies $I = J$ [1]. By [21, Corollary 1 to Theorem 9], finitely generated faithful multiplication modules are cancellation modules. It follows that if M is a finitely generated faithful multiplication R-module, then $(IN :_R M) = I(N :_R M)$ for all ideals I of R and all submodules N of M. If R is an integral domain and M a faithful multiplication R-module, then M is a finitely generated R-module [17]. Let M be an R-module and set

$$\begin{aligned} T &= \{t \in S : \text{ for all } m \in M,\, tm = 0 \text{ implies } m = 0\} \\ &= (R \setminus Z(M)) \cap (R \setminus Z(R)). \end{aligned}$$

Then T is a multiplicatively closed subset of R with $T \subseteq S$, and if M is torsion-free then $T = S$. In particular, $T = S$ if M is a faithful multiplication R-module [17, Lemma 4.1]. Let N be a nonzero submodule of M. Then we write $N^{-1} = (M :_{R_T} N) = \{x \in R_T : xN \subseteq M\}$. Then N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. We say that N is invertible in M if $NN^{-1} = M$. Clearly $0 \neq M$ is invertible in M. An R-module M is called a Dedekind module if every nonzero submodule of M is invertible [20]. An R-module M is called a valuation module if for all $m, n \in M$, either $Rm \subseteq Rn$ or $Rn \subseteq Rm$. Equivalently, M is a valuation module if for all submodules N and K of M, either $N \subseteq K$ or $K \subseteq N$ [3]. The $\nu-$closure of N is the submodule $N_\nu = (N^{-1})^{-1}$ and N is called $\nu-$submodule if $N = N_\nu M$ [23] and [3]. If M is a finitely generated faithful multiplication R-module, then $N_\nu = (N :_R M)$. Consequently, $M_\nu = R$. Let M be a finitely generated faithful multiplication R-module, N a submodule of M and I an ideal of R. Then N is a ν-submodule of M if and only if $(N :_R M)$ is a ν-ideal of R. Also I is ν-ideal of R if and only if IM is a ν-submodule of M [2]. If N is an invertible submodule of a faithful multiplication module M over an integral domain R, then $(N :_R M)$ is invertible and hence is a ν-ideal of R. So N is a ν-submodule of M [2]. If R is an integral domain, M a faithful multiplication R-module and N a nonzero submodule of M, then $N_\nu = (N :_R M)_\nu$ [2, Lemma 1].

Let M be an R-module. An element $r \in R$ is said to be zero divisor on M if $rm = 0$ for some $0 \neq m \in M$. The set of zero divisors of M is denoted by $Z_R(M)$ (briefly, $Z(M)$). It is easy to see that $Z(M)$ is not necessarily an ideal of R, but it has the property that if $a, b \in R$ with $ab \in Z(M)$, then either $a \in Z(M)$ or $b \in Z(M)$. A submodule N of M is called a nilpotent submodule if $[N :_R M]^n N = 0$ for some positive integer n. An element $m \in M$ is said to be nilpotent if Rm is a nilpotent submodule of M [4]. We let $\mathrm{Nil}(M)$ to denote the set of all nilpotent elements of M; then $\mathrm{Nil}(M)$ is a submodule of M provided that M is a faithful module, and if in addition M is multiplication, then $\mathrm{Nil}(M) = \mathrm{Nil}(R)M = \bigcap P$, where the intersection runs over all prime submodules of M, [4, Theorem 6]. If M contains no nonzero nilpotent elements, then M is called a reduced R-module. A submodule N of M is said to be a nonnil submodule if $N \nsubseteq \mathrm{Nil}(M)$. Recall that a submodule N of M is prime if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $rM \subseteq N$. If N is a prime submodule of M, then $p := [N :_R M]$ is a prime ideal of R. In this case we say that N is a p-prime submodule of M. Let N be a submodule of multiplication R-module M, then N is a prime submodule of M if and only if $[N :_R M]$ is a prime ideal of R if and only if $N = pM$ for some prime ideal p of R with $[0 :_R M] \subseteq p$, [17, Corollary 2.11]. Recall from [3] that a prime submodule P of M is called a divided prime submodule if $P \subset Rm$ for every $m \in M \setminus P$; thus a divided prime submodule is comparable to every submodule of M.

