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Co-intersection graph of submodules of a module

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Abstract. Let M be a unitary left R-module where R is a ring with identity. The co-intersection graph of proper submodules of M, denoted by $\Omega(M)$, is an undirected simple graph whose the vertex set $V(\Omega)$ is a set of all non-trivial submodules of M and there is an edge between two distinct vertices N and K if and only if $N + K \neq M$. In this paper we investigate connections between the graph-theoretic properties of $\Omega(M)$ and some algebraic properties of modules . We characterize all of modules for which the co-intersection graph of submodules is connected. Also the diameter and the girth of $\Omega(M)$ are determined. We study the clique number and the chromatic number of $\Omega(M)$.

1. Introduction

The investigation of the interplay between the algebraic structures-theoretic properties and the graph-theoretic properties has been studied by several authors. As a pioneer, J. Bosak [4] in 1964 defined the graph of semigroups. Inspired by his work, B. Csakany and G. Pollak [7] in 1969, studied the graph of subgroups of a finite group. The Intersection graphs of finite abelian groups studied by B. Zelinka [11] in 1975. Recently, in 2009, the intersection graph of ideals of a ring, was considered by I. Chakrabarty et. al. in [5]. In 2012, on a graph of ideals researched

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by A. Amini et. al. in [2] and Also, intersection graph of submodules of a module introduced by S. Akbari et. al. in [1]. Motivated by previous studies on the intersection graph of algebraic structures, in this paper we define the *co-intersection graph* of submodules of a module. Our main goal is to study the connection between the algebraic properties of a module and the graph theoretic properties of the graph associated to it.

Throughout this paper R is a ring with identity and M is a unitary left R-module. We mean from a non-trivial submodule of M is a nonzero proper left submodule of M.

The co-intersection graph of an R-module M, denoted by $\Omega(M)$, is defined the undirected simple graph with the vertices set $V(\Omega)$ whose vertices are in one to one correspondence with all non-trivial submodules of M and two *distinct* vertices are adjacent if and only if the sum of the corresponding submodules of M is not-equal M.

A submodule N of an R-module M is called *superfluous* or*small* in M (we write $N \ll M$), if for every submodule $X \subseteq M$, the equality $N + X = M$ implies $X = M$, i.e., a submodule N of M is called small in M, if $N + L \neq M$ for every proper submodule L of M. The radical of R-module M written $\mathrm{Rad}(M)$, is sum of all small submodules of M.

A non-zero R-module M is called *hollow*, if every proper submodule of M is small in M.

A non-zero R-module M is called *local*, if has a largest submodule, i.e., a proper submodule which contains all other proper submodules.

An R-module M is said to be A-*projective* if for every epimorphism $g : A \to B$ and homomorphism $f : M \to B$, there exists a homomorphism $h : M \to A$, such that $gh = f$. A module P is projective if P is A-projective for every R-module A. If P is P-projective, then P is also called self-(or quasi-)projective.

A non-zero R-module M is said to be *simple*, if it has no non-trivial submodule. A nonzero R-module M is called *indecomposable*, if it is not a direct sum of two non-zero submodules. For an R-module M, the length of M is the length of composition series of M, denoted by $l_R(M)$.

An R-module M has finite length if $l_R(M) < \infty$, i.e., M is Noetherian and Artinian. The ring of all endomorphisms of an R-module M is denoted by $\mathrm{End}_R(M)$.

Let Ω=($V(\Omega)$,$E(\Omega)$) be a graph with vertex set $V(\Omega)$ and edge set $E(\Omega)$ where an edge is an unordered pair of distinct vertices of Ω. Graph Ω is finite, if $\mathrm{Card}(V(\Omega)) < \infty$, otherwise Ω is infinite. A subgraph of a graph Ω is a graph Γ such that $V(\Gamma) \subseteq V(\Omega)$ and $E(\Gamma) \subseteq E(\Omega)$. By order of Ω, we mean the number of vertices of Ω and we denoted it by $|\Omega|$. If X and Y are two adjacent vertices of Ω, then we write $X \leftrightarrow Y$.

The *degree* of a vertex v in a graph Ω, denoted by $\deg(v)$, is the number of edges incident with v. A vertex v is called *isolated* if $\deg(v) = 0$. Let U and V be two distinct vertices of Ω. An U, V-path is a path with starting vertex U and ending vertex V. For distinct vertices U and V, $d(U, V)$ is the least length of an U, V-path. If Ω has no such a path, then $d(U, V) = \infty$. The *diameter* of Ω, denoted by $\mathrm{diam}(\Omega)$ is the supremum of the set $\{d(U, V)\colon U$ and V are distinct vertices of $\Omega\}$.

A *cycle* in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. We mean of (X, Y, Z) is a cycle of length 3. The *girth* of a graph is the length of its shortest cycle. A graph with no-cycle has infinite girth.

