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RESEARCH ARTICLE

Balleans of bounded geometry and G-spaces

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Abstract. A ballean (or a coarse structure) is a set endowed with some family of subsets which are called the balls. The properties of the family of balls are postulated in such a way that a ballean can be considered as an asymptotical counterpart of a uniform topological space.

We prove that every ballean of bounded geometry is coarsely equivalent to a ballean on some set X determined by some group of permutations of X.

1. Ball structures and balleans

A *ball structure* is a triple $\mathcal{B} = (X, P, B)$, where X, P are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of X which is called a *ball of radius* α around x. It is supposed that $x \in B(x, \alpha)$ for all $x \in X, \alpha \in P$. The set X is called the *support* of $\mathcal{B}$, P is called the *set of radii.* Given any $x \in X, A \subseteq X, \alpha \in P$ we put

$$B^*(x,\alpha) = \{y \in X : x \in B(y,\alpha)\},\ B(A,\alpha) = \bigcup_{a\in A} B(a,\alpha)$$

A ball structure is called

- *lower symmetric* if, for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,

$$B^*(x,\alpha') \subseteq B(x,\alpha),\ B(x,\beta') \subseteq B^*(x,\beta);$$

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- *upper symmetric* if, for any $\alpha, \beta \in P$, there exist α', β' such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'),\ B^*(x, \beta) \subseteq B(x, \beta');$$

- *lower multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \gamma), \gamma) \subseteq B(x, \alpha) \cap B(x, \beta);$$

- *upper multiplicative* if, for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

Let $\mathcal{B} = (X, P, B)$ be a lower symmetric and lower multiplicative ball structure. Then the family

$$\left\{ \bigcup_{x \in X} B(x, \alpha) \times B(x, \alpha) : \alpha \in P \right\}$$

is a base of entourages for some (uniquely determined) uniformity on X. On the other hand, if $\mathcal{U} \subseteq X \times X$ is a uniformity on X, then the ball structure $(X, \mathcal{U}, B)$ is lower symmetric and lower multiplicative, where $B(x, U) = \{y \in X : (x, y) \in U\}$. Thus, the lower symmetric and lower multiplicative ball structures can be identified with the uniform topological spaces.

We say that a ball structure $\mathcal{B}$ is a *ballean* if $\mathcal{B}$ is upper symmetric and upper multiplicative. In this paper we follow terminology from [6, 7]. A structure on X, equivalent to a ballean, can also be defined in terminology of entourages. In this case it is called a coarse structure [8] or a uniformly bounded space [5]. For motivations to study balleans see also [1, 2, 4].

2. Morphisms

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\prec$-*mapping* if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,

$$f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta)$$

A bijection $f : X_1 \to X_2$ is called an *asymorphism* between $\mathcal{B}_1$ and $\mathcal{B}_2$ if f and f^{-1} are $\prec$-mappings.

Let $\mathcal{B} = (X, P, B)$ be a ballean, S be a set. Two mappings $f, f' : S \to X$ are called *close* if there exists $\alpha \in P$ such that $f'(s) \in B(f(s), \alpha)$ for every $s \in S$.

Two balleans $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ are called *coarsely equivalent* if there exist the $\prec$-mappings $f_1 : X_1 \to X_2$, $f_2 : X_2 \to X_1$ such that $f_1 \circ f_2$, $f_2 \circ f_1$ are close to the identity mappings id_{X_1}, id_{X_2}.

Let $\mathcal{B} = (X, P, B)$ be a ballean. Every non-empty subset $Y \subseteq X$ determines the subballean $\mathcal{B}_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) = B(Y, \alpha) \cap Y$, $y \in Y$, $\alpha \in P$. A subset Y is called *large* if there exists $\gamma \in P$ such that $B(Y, \gamma) = X$. If Y is large, then $\mathcal{B}_Y$ and $\mathcal{B}$ are coarsely equivalent. We shall use also the following observations. Two balleans $\mathcal{B}_1 = (X_1, P_1, B_1)$ and $\mathcal{B}_2 = (X_2, P_2, B_2)$ are coarsely equivalent if and only if there exist the large subsets $Y_1 \subseteq X_1$,$Y_2 \subseteq X_2$ such that the subballeans $\mathcal{B}_{Y_1}$ and $\mathcal{B}_{Y_2}$ are asymorphic.