Now assume that $T^{-1}(M) = \mathfrak{T}(M)$. Set

$$\mathbb{H} = \{M \mid M \text{ is an } R\text{-module and } \mathrm{Nil}(M) \text{ is a divided prime submodule of } M\}.$$

For an R-module $M \in \mathbb{H}$, $\mathrm{Nil}(M)$ is a prime submodule of M. So $P := [\mathrm{Nil}(M) :_R M]$ is a prime ideal of R. If M is an R-module and $\mathrm{Nil}(M)$ is a proper submodule of M, then $[\mathrm{Nil}(M) :_R M] \subseteq Z(R)$. Consequently, $R \setminus Z(R) \subseteq R \setminus [\mathrm{Nil}(M) :_R M]$. In particular, $T \subseteq R \setminus [\mathrm{Nil}(M) :_R M]$ [22]. Recall from [22] that we can define a mapping $\Phi : \mathfrak{T}(M) \longrightarrow M_P$ given by $\Phi(x/s) = x/s$ which is clearly an R-module homomorphism. The restriction of Φ to M is also an R-module homomorphism from M in to M_P given by $\Phi(m/1) = m/1$ for every $m \in M$. A nonnil submodule

N of M is said to be Φ-invertible if $\Phi(N)$ is an invertible submodule of $\Phi(M)$ [26]. Let $M \in \mathbb{H}$. Then M is a Φ-Dedekind R-module if every nonnil submodule of M is Φ-invertible [26]. In this paper we introduce a generalization of ϕ-sharp rings and give some properties of this class of modules.

2. Φ-sharp modules

Definition 2.1. Let R be a ring and $M \in \mathbb{H}$ be an R-module. Then M is called a Φ-sharp module if for every nonnil submodules N, L of M and every nonnil ideal I of R with $N \supseteq IL$, there exist a nonnil ideal $I' \supseteq I$ of R and a submodule $L' \supseteq L$ of M such that $N = I'L'$.

Theorem 2.2. *Let R be a ring and $M \in \mathbb{H}$ with $\operatorname{Nil}(M) = Z(R)M$. Then M is a Φ-sharp module if and only if $M/\operatorname{Nil}(M)$ is a sharp module.*

Proof. Since $\operatorname{Nil}(M) = Z(R)M$, then $\operatorname{Nil}(R) = (\operatorname{Nil}(M) :_R M) = (Z(R)M :_R M) = Z(R)$ by [22, Proposition 1]. Let M be a Φ-sharp module and let $N/\operatorname{Nil}(M), L/\operatorname{Nil}(M)$ be nonzero submodules of $M/\operatorname{Nil}(M)$ and I be a nonzero ideal of R with $N/\operatorname{Nil}(M) \supseteq I(L/\operatorname{Nil}(M))$. Then $N \supseteq IL$ and so there exist a nonnil ideal $I' \supseteq I$ of R and a submodule $L' \supseteq L$ of M such that $N = I'L'$. Thus $N/\operatorname{Nil}(M) = I'((L'/\operatorname{Nil}(M))$ for nonzero ideal $I' \supseteq I$ of R and for a nonzero submodule $L/\operatorname{Nil}(M) \supseteq L'/\operatorname{Nil}(M)$ of $M/\operatorname{Nil}(M)$ as well.

