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Partitions of groups into thin subsets

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Abstract. Let G be an infinite group with the identity e, κ be an infinite cardinal $\leqslant |G|$. A subset $A \subset G$ is called κ-thin if $|gA \cap A| \leqslant \kappa$ for every $g \in G \setminus \{e\}$. We calculate the minimal cardinal $\mu(G, \kappa)$ such that G can be partitioned in $\mu(G, \kappa)$ κ-thin subsets. In particular, we show that the statement $\mu(\mathbb{R}, \aleph_0) = \aleph_0$ is equivalent to the Continuum Hypothesis.

Let G be an infinite group with the identity e, κ be an infinite cardinal $\leqslant |G|$. By $cf|G|$ we denote the cofinality of $|G|$, κ^+ is the cardinal-successor of κ, $[G]^{<\kappa} = \{F \subseteq G : |F| < \kappa\}$.

We say that a subset $A \subseteq G$ is

- κ-*large* if there exists $F \in [G]^{<\kappa}$ such that $G = FA$;
- κ-*small* if $L \setminus A$ is κ-large for every κ-large subset L;
- κ-*thick* if, for every $F \in [G]^{<\kappa}$, there exists $a \in A$ such that $Fa \subseteq A$;
- κ-*thin* if $|gA \cap A| < \kappa$ for every $g \in G \setminus \{e\}$.

We note that $\aleph_0$-large and $\aleph_0$-small subsets were introduced in [2] under name large and small subsets. Following [6, Chapter 9], κ-large, κ-small, κ-thick and κ-thin subsets of a group can be considered as an asymptotic counterparts of dense, nowhere dense, open and discrete subsets of a topological space.

By [4, Theorem 4.2], G can be partitioned in κ κ-large subsets. By [4, Theorem 4.1], G can be partitioned in $\aleph_0$ subsets which are κ-small for each cardinal κ such that $\aleph_0 \leqslant \kappa \leqslant cf|G|$. By [4, Theorem 4.3], if either $\kappa < |G|$ or $\kappa = |G|$ and κ is regular then G can be partitioned in $|G|$ κ-thick subsets.

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For κ-thin subsets, its modifications, applications and references see [3]. For a subset A of G, we put $cov(A) = \min\{|S| : S \subseteq G, G = SA\}$. By [5, Theorem 5], if A is κ-thin and $\kappa < |G|$ then $cov(A) = |G|$, if $\kappa = |G|$ then $cov(A) \geqslant cf A$. In contrast to κ-thin subsets, every subgroup A of index κ is κ-small and $cov(A) = \kappa$.

Given an infinite group G and infinite cardinal κ, $\kappa \leqslant |G|$, we denote by $\mu(G, \kappa)$ the minimal cardinal μ such that G can be partitioned in μ κ-thin subsets.

In the following theorem we calculate exact values of $\mu(G, \kappa)$ for all G and κ with only one exception: $|G|$ is singular, $\kappa = |G|$ and $cf|G|$ is a non-limit cardinal.

Theorem. *For every infinite group G and every infinite cardinal κ, $\kappa \leqslant |G|$, we have*

$$\mu(G,\kappa) = \begin{cases} \gamma & \text{if } |G| \text{ is non-limit cardinal and } |G| = \gamma^+; \\ |G| & \text{if } |G| \text{ is a limit cardinal and either} \\ & \kappa < |G| \text{ or } |G| \text{ is regular;} \\ cf|G| & \text{if } |G| \text{ is singular}, \kappa = |G| \text{ and } cf|G| \text{ is} \\ & \text{a limit cardinal.} \end{cases}$$

If $|G|$ is singular, $\kappa = |G|$ and $cf|G|$ is a non-limit cardinal, $cf|G| = \gamma^+$, then $\mu(G, \kappa) \in \{\gamma, \gamma^+\}$.

To prove this theorem we need two lemmata.

Lemma 1. *For an infinite group G of cardinality κ, we have*

$$(\mu(G, \kappa))^+ \geqslant cf\kappa.$$

Proof. On the contrary, we assume that, for some cardinal μ such that $\mu^+ < cf\kappa$, there is a partition $\mathcal{P}$ of G in μ κ-thin subsets. We fix a subset A of G, $|A| = \mu^+$ and, for each $g \in G$, pick $P_g \in \mathcal{P}$ such that $|Ag \cap P_g| > 1$. Then we choose distinct elements $x_g, y_g \in A$ such that $x_g g, y_g g \in P_g$. Since $|\mathcal{P}| = \mu$, $|A| = \mu^+$ and $\mu^+ < cf\kappa$, there exist $P \in \mathcal{P}$, distinct elements $x, y \in A$ and a subset X of G such that $|X| = \kappa$ and $xg, yg \in P$ for each $g \in X$. Then $|xy^{-1}P \cap P| = \kappa$ so P is not κ-thin and we get a contradiction. □

Lemma 2. *Let γ be an infinite cardinal, G be a group of cardinality γ^+. Then there exists a partition $\mathcal{P}$ of G such that $|\mathcal{P}| = \gamma$ and $|gP \cap P| \leqslant 2$ for all $P \in \mathcal{P}$, $g \in G \setminus \{e\}$.*

Proof. Since G is uncountable, we can choose a family $\{G_\alpha : \alpha < \gamma^+\}$ of subgroups of G such that

(i) $G_0 = \{e\}$, $G = \bigcup\{G_\alpha : \alpha < \gamma^+\}$;

(ii) $G_\alpha \subseteq G_\beta$ for all $\alpha < \beta < \gamma^+$;

(iii) $\bigcup\{G_\alpha : \alpha < \beta\} = G_\beta$ for every limit ordinal $\beta < \gamma^+$;

(iv) $|G_\alpha| \leqslant \gamma$ for each $\alpha < \gamma^+$.