By a *null graph*, we mean a graph with no edges. A graph is said to be *connected* if there is a path between every pair of vertices of the graph.

A *tree* is a connected graph which does not contain a cycle.

A *star graph* is a tree consisting of one vertex adjacent to all the others.

A *complete* graph is a graph in which every pair of distinct vertices are adjacent. The complete graph with n distinct vertices, denoted by K_n.

By a *clique* in a graph Ω, we mean a complete subgraph of Ω and the number of vertices in a largest clique of Ω, is called the clique number of Ω and is denoted by $\omega(\Omega)$.

An *independent set* in a graph is a set of pairwise non-adjacent vertices. An *independence number* of Ω, written $\alpha(\Omega)$, is the maximum size of an independent set.

For a graph Ω, let $\chi(\Omega)$, denote the *chromatic number* of Ω, i.e., the minimum number of colors which can be assigned to the vertices of Ω such that every two adjacent vertices have different colors.

2. Connectivity, diameter and girth of $\Omega(M)$

In this section, we characterize all modules for which the co-intersection graph of submodules is not connected. Also the diameter and the girth of $\Omega(M)$ are determined. Finally we study some modules whose co-intersection graphs are complete.

Theorem 2.1. *Let M be an R-module. Then the graph $\Omega(M)$ is not connected if and only if M is a direct sum of two simple R-modules.*

Proof. Assume that $\Omega(M)$ is not connected. Suppose that Ω_1 and Ω_2 are two components of $\Omega(M)$. Let X and Y be two submodules of M such that $X \in \Omega_1$ and $Y \in \Omega_2$. Since there is no X,Y-path, then $M = X + Y$. Now, if $X \cap Y \neq (0)$, then by

$$X \cap Y + X = X \neq M \quad \text{and} \quad X \cap Y + Y = Y \neq M$$

implies that there is a X,Y-path by $X \cap Y$, to form $X \leftrightarrow X \cap Y \leftrightarrow Y$, a contradiction. Hence, $X \cap Y = (0)$ and $M = X \oplus Y$. Now, we show that X and Y are minimal submodules of M. To see this, let Z be a submodule of M such that $(0) \neq Z \subseteq X$ then $Z + X = X \neq M$. Hence Z and X are adjacent vertices, which implies that $Z \in \Omega_1$. Hence there is no Z,Y-path and by arguing as above, we have $M = Z + Y$, since Z and Y are not adjacent vertices. But since

$$X = X \cap M = X \cap (Z + Y) = Z + X \cap Y = Z$$

by Modularity condition, X is a minimal submodule of M.

A similar argument shows that Y is also a minimal submodule of M and in fact every non-trivial submodule of M is a minimal submodule, which yields that every non-trivial submodule is also maximal. But, minimality of X and Y implies that, they are simple R-modules and since $M = X \oplus Y$, we are done.

Conversely, suppose that $\Omega(M)$ is connected. Let $M = X \oplus Y$, where X and Y are simple R-modules. Let $M_1 = X \times \{0\}$ and $M_2 = \{0\} \times Y$. Then M_1 and M_2 are minimal submodules of M. Moreover, M_1 and M_2 are simple R-modules. But, $M = M_1 \oplus M_2$ and $M_1 \cong M/M_2$ and $M_2 \cong M/M_1$. Consequently, M_1 and M_2 are maximal submodules of M. Therefore, M_1 and M_2 are two maximal and minimal submodules of M. We show that M_1 is an isolated vertex in $\Omega(M)$. To see this, let N be a vertex in $\Omega(M)$, with $N + M_1 \neq M$. Then, maximality of M_1 implies that $N + M_1 = M_1$, and hence $N \subseteq M_1$. Then, minimality of M_1 implies that $M_1 = N$. Hence, M_1 is an isolated vertex in $\Omega(M)$. Thus, $\Omega(M)$ is not connected, a contradiction. This completes the proof. $\square$

Example 2.2. Let $\mathbb{Z}_{pq}$ be a $\mathbb{Z}$-module, such that p and q are two distinct prime numbers. Then $\Omega(\mathbb{Z}_{pq})$ is not connected. Because, $\mathbb{Z}_p$ and $\mathbb{Z}_q$ are simple $\mathbb{Z}$-modules and by Theorem 2.1, $\Omega(\mathbb{Z}_p \oplus \mathbb{Z}_q)$ is not connected. Since $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$, $\Omega(\mathbb{Z}_{pq})$ is not connected. But, we consider $\mathbb{Z}_{p_1p_2p_3}$ as $\mathbb{Z}$-module, such that p_i is a prime number, for $i = 1, 2, 3$. We know $M_1 = p_1\mathbb{Z}_{p_1p_2p_3}$, $M_2 = p_2\mathbb{Z}_{p_1p_2p_3}$ and $M_3 = p_3\mathbb{Z}_{p_1p_2p_3}$ are the only maximal submodules of $\mathbb{Z}_{p_1p_2p_3}$. Also, $K = p_1p_2\mathbb{Z}_{p_1p_2p_3}$, $L = p_1p_3\mathbb{Z}_{p_1p_2p_3}$ and $N = p_2p_3\mathbb{Z}_{p_1p_2p_3}$ are the other submodules of $\mathbb{Z}_{p_1p_2p_3}$. Hence, $\Omega(\mathbb{Z}_{p_1p_2p_3})$ is connected (see Fig. 1).