3. Density and capacity

Let $\mathcal{B} = (X, P, B)$ be a ballean, $Y \subseteq X$, $S \subseteq Y$, $\alpha \in P$. We say that a subset S is *α-dense* in Y if $Y \subseteq B(S, \alpha)$. An *α-density* of Y is the cardinal

$$den_\alpha(Y) = min\{|S| : S \textit{ is an } \alpha - \textit{dense subset of } Y\}.$$

A subset S of X is called *α-separated* if $B(x, \alpha) \cap B(y, \alpha) = \varnothing$ for all distinct $x, y \in S$. An *α-capacity* of Y is the cardinal

$$cap_\alpha(Y) = sup\{|S| : S \textit{ is an } \alpha - \textit{separated subset of } Y\}.$$

Let $\mathcal{B} = (X, P, B)$ be an arbitrary ballean. Replacing every ball $B(x, \alpha)$ to $B'(x, \alpha) = B(x, \alpha) \cap B^*(x, \alpha)$, we get the asymorphic ballean $\mathcal{B}' = (X, P, B')$ with $(B')^* = B'$. Thus, in what follows we may suppose that $B^*(x, \alpha) = B(x, \alpha)$ for all $x \in X$, $\alpha \in P$.

Lemma 1. *Let $\mathcal{B} = (X, P, B)$ be a ballean, $Y \subseteq X$, $\alpha, \beta \in P$ and $B(B(x, \alpha)) \subseteq B(x, \beta)$ for every $x \in X$. Then the following statements hold*

(i) $den_\beta(Y) \leqslant cap_\alpha(Y) \leqslant den_\alpha(Y)$;

(ii) if $Z \subseteq X$ and $Y \subseteq B(Z, \alpha)$, then $den_\beta(Y) \leqslant |Z|$.

Proof. (i) Let S be an α-separated subset of Y, D be an α-dense subset of Y. Then every ball $B(x, \alpha)$, $x \in D$ has at most one point of S. Since $S \subseteq Y \subseteq \bigcup_{x \in D} B(x, \alpha)$, we have $|S| \leqslant |D|$, so $cap_\alpha(Y) \leqslant den_\alpha(Y)$.

Let S be a maximal by inclusion α-separated subset of Y. Then every ball $B(x,\alpha)$, $x \in Y$ meets at least one ball $B(y,\alpha)$, $y \in S$. It follows that $Y \subseteq \bigcup_{x\in S} B(x,\beta)$, so S is β-dense in Y and $den_\beta(Y) \leqslant cap_\alpha(Y)$.

(ii) We put $Z' = \{z \in Z : B(z,\alpha) \cap Y \neq \varnothing\}$ and, for every $z \in Z'$, pick some point $y_z \in B(z,\alpha) \cap Y$. Then the subset $\{y_z : z \in Z'\}$ of Y is β-dense in Y, so $den_\beta(Y) \leqslant |Z'| \leqslant |Z|$. □

4. Locally finite balleans

A ballean $\mathcal{B} = (X,P,B)$ is called *locally finite* if every ball $B(x,\alpha)$, $x \in X$, $\alpha \in P$ is finite.

Let $\mathcal{B} = (X,P,B)$, $\mathcal{B}' = (X',P',B')$ be balleans, $f : X \to X'$ be an injective $\prec$-mapping. If $\mathcal{B}'$ is locally finite then $\mathcal{B}$ is locally finite. In particular, every ballean asymorphic to a locally finite ballean is locally finite.

We say that a ballean $\mathcal{B}$ is *coarsely locally finite* if $\mathcal{B}$ is coarsely equivalent to some locally finite ballean.

Proposition 1. *A ballean $\mathcal{B} = (X,P,B)$ is coarsely locally finite if and only if there exists $\beta \in P$ such that β-capacity of every ball $B(x,\gamma)$,$x \in X$, $\gamma \in P$ is finite.*

Proof. Let $\mathcal{B}' = (X',P',B')$ be a locally finite ballean coarsely equivalent to $\mathcal{B}$. Then there exist the large subsets $Y \subseteq X$, $Y' \subseteq X'$ such that the subballeans $\mathcal{B}_Y$ and $\mathcal{B}_{Y'}$ are asymorphic. We choose $\alpha \in P$ such that $B(Y,\alpha) = X$ and take an arbitrary $x \in X$, $\gamma \in P$. Since $\mathcal{B}_Y$ is locally finite then the subset $Z = B(B(x,\gamma),\alpha) \cap Y$ is finite. Since $B(x,\gamma) \subseteq B(Z,\alpha)$, by Lemma 1 (ii), $den_\beta(B(x,\gamma)) \leqslant |Z|$. Since Z is finite, by Lemma 1(i), β-capacity of $B(x,\gamma)$ is finite.