Using (iv), for every $\alpha < \gamma^+$, we fix an injective mapping $\chi_\alpha : G_{\alpha+1} \setminus G_\alpha \to \gamma$ and define a mapping $\chi : G \to \gamma$ by the rule $\chi(e) = 0$ and $\chi|G_{\alpha+1} \setminus G_\alpha = \chi_\alpha$, $\alpha < \gamma^+$. We show that $\{\chi^{-1}(\lambda) : \lambda < \gamma\}$ is the desired partition $\mathcal{P}$. On the contrary, suppose that there are $g \in G \setminus \{e\}$ and $\lambda < \gamma$ such that $|g\chi^{-1}(\lambda) \cap \chi^{-1}(\lambda)| > 2$. Let x_1, x_2, x_3 be distinct elements from $\chi^{-1}(\lambda)$ such that $gx_1, gx_2, gx_3 \in \chi^{-1}(\lambda)$. We choose the ordinals $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 < \gamma^+$ such that

$$x_{\alpha_i} \in G_{\alpha_i+1} \setminus G_{\alpha_i},\ gx_i \in G_{\beta_i+1} \setminus G_{\beta_i},\ i \in \{1, 2, 3\}.$$

By the definition of χ, $|\chi^{-1}(\lambda) \cap (G_{\alpha+1} \setminus G_\alpha)| \leqslant 1$ for every $\alpha < \gamma^+$ so $\alpha_i \neq \beta_i$, $i \in \{1, 2, 3\}$. By the pigeonhole principle, there are distinct $k, l \in \{1, 2, 3\}$ such that either $\alpha_k < \beta_k$, $\alpha_l < \beta_l$ or $\alpha_k > \beta_k$, $\alpha_l > \beta_l$. In the first case, we have $x_{\beta_k} x_{\alpha_k}^{-1} \in G_{\beta_k+1} \setminus G_{\beta_k}$, $x_{\beta_l} x_{\alpha_l}^{-1} \in G_{\beta_l+1} \setminus G_{\beta_l}$ and $g = x_{\beta_k} x_{\alpha_k}^{-1} = x_{\beta_l} x_{\alpha_l}^{-1}$ which is impossible because $(G_{\beta_k+1} \setminus G_{\beta_k}) \cap (G_{\beta_l+1} \setminus G_{\beta_l}) = \varnothing$. The second case is reduced to the first if we replace g to g^{-1}. □

Proof of Theorem. Assume that $|G|$ is a non-limit cardinal, $|G| = \gamma^+$. Since $\mu(G, \kappa) \geqslant \mu(G, |G|)$, by Lemma 1, $\mu(G, \kappa) \geqslant \gamma$. By Lemma 2, $\mu(G, \kappa) \leqslant \gamma$.

Assume that $|G|$ is a limit cardinal. If $|G|$ is regular, by Lemma 1, $\mu(G, |G|) = |G|$ so $\mu(G, \kappa) = |G|$. If $\kappa < |G|$ and $\lambda < |G|$, we take a subgroup H of G such that $\kappa < |H|, \lambda < |H|$ and $|H|$ is a non-limit cardinal, say, $|H| = \gamma^+$. By above paragraph $\mu(H, \kappa) = \gamma$. Hence, $\mu(G, \kappa) = |G|$.

Assume that $|G|$ is singular and $|G| = \kappa$. Since G can be partitioned in $cf|G|$ subsets of cardinality $< \kappa$ and each subset of cardinality $< \kappa$ is κ-thin, we have $\mu(G, \kappa) \leqslant cf|G|$. If $cf|G|$ is a limit cardinal, by Lemma 1, $\mu(G, \kappa) \geqslant cf|G|$ so $\mu(G, \kappa) = cf|G|$. If $cf|G| = \gamma^+$, by Lemma 1, $\mu(G, \kappa) \geqslant \gamma$ so $\mu(G, \kappa) \in \{\gamma, \gamma^+\}$. □

Applying Theorem, we conclude that the statement $\mu(\mathbb{R}, \aleph_0) = \aleph_0$ is equivalent to CH. In the seminal version of this paper, I asked whether

$\mathbb{R} \setminus \{0\}$ can be partitioned in $\aleph_0$ subsets linearly independent over $\mathbb{Q}$. By [1, Theorem 4], this statement is also equivalent to CH.

Let G be a group, X be a G-space with the action $G \times X \to X$, $(g, x) \mapsto gx$. For a cardinal $\kappa \leqslant X$, we say that a subset T of X is κ-thin if

$$|\{x \in T : gx \neq x, gx \in T\}| < \kappa$$

for every $g \in G$.

Problem. Calculate the minimal cardinal $\mu(G, X, \kappa)$ such that X can be partitioned in $\mu(G, X, \kappa)$ κ-thin subsets.

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