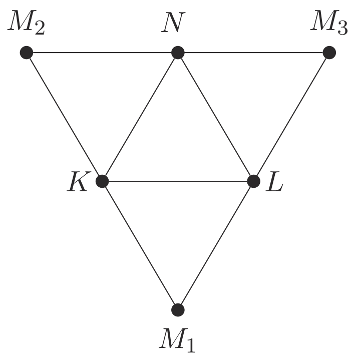

Figure 1. $\Omega(\mathbb{Z}_{p_1p_2p_3})$.

Corollary 2.3. *Let M be an R-module. If $\Omega(M)$ is connected, then the following hold:*

(1) *every pair of maximal submodules of M, have non-trivial intersection, and there exists a path between them;*

(2) *every pair of minimal submodules of M, have non-trivial sum, and there is an edge between them.*

Proof. (1) Let M_1 and M_2 be two maximal submodules of M. Clearly, $M_1 \cap M_2 \neq M$. Let $M_1 \cap M_2 = (0)$. Since $M = M_1 + M_2$, $M = M_1 \oplus M_2$. So $M/M_1 \cong M_2$ and $M/M_2 \cong M_1$, hence M_1 and M_2 are two simple R-modules. Now, by Theorem 2.1, $\Omega(M)$ is not connected, which is a contradiction by hypothesis. Hence $M_1 \cap M_2 \neq (0)$, and there exists a path to form $M_1 \leftrightarrow M_1 \cap M_2 \leftrightarrow M_2$ between them.

(2) Let M_1 and M_2 be two minimal submodules of M such that $M = M_1 + M_2$. If $M_1 \cap M_2 = (0)$, then $M = M_1 \oplus M_2$, such that M_1 and M_2 are two simple R-modules, then by Theorem 2.1, $\Omega(M)$ is not connected, which is a contradiction by hypothesis. Also if $M_1 \cap M_2 \neq (0)$, since $(0) \subsetneqq M_1 \cap M_2 \subseteq M_i \subsetneqq M$, for $i = 1, 2$, by minimality of M_1 and M_2 implies that $M_1 \cap M_2 = M_1 = M_2$, which is a contradiction by hypothesis $M_1 \neq M_2$. Therefore, $M \neq M_1 + M_2$, and there is an edge between them. $\square$

Corollary 2.4. *Let M be an R-module. If $|\Omega(M)| \geqslant 2$, and $\Omega(M)$ is not connected, then the following hold:*
(1) *$\Omega(M)$ is a null graph;*
(2) *$l_R(M) = 2$.*

Proof. (1) Suppose that $\Omega(M)$ is not connected, then by Theorem 2.1, $M = M_1 \oplus M_2$, such that M_1 and M_2 are two simple R-modules. So any non-trivial submodule of M is simple. In fact any non-trivial submodule of M is minimal and consequently a maximal submodule. Hence for each two distinct non-trivial submodules K and L of M, we have $M = K + L$, thus there is no edge between two distinct vertices K and L of the graph $\Omega(M)$. Therefore, $\Omega(M)$ is a null graph.

(2) It is clear by Theorem 2.1. □

Theorem 2.5. *Let M be an R-module. If $\Omega(M)$ is connected, then $diam(\Omega(M)) \leqslant 3$.*

Proof. Let A and B be two non-trivial distinct submodules of M. If $A + B \neq M$ then A and B are adjacent vertices of $\Omega(M)$, so $d(A, B) = 1$. Suppose that $A + B = M$. If $A \cap B \neq (0)$, then there exists a path $A \leftrightarrow A \cap B \leftrightarrow B$ of length 2, so $d(A, B) = 2$. Now, if $A \cap B = (0)$, then $M = A \oplus B$, and since $\Omega(M)$ is connected, by Corollary 2.3(1), implies that at least one of A and B should be non-maximal. Assume that B is not maximal. Hence there exists a submodule X of M such that $B \subsetneqq X \subsetneqq M$, and $B + X = X \neq M$. Now, if $A + X \neq M$, then there exists a path $A \leftrightarrow X \leftrightarrow B$ of length 2, then $d(A, B) = 2$. But, if $A + X = M$, then by Modularity condition, $X = X \cap (A \oplus B) = (X \cap A) \oplus B$. Now, if $X \cap A = (0)$, then $X = B$, a contradiction with existence X. Also, if $X \cap A \neq (0)$, then there exists a path $A \leftrightarrow X \cap A \leftrightarrow X \leftrightarrow B$ of length 3, so $d(A, B) \leqslant 3$. Therefore, $\mathrm{diam}(\Omega(M)) \leqslant 3$. □

Remark 2.6. Let R be an integral domain. Then $\Omega(R)$ is a connected graph with $\mathrm{diam}(\Omega(R)) = 2$.

