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Prime radical of Ore extensions over δ-rigid rings

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Abstract. Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. We say that R is a δ-rigid ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, $a \in R$; where P(R) is the prime radical of R. In this article, we find a relation between the prime radical of a δ-rigid ring R and that of $R[x, \sigma, \delta]$. We generalize the result for a Noetherian Q-algebra (Q is the field of rational numbers).

1. Introduction

A ring R always means an associative ring. Q denotes the field of rational numbers, and Z denotes the ring of integers unless other wise stated. Spec(R) denotes the set of all prime ideals of R. Min.Spec(R) denotes the sets of all minimal prime ideals of R. P(R) and N(R) denote the prime radical and the set of all nilpotent elements of R respectively.

Let R be a ring. Let σ be an automorphism and δ be a σ-derivation of R. Recall that $R[x, \sigma, \delta]$ is the usual polynomial ring with coefficients in R and we consider any $f(x) \in R[x, \sigma, \delta]$ to be of the form f(x) $= \sum x^i a_i$, $0 \leq i \leq n$. Multiplication in $R[x, \sigma, \delta]$ is subject to the relation ax $= x\sigma(a) + \delta(a)$ for $a \in R$.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [1, 2, 4, 5, 7, 11, 12]. In [1] associated prime ideals of skew polynomial rings have been discussed. In [5] it is shown that if R is embeddable in a right Artinian ring and if characteristic of R is zero, then the differential operator ring

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$R[x, \delta]$ embeds in a right Artinian ring, where δ is a derivation of R. It is also shown in [5] that if R is a commutative Noetherian ring and σ is an automorphism of R, then the skew-polynomial ring $R[x, \sigma]$ embeds in an Artinian ring. In [2] it is proved that if R is a ring which is an order in an Artinian ring, then $R[x, \sigma, \delta]$ is also an order in an Artinian ring.

Some authors have worked on $R[x, \sigma, \delta]$ when R is 2-primal. Recall that a ring R is 2-primal if N(R) = P(R). R is 2-primal if and only if P(R) is completely semiprime (i.e. $a^2 \in P(R)$ implies $a \in P(R)$, $a \in R$). We note that any reduced ring is 2-primal, and any commutative ring is also 2-primal. The nature of nil radical, prime ideals, minimal prime ideals, prime radical of $R[x, \sigma, \delta]$ has been investigated, and relations between R and $R[x, \sigma, \delta]$ have been obtained in some cases. For further details on 2-primal rings, we refer the reader to [4, 6, 8, 10, 13].

Recall that in [11], a ring R is called σ-rigid if a$\sigma(a) = 0$ implies that a $= 0$ for $a \in R$. In [12], a ring R is called a $\sigma(*)$-ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ and a relation has been established between a $\sigma(*)$-ring and a 2-primal ring. The property is also extended to $R[x, \sigma]$.

Motivated by these developments, in this article, we define a δ-rigid ring (Definition (2.1)), and establish a relation between a δ-rigid ring and a 2-primal ring. We also find a relation between the prime radical of a δ-rigid ring R and that of $R[x, \sigma, \delta]$. We also discuss completely prime ideals and the prime radical of a 2-primal ring R and try to relate completely prime ideals of a ring R with the completely prime ideals of $R[x, \sigma, \delta]$. This is given in Proposition (2.4). We also find a relation between the prime radical of a 2-primal ring R and that of $R[x, \sigma, \delta]$. This is given in Theorem (2.6). We generalize this result for a Noetherian Q-algebra R. This is given in Corollary (2.8).

2. Main Result

Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of a ring R. Recall that an ideal I of a ring R is called σ-invariant if $\sigma(I)$ = I and is called δ-invariant if $\delta(I) \subseteq I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. With this we have the following definition:

Definition 2.1. Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. We say that R is a δ-rigid ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, $a \in R$. We note that a ring R with identity 1 is not a δ-rigid ring as $1\delta(1) = 0$

We note that all σ-derivations need not satisfy the property ($a\delta(a) \in P(R)$ implies $a \in P(R)$, $a \in R$). For example the following:

Consider R $= (a_{ij})_{2,2}$, the set of all 2x2 matrices over the ring nZ, $n > 1$ with $a_{21} = 0$. Define σ: R $\rightarrow R$ by $\sigma(a_{ij}) = (b_{ij})$, where $b_{ij} = a_{ij}$ except that $b_{12} = -a_{12}$. Then it can be seen that δ is an automorphism of R. Now define $\delta : R \rightarrow R$ by $\delta(a_{ij}) = (c_{ij})$, where $c_{ij} = 0$ except that $c_{12} = 2a_{12} + a_{22} - a_{11}$. Then it can be seen that δ is a σ-derivation of R. But R is not a δ-rigid ring, as for A $= (a_{ij})_{2,2}$, with a_{ij} =0 except $a_{22} = 1$, $A\delta(A) = (0)$.

Proposition 2.2. *Let R be a 2-primal ring. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. Let $P \in Min.Spec(R)$ be such that $\sigma(P) = P$. Then $\delta(P) \subseteq P$.*

Proof. The proof follows from Theorem (3.6) and Lemma (3.2) of [9]. We give a sketch of the proof.

Let $P \in Min.Spec(R)$ with $\sigma(P) =$ P. Let $a \in P$. Then there exists $b \notin P$ such that $ab \in P(R)$ by Corollary (1.10) of [11]. Now we have $\delta(P(R)) \subseteq P(R)$. Therefore $\delta(ab) = \delta(a)\sigma(b) + a\sigma(b) \in P(R) \subseteq P$. So we have $\delta(a)\sigma(b) \in P$. But $\sigma(b) \notin P$, and therefore $\delta(a) \in P$ as by Proposition (1.11) of [12] P is completely prime. Hence $\delta(P) \subseteq P$. □

We now give a relation between a δ-rigid ring and a 2-primal ring.

Theorem 2.3. *Let R be a δ-rigid ring. Let σ be an automorphism of R such that $\sigma(P(R)) = P(R)$, and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof. Define a map $\partial : R/P(R) \rightarrow R/P(R)$ by $\partial(a + P(R)) = \delta(a) + P(R)$ for $a \in R$ and $\tau : R/P(R) \rightarrow R/P(R)$ a map by $\tau(a + P(R))$ $= \sigma(a) + P(R)$ for $a \in R$. Now it is easy to see that that τ is an automorphism of $R/P(R)$. Also for any a + P(R), $b + P(R) \in R/P(R)$; $\partial((a + P(R))(b + P(R)) = \partial(ab + P(R)) = \delta(ab) + P(R) = \delta(a)\sigma(b) + a\delta(b) + P(R) = (\delta(a) + P(R))(\sigma(b) + P(R)) + (a + P(R))(\delta(b) + P(R))$ $= \partial(a + P(R))\tau(b + P(R)) + (a + P(R))\partial(b + P(R))$, and it is obvious that $\partial(a + P(R) + b + P(R)) = \partial(a + P(R)) + \partial(b + P(R))$. Therefore ∂ is a $\tau - derivation$ of R/P(R). Now a $\delta(a) \in P(R)$ if and only if $(a + P(R))\partial(a + P(R)) =$ P(R) in R/P(R). Thus, as in Proposition (5) of [7], R is a reduced ring and hence R is 2-primal. □

We notice that a 2-primal ring need not be a δ-rigid ring, as can be seen from the following example.

Consider R $= Z_2 \oplus Z_2$. Then R is a commutative reduced ring, and so is a 2-primal ring. Define a map $\sigma : R \rightarrow R$ by $\sigma(a, b) =$ (b, a). Then σ is an automorphism of R. Now define a map $\delta : R \rightarrow R$ by $\delta(a, b) =$ (a-b, 0). Then δ is a σ-derivation of R. But R is not a δ-rigid ring, as (0, 1)$\delta(0, 1) = (0, 0)$.

Proposition 2.4. *Let R be a ring. Let σ be an automorphism of R and δ be a σ-derivation of R. Then:*

1. *For any completely prime ideal P of R with $\delta(P) \subseteq P$ and $\sigma(P) = P$, $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.*
2. *For any completely prime ideal Q of $R[x, \sigma, \delta]$, $Q \cap R$ is a completely prime ideal of R.*

Proof. See Proposition (2.5) of [3]. □

The above discussion leads to the following question:

Is $\delta(Q \cap R) \subseteq Q \cap R$ in Proposition (2.4)? If so, is Q = $(Q \cap R)[x, \sigma, \delta]$? The question remains to be answered, but in this connection we note that σ and δ can be extended to $R[x, \sigma, \delta]$ by taking $\sigma(x) =$ x and $\delta(x) = 0$. In other words, $\sigma(xa) = x\sigma(a)$ and $\delta(xa) = x\delta(a)$ for all $a \in R$.