Conversely, let $M/\operatorname{Nil}(M)$ be a sharp module and let N, L be nonnil submodules of M and I a nonnil ideal of R such that $N \supseteq IL$. Then $N/\operatorname{Nil}(M), L/\operatorname{Nil}(M)$ are nonzero submodules of $M/\operatorname{Nil}(M)$ and I is a nonzero ideal of R with $N/\operatorname{Nil}(M) \supseteq I(L/\operatorname{Nil}(M))$. So, $N/\operatorname{Nil}(M) = I'((L'/\operatorname{Nil}(M))$ for nonzero ideal $I' \supseteq I$ of R and for a nonzero submodule $L/\operatorname{Nil}(M) \supseteq L'/\operatorname{Nil}(M)$ of $M/\operatorname{Nil}(M)$. Therefore $N = I'L'$ for a nonnil ideal $I' \supseteq I$ of R and for a submodule $L' \supseteq L$ of M. Thus M is a Φ-sharp module. □

Lemma 2.3. *([26, Lemma 2.6]) Let R be a ring and M a finitely generated faithful multiplication R-module with $M \in \mathbb{H}$. Then $\frac{M}{\operatorname{Nil}(M)}$ is isomorphic to $\frac{\Phi(M)}{\operatorname{Nil}(\Phi(M))}$ as R-module.*

Corollary 2.4. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $\operatorname{Nil}(M) = Z(R)M$. Then M is a Φ-sharp module if and only if $\frac{\Phi(M)}{\operatorname{Nil}(\Phi(M))}$ is a sharp module.*

Theorem 2.5. *Let R be a ring and $M \in \mathbb{H}$ with $\mathrm{Nil}(M) = Z(R)M$. Then M is a Φ-sharp module if and only if $\Phi(M)$ is a sharp module.*

Proof. Let M be a Φ-sharp module and let $\Phi(N) \supseteq I\Phi(L)$ for nonnil submodules N, L of M and nonnil ideal I of R. Since $\mathrm{Nil}(M)$ is a divided prime submodule of M and N, L properly contain $\mathrm{Nil}(M)$, so both contain $Ker(\Phi)$ by [26, Propoition 2.1]. Therefore $N \supseteq IL$ and hence $N = I'L'$ for a nonnil submodule $L' \supseteq L$ of M and a nonnil ideal $I' \supseteq I$ of R. Thus $\Phi(N) = I'\Phi(L')$ for a submodule $\Phi(L') \supseteq \Phi(L)$ and an ideal $I' \supseteq I$. So $\Phi(M)$ is a sharp module.

Converesly, Let $\Phi(M)$ be a sharp module and let N, L be nonnil submodules of M and I an ideal of R with $N \supseteq IL$. Thus $\Phi(N) \supseteq I\Phi(L)$ and so $\Phi(N) = I'\Phi(L')$ for a submodule $\Phi(L') \supseteq \Phi(L)$ and an ideal $I' \supseteq I$. By the same reason as above, we have $N = I'L'$ for a nonnil submodule $L' \supseteq L$ of M and a nonnil ideal $I' \supseteq I$ of R. Hence M is a Φ-sharp module. □

Corollary 2.6. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $\mathrm{Nil}(M) = Z(R)M$. The following statements are equivalent:*

(1) *M is a Φ-sharp module;*
(2) *$M/\mathrm{Nil}(M)$ is a sharp module;*
(3) *$\frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$ is a sharp module;*
(4) *$\Phi(M)$ is a sharp module.*

Proposition 2.7. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with $\mathrm{Nil}(M) = Z(R)M$. If M is a Φ-Dedekind module, then M is a Φ-sharp module.*

Proof. If M is a Φ-Dedekind module, then $M/\mathrm{Nil}(M)$ is a Dedekind module by [26, Theorem 2.10]. So, by [23, Corollary 3.5], $M/\mathrm{Nil}(M)$ is a sharp module. Therefore, by Theorem 2.2, M is a Φ-sharp module. □

In [26] it is shown that for each prime ideal P of R, $(M/\mathrm{Nil}(M))_P = M_P/(\mathrm{Nil}(M))_P = M_P/\mathrm{Nil}(M_P)$ and $M_P \in \mathbb{H}$.