On the other hand, let β-capacity of every ball $B(x,\gamma)$ is finite. We choose a maximal by inclusion β-separated subset Y of X. Clearly, Y is large in X, so $\mathcal{B}_Y$ is coarsely equivalent to $\mathcal{B}$. Since $cap_\beta B(x,\gamma)$ is finite, then $B(x,\gamma) \cap Y$ is finite. Hence, $\mathcal{B}_Y$ is locally finite. □

Every metric space (X,d) determines the metric ballean $\mathcal{B}(X,d) = (X,\mathbb{R}^+,B_d)$, where $B_d(x,r) = \{y \in X : d(x,y) \leqslant r\}$. For criterion of metrizability of balleans see [7, Theorem 2.1.1]. A metric space is called *proper* if every ball $B_d(x,r)$ is compact.

Corollary 1. *Let (X,d) be a proper metric space. Then the metric ballean $\mathcal{B}(X,d)$ is coarsely locally finite.*

Proof. It suffices to note that an 1-capacity of every ball in (X,d) is finite, and apply Proposition 1. □

5. Uniformly locally finite balleans

A ballean $\mathcal{B} = (X, P, B)$ is called *uniformly locally finite* if there exists a function $h : P \to \omega$ such that $|B(x, \alpha)| \leqslant h(\alpha)$ for all $x \in X$, $\alpha \in P$.

Let $\mathcal{B} = (X, P, B)$, $\mathcal{B}' = (X', P', B')$ be balleans, $f : X \to X'$ be an injective $\prec$-mapping. If $\mathcal{B}'$ is uniformly locally finite then so is $\mathcal{B}$. In particular, every ballean asymorphic to an uniformly locally finite ballean is uniformly locally finite.

We say that a ballean $\mathcal{B} = (X, P, B)$ has *bounded geometry* if there exist $\beta \in P$ and a function $h : P \to \omega$ such that $cap_\beta B(x, \alpha) \leqslant h(\alpha)$ for all $x \in X$, $\alpha \in P$.

Repeating the arguments proving Proposition 1 we get the following statements.

Proposition 2. *A ballean $\mathcal{B} = (X, P, B)$ has bounded geometry if and only if $\mathcal{B}$ is coarsely equivalent to some uniformly locally finite ballean.*

Example 1. Let $\Gamma(V, E)$ be a connected graph with the set of vertices V and the set of edges E. Given any $u, v \in V$, we denote by $d(u, v)$ the length of a shortest path between u and v. Then we get the metric space (V, d) associated with $\Gamma(V, E)$ and the metric ballean $\mathcal{B}(V, d)$. Clearly, $\mathcal{B}(V, d)$ is uniformly locally finite if and only if there exists a natural number r such that $|B_d(v, 1)| \leqslant r$ for every $v \in V$.

Example 2. Let G be a finitely generated group with the identity e, F be a symmetric $(F = F^{-1})$ set of generators of G such that $e \notin F$. The Cayley graph $Cay(G, F)$ is a graph with the set of vertices G and set of edges $\{\{u, v\} : uv^{-1} \in F\}$. Let d_F be a path metric on $Cay(G, F)$. Then the metric ballean $\mathcal{B}(G, d_F)$ is uniformly locally finite.

Example 3. Let G be an arbitrary group, $\mathcal{F}_e$ the family of all symmetric subsets of G containing e. Then we get a ballean $\mathcal{B}(G) = (G, \mathcal{F}_e, B)$, where $B(g, F) = Fg$. Clearly, $\mathcal{B}(G)$ is uniformly locally finite and in the case G is finitely generated, $\mathcal{B}(G)$ is asymorphic to the ballean $\mathcal{B}(G, d_F)$ determined in Example 2.