Corollary 2.5. *Let R be a δ-rigid ring. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. Let $P \in Min.Spec(R)$ be such that $\sigma(P) = P$. Then $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$.*

Proof. R is 2-primal by Theorem (2.3), and so by Proposition (2.2) $\delta(P) \subseteq P$. Further more P is a completely prime ideal of R by Proposition (1.11) of [12]. Now use Proposition (2.4). □

Theorem 2.6. *Let R be a δ-rigid ring. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in Min.Spec(R)$. Then $R[x, \sigma, \delta]$ is 2-primal if and only if $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.*

Proof. Let $R[x, \sigma, \delta]$ be 2-primal. Let $P \in Min.Spec(R)$. By Corollary (2.5) $P[x, \sigma, \delta]$ is a completely prime ideal of $R[x, \sigma, \delta]$, and therefore $P(R[x, \sigma, \delta]) \subseteq P(R)R[x, \sigma, \delta]$. One may see Proposition (3.8) of [9] also. Let f(x) = $\sum x^j a_j \in P(R)[x, \sigma, \delta]$, $0 \leq i \leq n$. Now R is a 2-primal sub ring of $R[x, \sigma, \delta]$ by Theorem (2.3). This implies that a_j is nilpotent and thus $a_j \in N(R[x, \sigma, \delta]) = P(R[x, \sigma, \delta]$, and so we have $x^j a_j \in P(R[x, \sigma, \delta])$ for each j. Therefore $f(x) \in P(R[x, \sigma, \delta])$. Hence we have $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.

Conversely suppose $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$. We will show that $R[x, \sigma, \delta]$ is 2-primal. Let g(x) = $\sum x^i b_i \in R[x, \sigma, \delta]$, $0 \leq i \leq n$ be such that $(g(x))^2 \in P(R[x, \sigma, \delta]) = P(R)[x, \sigma, \delta]$. Then by an easy induction and by using the fact that P(R) is completely semiprime by Theorem (2.3), it can be easily seen that $b_i \in P(R)$ for all b_i, $0 \leq i \leq n$. This means that $f(x) \in P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$. Therefore $P(R[x, \sigma, \delta]$ is completely semiprime. Hence $R[x, \sigma, \delta]$ is 2-primal. □

We now generalize the above result for a Noetherian Q-algebra R, and towards this we have the following:

Proposition 2.7. *Let R be a Noetherian Q-algebra. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for $a \in R$. Then:*

1. $\sigma(N(R)) = N(R)$
2. *If $P \in Min.Spec(R)$ such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof. **(1)** Denote N(R) by N. We have $\sigma(N) \subseteq N$ as $\sigma(N)$ is a nilpotent ideal of R. Now for any $n \in N$, there exists $a \in R$ such that $n = \sigma(a)$. So I= $\sigma^{-1}(N)$={$a \in R$ such that $\sigma(a) = n \in N$} is an ideal of R. Now I is nilpotent, therefore $I \subseteq N$, which implies that $N \subseteq \sigma(N)$. Hence $\sigma(N) = N$.

(2) Let T = $\{a \in P$ such that $\delta^k(a) \in P$ for all integers $k \geq 1\}$. Then T is a δ-invariant ideal of R. Now it can be seen that $T \in Spec(R)$, and since $P \in Min.Spec(R)$, we have T = P. Hence $\delta(P) \subseteq P$. □

Corollary 2.8. *Let R be a δ-rigid Noetherian Q-algebra. Let σ be an automorphism of R and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for $a \in R$. Let $\sigma(P) = P$ for all $P \in Min.Spec(R)$. Then $R[x, \sigma, \delta]$ is 2-primal if and only if $P(R)[x, \sigma, \delta] = P(R[x, \sigma, \delta])$.*

Proof. Use Theorems (2.6) and (2.7). □

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