Proposition 2.8. *Let R be a ring and $M \in \mathbb{H}$ be a Φ-sharp module with $\mathrm{Nil}(M) = Z(R)M$. Then M_P is a Φ-sharp module for each prime ideal P of R.*

Proof. We have $\mathrm{Nil}(R) \subseteq Ann(\frac{M}{\mathrm{Nil}(R)M}) = Ann(\frac{M}{\mathrm{Nil}(M)})$. If M is a Φ-sharp module, then by Theorem 2.2, $M/\mathrm{Nil}(M)$ is a sharp module. So, by

[23, Proposition 3.8], $(M/\operatorname{Nil}(M))_P = M_P/\operatorname{Nil}(M_P)$ is a sharp module. Therefore, by Theorem 2.2, M_P is a Φ-sharp module. □

Theorem 2.9. *Let R be a ring and M be a finitely generated faithful multiplication R-module. The following statements are equivalent:*

(1) *If $R \in \mathcal{H}$ is a ϕ-sharp ring, then M is a Φ-sharp module;*

(2) *If $M \in \mathbb{H}$ is a Φ-sharp module, then R is a ϕ-sharp ring.*

Proof. (1) ⇒ (2) Let $R \in \mathcal{H}$. Then, by [22, Proposition 3], $M \in \mathbb{H}$. Let R be a ϕ-sharp ring and let N, L be nonnil submodules of M and I be a nonnil ideal of R with $N \supseteq IL$. Then $(N :_R M), (L :_R M)$ are nonnil ideals of R such that $(N :_R M) \supseteq I(L :_R M)$. So $(N :_R M) = I'J'$ for nonnil ideals $I' \supseteq I$ and $J' \supseteq (L :_R M)$ of R. Thus $N = I'(J'M)$ for a nonnil ideal $I' \supseteq I$ of R and a nonnil submodule $J'M \supseteq L$ of M. Therefore M is a Φ-sharp module.

(2) ⇒ (1) Let $M \in \mathbb{H}$. Then, by [22, Proposition 3], $R \in \mathcal{H}$. Let M be a Φ-sharp module and let I, J, K be nonnil ideals of R with $K \supseteq IJ$. So KM, JM are nonnil submodules of M such that $KM \supseteq I(JM)$. Thus $KM = I'L'$ for a nonnil ideal $I' \supseteq I$ of R and a nonnil submodule $L' \supseteq JM$ of M. Therefore $K = I'(L' :_R M)$ for nonnil ideals $I' \supseteq I$ and $(L' :_R M) \supseteq J$ of R. So R is a ϕ-sharp ring. □

Definition 2.10. Let R be a ring and M be an R-module. Then M is said to be a Φ-pseudo-Dedekind module if the ν-closure of each nonnil submodule of M is Φ-invertible.

Theorem 2.11. *Let R be a ring and $M \in \mathbb{H}$ be an R-module. Then M is a Φ-pseudo-Dedekind module if and only if $M/\operatorname{Nil}(M)$ is a pseudo-Dedekind module.*

Proof. Let M be a Φ-pseudo-Dedekind module and $N/\operatorname{Nil}(M)$ be a nonzero submodule of $M/\operatorname{Nil}(M)$. Then N is a nonnil submodule of M and so the ν-closure of N is Φ-invertible, i.e, N_ν is Φ-invertible. Thus, by [24, Lemma 3.6], $(N/\operatorname{Nil}(M))_\nu = N_\nu/\operatorname{Nil}(M)$ is invertible as well.

Conversely, let $M/\operatorname{Nil}(M)$ be a pseudo-Dedekind module and N be a nonnil submodule of M. Thus $N/\operatorname{Nil}(M)$ is a nonzero submodule of $M/\operatorname{Nil}(M)$ and so $N_\nu/\operatorname{Nil}(M) = (N/\operatorname{Nil}(M))_\nu$ is invertible. So, by [24, Lemma 3.6], N_ν is Φ-invertible. Therefore, M is a Φ-pseudo-Dedekind module. □

By Lemma 2.3, we have the following theorem.