Example 4. Let G be a group and X be a G-space with the action of G on X defined by $(g, x) \mapsto g(x)$. We denote by $\mathcal{F}_e$ the family of all finite symmetric subsets of G containing e. Then we get the ballean $\mathcal{B}(G, X) = (X, \mathcal{F}_e, B)$, where $B(x, F) = \{g(x) : g \in F\}$, $x \in X$, $F \in \mathcal{F}_e$. Clearly, $\mathcal{B}(G, X)$ is uniformly locally finite.

Example 5. Let G be a gruppoid (=inverse semigroup) of partial bijections of a set X, $\mathcal{F}$ be a family of all finite subsets of G such that

$F = F^{-1}$ for every $F \in \mathcal{F}$. Given any $x \in X$ and $F \in \mathcal{F}$, we put $B(x, F) = \{x\} \cup \{g(x) : g \in F\}$ and get the uniformly locally finite ballean $\mathcal{B}(G, X)$.

Example 6. Let G be a locally compact topological group, C be the family of all compact symmetric subsets of G containing e. Then, by Proposition 5.1, the ballean $\mathcal{B}(G) = (G, C, B)$, where $B(x, C) = Cx$, is of bounded geometry.

Remark 1. Let G be a locally compact group. Does there exist a discrete group D such that the balleans $\mathcal{B}(G)$ and $\mathcal{B}(D)$ are coarsely equivalent? This is so if G is Abelian or a connected Lie group.

6. G-space realization

Let $\mathcal{B}$, $\mathcal{B}'$ be balleans with the same support X. We write $\mathcal{B} \prec \mathcal{B}'$ if the identity mapping $id : X \to X$ is a $\prec$-mapping from $\mathcal{B}$ to $\mathcal{B}'$. If $\mathcal{B} \prec \mathcal{B}'$ and $\mathcal{B}' \prec \mathcal{B}$, we identify $\mathcal{B}$ and $\mathcal{B}'$ and write $\mathcal{B} = \mathcal{B}'$.

Let $\mathcal{B}$ be a uniformly locally finite ballean with the support X. Applying Lemma 4.10 from [8], one can show that there exists a gruppoid G of partial bijections of X such that $\mathcal{B} = \mathcal{B}(G, X)$ where $\mathcal{B}(G, X)$ is a ballean determined in Example 5. Our next result states that instead of the gruppoid G we can take some group of permutations of X.

Theorem 1. *For every uniformly locally finite ballean $\mathcal{B} = (X, P, B)$, there exists a group G of permutations of X such that $\mathcal{B} = \mathcal{B}(G, X)$.*

Proof. We fix an arbitrary $\alpha \in P$ and choose $\beta \in P$ such that

$$B(B(x, \alpha), \alpha) \subseteq B(x, \beta)$$

for each $x \in X$. Then we define the graph Γ_β with the set of vertices X and the set of edges E_β defined by the rule: $\{x, y\} \in E_\beta$ if and only if $x \in B(y, \beta)$. Since $\mathcal{B}$ is uniformly locally finite, there exists a natural number $n(\alpha)$ such that the local degree of every vertex of Γ_β does not exceed $n(\alpha)$. By [3, Corollary 12.2], the chromatic number of Γ_β does not exceed $n(\alpha) + 1$. It follows that we can partition $X = X_1 \cup \ldots \cup X_{n(\alpha)+1}$ so that any two vertices from X_j are non-adjacent, in particular, every subset X_i is α-separated.

Now we fix $i \in \{1, \ldots, n(\alpha) + 1\}$ and, for every vertex $x \in X_i$, enumerate the set $B(x, \alpha) \setminus \{x\} = \{x(1), \ldots, x(n_x)\}$, where $n_x \leqslant n(\alpha)$. Then we define the set $S_i(\alpha)$ of $n(\alpha)$ permutations of X as follows. For each $j \in \{1, \ldots, n(\alpha)\}$ and $x \in X_i$, we put $\pi_j(x) = x(j)$, $\pi_j(x(j)) = x$ if $j \leqslant n_x$, and $\pi_j(x) = x$ otherwise. Then we extend π to X putting

$\pi_j(y) = y$ for all $y \in X \setminus \bigcup_{x \in X_i} \{x, x(j)\}$. Since X_i is α-separated, this definition is correct. Thus, we get the set $S_i(\alpha) = \{\pi_1, \ldots, \pi_{n(\alpha)}\}$ of permutations of X. We put $S(\alpha) = S_1(\alpha) \cup \ldots \cup S_{n(\alpha)+1}(\alpha)$ and denote by G the group of permutations of X generated by $\bigcup_{\alpha \in P} S(\alpha)$.

At last we show that the identity mapping $id : X \to X$ is an asymorphism between $\mathcal{B}$ and the ballean $\mathcal{B}(G, X) = (X, \mathcal{F}_e, B')$ determined in Example 5.4. Given any $\alpha \in P$ and $x \in X$, we have $B(x, \alpha) \subseteq B'(x, S_\alpha)$. On the other hand, let F be a finite subset of G, $g \in F$. Then there exists $\alpha_1, \ldots, \alpha_m \in P$ and $s(\alpha_1) \in S(\alpha_1), \ldots, s(\alpha_m) \in S(\alpha_m)$ such that $g = s(\alpha_m) \ldots s(\alpha_1)$. We choose $\gamma_g \in P$ such that

$$B(\ldots(B(B(x, \alpha_1), \alpha_2), \ldots), \alpha_m) \subseteq B(x, \gamma_g)$$

for every $x \in X$. Then $B'(x, \{g\}) \subseteq B(x, \gamma_g)$ for every $x \in X$. Since F is finite, there exists $\gamma \in P$ such that, for each $x \in X$, we have $B'(x, F) \subseteq B(x, \gamma)$. □

Sticking together Proposition 1 and Theorem 1 we get the following statement.

Theorem 2. *Every ballean of bounded geometry is coarsely equivalent to some ballean $\mathcal{B}(G, X)$ of G-space X.*

We conclude our paper with two applications of Theorem 1.

Theorem 3. *Let X be a set, S_X be a group of all permutations of X. Then $\mathcal{B}(S_X, X)$ is the strongest uniformly locally finite ballean on X.*

Proof. Let $\mathcal{B}'$ be a uniformly locally finite ballean on X. Using Theorem 1, we choose a group G of permutations of X such that $\mathcal{B}' = \mathcal{B}(G, X)$. Since G is a subgroup of S_X, we have $\mathcal{B}' \prec \mathcal{B}(S_X, X)$. □

A ballean $\mathcal{B} = (X, P, B)$ is called *connected* if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. Clearly, a ballean $\mathcal{B}(G, X)$ of a G-space is connected if and only if G acts transitively on X.

Let $\mathcal{B}_1 = (X_1, P_1, B_1)$, $\mathcal{B}_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f : X_1 \to X_2$ is called a $\succ$*-mapping* if, for every $\beta \in P_2$, there exists $\alpha \in P_1$ such that $B_2(f(x), \beta)) \subseteq f(B_1(x, \alpha))$ for each $x \in X_1$. A bijection $f : X_1 \to X_2$ is a $\succ$-mapping if and only if f^{-1} is a $\prec$-mapping. Thus, $\mathcal{B}_1$ and $\mathcal{B}_2$ are asymorphic if and only if there is a bijection $f : X_1 \to X_2$ which is a $\prec$-mapping and a $\succ$-mapping.

Theorem 4. *For every connected uniformly locally finite ballean $\mathcal{B}$ on a set X, there exist a group G of permutations of X and a surjective mapping $f : G \to X$ which is a $\prec$-mapping and a $\succ$-mapping from $\mathcal{B}(G)$ to $\mathcal{B}$.*

Proof. Applying Theorem 1, we identify $\mathcal{B}$ with $\mathcal{B}(G, X)$ for some group G of permutations of X. Then we fix $x_0 \in X$ and, for every $g \in G$, put $f(g) = g(x_0)$. Since $\mathcal{B}$ is connected, (G, X) is a transitive G-space, so f is surjective. For any finite subset F of G, we have $f(Fg) = Fg(x_0) = F(g(x_0)) = F(f(g))$. It follows that f is a $\prec$-mapping and a $\succ$-mapping.

Let (G, X) be a transitive G-space, $x_0 \in X$. If $St(x_0) = \{g \in G : g(x_0) = x_0\}$ is finite, applying Theorem 4, it is easy to show that the balleans $\mathcal{B}(G)$ and $\mathcal{B}(G, X)$ are coarsely equivalent. □

Remark 2. Let (G, X) be a transitive G-space. How to detect whether the ballean $\mathcal{B}(G, X)$ is asymorphic (coarsely equivalent) to the ballean $\mathcal{B}(H)$ of some group H?

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