Corollary 2.12. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module. Then M is a Φ-pseudo-Dedekind module if and only if $\frac{\Phi(M)}{\mathrm{Nil}(\Phi(M))}$ is a pseudo-Dedekind module.*

Theorem 2.13. *Let R be a ring and $M \in \mathbb{H}$ be an R-module. Then M is a Φ-pseudo-Dedekind module if and only if $\Phi(M)$ is a pseudo-Dedekind module.*

Proof. Let M be a Φ-pseudo-Dedekind module and $\Phi(N)$ be a submodule of $\Phi(M)$ for a nonnil submodule N of M. Thus N_ν is Φ-invertible. Hence $\Phi(N_\nu) = (\Phi(N))_\nu$ is invertible.

Conversely, let $\Phi(M)$ be a pseudo-Dedekind module and N be a nonnil submodule of M. Then $\Phi(N)$ is a submodule of $\Phi(M)$ and so $(\Phi(N))_\nu = \Phi(N_\nu)$ is invertible submodule of $\Phi(M)$. Therefore N_ν is Φ-invertible. □

Corollary 2.14. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module. The following are equivalent:*

(1) *M is a Φ-pseudo-Dedekind module;*

(2) *$M/\mathrm{Nil}(M)$ is a pseudo-Dedekind module;*

(3) *$\Phi(M)/\mathrm{Nil}(\Phi(M))$ is a pseudo-Dedekind module;*

(4) *$\Phi(M)$ is a pseudo-Dedekind module.*

Theorem 2.15. *Let R be a ring and M be a finitely generated faithful multiplication R-module. The following statements are equivalent:*

(1) *If $R \in \mathcal{H}$ is a ϕ-pseudo-Dedekind ring, then M is a Φ-pseudo-Dedekind module;*

(2) *If $M \in \mathbb{H}$ is a Φ-pseudo-Dedekind module, then R is a ϕ-pseudo-Dedekind ring.*

Proof. Since $\mathrm{Nil}(R) \subseteq Ann(\frac{M}{\mathrm{Nil}(R)M}) = Ann(\frac{M}{\mathrm{Nil}(M)})$, we have:

(1) ⇒ (2) Let $R \in \mathcal{H}$. Then, by [22, Proposition 3], $M \in \mathbb{H}$. If R is a ϕ-pseudo-Dedekind ring, then by [25, Theorem 2.10], $\frac{R}{\mathrm{Nil}(R)}$ is a pseudo-Dedekind domain. So, by [23, Theorem 3.12], $\frac{M}{\mathrm{Nil}(M)}$ is a pseudo-Dedekind module. Therefore, by Theorem 2.11, M is a Φ-pseudo-Dedekind module.

(2) ⇒ (1) Let $M \in \mathbb{H}$. Then, by [22, Proposition 3], $R \in \mathcal{H}$. If M is a Φ-pseudo-Dedekind module, then by Theorem 2.11, $\frac{M}{\mathrm{Nil}(M)}$ is a pseudo-Dedekind module. So, by [23, Theorem 3.12], $\frac{R}{\mathrm{Nil}(R)}$ is a pseudo-Dedekind domain. Therefore, by [25, Theorem 2.10], R is a ϕ-pseudo-Dedekind ring. □

Proposition 2.16. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module. If M is a Φ-sharp module, then M is a Φ-pseudo-Dedekind module.*

Proof. Let M be a Φ-sharp module. Then, by Theorem 2.2, $M/\operatorname{Nil}(M)$ is a sharp module. So, by [23, Lemma 3.11], $M/\operatorname{Nil}(M)$ is a pseudo-Dedekind module. Therefore, by Theorem 2.11, M is a Φ-pseudo-Dedekind module. □

Recall from [26], an R-module $M \in \mathbb{H}$ is called a Φ-valuation module if for every $u \in R_{(\operatorname{Nil}(R):_R M)}$, we have $u\Phi(M) \subseteq \Phi(M)$ or $u^{-1}\Phi(M) \subseteq \Phi(M)$; equivalently, for every $a, b \notin (\operatorname{Nil}(R) :_R M)$, either, $a\Phi(M) \subseteq b\Phi(M)$ or $b\Phi(M) \subseteq a\Phi(M)$.

Theorem 2.17. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication Φ-valuation R-module. Then the following are equivalent:*

(1) *M is a Φ-sharp module;*

(2) *M is a Φ-pseudo-Dedekind module.*

Proof. (1) $\Rightarrow$ (2) is given by Proposition 2.16.

(2) $\Rightarrow$ (1) Let M is a Φ-pseudo-Dedekind module. Then, by Theorem 2.11, $M/\operatorname{Nil}(M)$ is a pseudo-Dedekind-module. Since M is a Φ-valuation module, then by [26, Theorem 2.13], $M/\operatorname{Nil}(M)$ is a Valuation module. So $M/\operatorname{Nil}(M)$ is sharp module by [23, Proposition 3.14]. Therefore, by Theorem 2.11, M is a Φ-sharp module. □

Definition 2.18. Let R be a ring and $M \in \mathbb{H}$ be an R-module. A nonnil submodule N of M is called a Φ-t-submodule of M if $\Phi(N)$ is a t-submodule of $\Phi(M)$.

It is worthwhile to note that $N/\operatorname{Nil}(M)$ is a t-submodule of $M/\operatorname{Nil}(M)$ if and only if $\Phi(N)/\operatorname{Nil}(\Phi(M))$ is a t-submodule of $\Phi(M)/\operatorname{Nil}(\Phi(M))$.

Lemma 2.19. *Let R be a ring and $M \in \mathbb{H}$ be an R-module and let N be a nonnil submodule of M. Then N is a Φ-t-submodule of M if and only if $N/\operatorname{Nil}(M)$ is a t-submodule of $M/\operatorname{Nil}(M)$.*

Proof. Let N be a Φ-t-submodule of M. Then $\Phi(N)$ is a t-submodule of $\Phi(M)$. Thus $\Phi(N) = \Phi(N)_\nu\Phi(M)$ and so

$$\Phi(N)/\operatorname{Nil}(\Phi(M)) = (\Phi(N)_\nu/\operatorname{Nil}(\Phi(M)))(\Phi(M)/\operatorname{Nil}(\Phi(M))).$$

Therefore $\Phi(N)/\operatorname{Nil}(\Phi(M))$ is a t-submodule of $\Phi(M)/\operatorname{Nil}(\Phi(M))$. Hence $N/\operatorname{Nil}(M)$ is a t-submodule of $M/\operatorname{Nil}(M)$. Conversely is same. □

Definition 2.20. Let R be a ring and $M \in \mathbb{H}$ be an R-module. Then M is said to be a Φ-TV module if every Φ-t-submodule is a Φ-ν-submodule.

Theorem 2.21. *Let R be a ring and $M \in \mathbb{H}$ be an R-module. Then M is a Φ-TV module if and only if $M/\operatorname{Nil}(M)$ is a TV-module.*

Proof. Let M be a Φ-TV module and $N/\operatorname{Nil}(M)$ be a t-submodule of $M/\operatorname{Nil}(M)$. Then, by Lemma 2.19, N a is Φ-t-submodule of M and so N is a Φ-ν-submodule of M. Hence, by [24, Lemma 3.6], $N/\operatorname{Nil}(M)$ is a ν-submodule of $M/\operatorname{Nil}(M)$. Thus $M/\operatorname{Nil}(M)$ is a TV-module.

Conversely, let $M/\operatorname{Nil}(M)$ be a TV-module and N be a Φ-t-submodule of M. Then, by Lemma 2.19, $N/\operatorname{Nil}(M)$ is a t-submodule of $M/\operatorname{Nil}(M)$ and so $N/\operatorname{Nil}(M)$ is a ν-submodule of $M/\operatorname{Nil}(M)$. Therefore, by [24, Lemma 3.6], N is a Φ-t-submodule of M as well. □

Corollary 2.22. *Let R be a ring and $M \in \mathbb{H}$ be an R-module. Then M is a Φ-TV module if and only if $\Phi(M)/\operatorname{Nil}(\Phi(M))$ is a TV-module.*

Theorem 2.23. *Let R be a ring and $M \in \mathbb{H}$ be an R-module. Then M is a Φ-TV module if and only if $\Phi(M)$ is a TV module.*

Proof. Let M be a Φ-TV module and $\Phi(N)$ be a t-submodule of $\Phi(M)$. Then N is a Φ-t-submodule of M and so N is a Φ-ν-submodule of M. Therefore, $\Phi(N)$ is a ν-submodule of $\Phi(M)$. Hence $\Phi(M)$ is a TV module.

Conversely, let $\Phi(M)$ be a TV module and N be a Φ-t-submodule of M. Then $\Phi(N)$ is a t-submodule of $\Phi(M)$ and so $\Phi(N)$ is a ν-submodule of $\Phi(M)$. Thus N is a Φ-ν-submodule of M. Therefore M is a Φ-TV module. □

Corollary 2.24. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module. The following are equivalent:*

(1) *M is a Φ-TV module;*
(2) *$M/\operatorname{Nil}(M)$ is a TV module;*
(3) *$\Phi(M)/\operatorname{Nil}(\Phi(M))$ is a TV module;*
(4) *$\Phi(M)$ is a TV module.*

Theorem 2.25. *Let R be a ring and M be a finitely generated faithful multiplication R-module. The following statements are equivalent:*

(1) *If $R \in \mathcal{H}$ is a ϕ-TV ring, then M is a Φ-TV module;*
(2) *If $M \in \mathbb{H}$ is a Φ-TV module, then R is a ϕ-TV ring.*

Proof. By [22], [23] and [25], the proof is the same of the proof of Theorem 2.15. □

The notion of a Φ-sharp-TV module means that a module that is both a Φ-sharp module and a Φ-TV module.

Theorem 2.26. *Let R be a ring and $M \in \mathbb{H}$ be a finitely generated faithful multiplication R-module with* $\text{Nil}(M) = Z(R)M$. *If M is a Φ-sharp TV module, then M is a Φ-Dedekind module.*

Proof. Let M be a Φ-sharp TV module. Then, by Theorem 2.2 and Theorem 2.21, $M/\text{Nil}(M)$ is a sharp TV module. So, by [23, Corollary 3.21], $M/\text{Nil}(M)$ is a Dedekind module. Therefore M is a Φ-Dedekind module by [26, Theorem2.10]. □

Theorem 2.27. *Let R be a countable ring and $M \in \mathbb{H}$ be an R-module with* $\text{Nil}(M) = Z(R)M$. *If M is a Φ-sharp module, then M is a Φ-Dedekind module.*

Proof. If M is a Φ-sharp module, then $M/\text{Nil}(M)$ is a sharp module by Theorem 2.2. So, by [23, Theorem 3.7], R is a sharp domain and hence by [27, Corollary 17], R is a Dedekind domain. Thus $M/\text{Nil}(M)$ is a Dedekind domain. Therefore, by [26, Theorem2.10], M is a Φ-Dedekind module. □

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Contact information

A. Yousefian Darani, M. Rahmatinia

Department of Mathematics and Applications, University of Mohaghegh Ardabili, P. O. Box 179, Ardabil, Iran

E-Mail(s): yousefian@uma.ac.ir, m.rahmati@uma.ac.ir

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