Proof. Suppose that I and J are two ideals of integral domain R. Now, if $I + J \neq R$, then I and J are adjacent vertices, then $d(I, J) = 1$. But, if $I + J = R$, there exist two possible cases $I \cap J = (0)$ or $I \cap J \neq (0)$. The first case implies that $R = I \oplus J$, then there is idempotent e in R, such that $I = Re$ and $J = R(1 - e)$. Since integral domain R has no zero divisor, then $e = 0$ or $e = 1$, thus $I = (0)$ and $J = R$ or $I = R$ and $J = (0)$, this is a contradiction. In second case, since $IJ = I \cap J \neq (0)$

and $I + IJ = I \neq R$, $J + IJ = J \neq R$, then there exists a path to form $I \leftrightarrow IJ \leftrightarrow J$, then $d(I, J) = 2$. Consequently, $\Omega(R)$ is a connected graph and $\mathrm{diam}(\Omega(R)) = 2$. $\square$

Theorem 2.7. *Let M be an R-module, and $\Omega(M)$ a graph, which contains a cycle. Then girth$(\Omega(M)) = 3$.*

Proof. On the contrary, assume that $\mathrm{girth}(\Omega(M)) \geqslant 4$. This implies that every pair of distinct non-trivial submodules M_1 and M_2 of M with $M_1 + M_2 \neq M$ should be comparable. Because, if X and Y are two distinct non-trivial submodules of M with $X + Y \neq M$ such that $X \nsubseteq Y$ and $Y \nsubseteq X$, then $X \subsetneqq X+Y$ and $Y \subsetneqq X+Y$. As $X+Y+X = X+Y \neq M$ and $Y + X + Y = X + Y \neq M$, hence $\Omega(M)$ has a cycle to form $(X, X+Y, Y)$ of length 3, a contradiction. Now, since $\mathrm{girth}(\Omega(M)) \geqslant 4$, $\Omega(M)$ contains a path of length 3, say $A \leftrightarrow B \leftrightarrow C \leftrightarrow D$. Since every two submodules in this path are comparable and every chain of non-trivial submodules of length 2 induces a cycle of length 3 in $\Omega(M)$, the only two possible cases are $A \subseteq B$, $C \subseteq B$ or $B \subseteq A$, $B \subseteq C$, $D \subseteq C$. The first case yields $A + B = B \neq M$, $C + B = B \neq M$, $A + C \subseteq B \neq M$, then (A, B, C) is a cycle of length 3 in $\Omega(M)$, a contradiction. In the second case, we have $B + A = A \neq M$, $B + C = C \neq M$, $B + D \subseteq C \neq M$ and $C + D = C \neq M$, then (B, C, D) is a cycle of length 3 in $\Omega(M)$, which again this is a contradiction. Consequently, $\mathrm{girth}(\Omega(M)) = 3$, and the proof is complete. $\square$

Example 2.8. Since $\mathbb{Z}$ is an integral domain, then by Remark 2.6, $\Omega(\mathbb{Z})$ is a connected graph and contains a cycle $(2\mathbb{Z},\ 4\mathbb{Z},\ 6\mathbb{Z})$, then by Theorem 2.7, $\mathrm{girth}(\Omega(\mathbb{Z})) = 3$.

Theorem 2.9. *Let M be a Noetherian R-module. Then, $\Omega(M)$ is complete if and only if M contains a unique maximal submodule.*

Proof. Suppose that M is a Noetherian R-module, then M has at least one maximal submodule. Moreover every nonzero submodule of M contained in a maximal submodule. Therefore, if M possesses a unique maximal submodule, say U, then U contains every nonzero submodule of M. Assume that K and L are two distinct vertices of $\Omega(M)$. Then $K \subseteq U$ and $L \subseteq U$, hence $K + L \subseteq U \neq M$. Therefore, $\Omega(M)$ is complete.

Conversely, suppose that $\Omega(M)$ is complete. Let X and Y be two distinct maximal submodules of M. Then $X + Y \neq M$, since $X \subseteq X + Y$ and $Y \subseteq X + Y$, by maximality of X and Y, we have $X + Y = X = Y$, a contradiction. Consequently, M contains a unique maximal submodule, and the proof is complete. $\square$

Theorem 2.10. *Let M be an Artinian R-module. Then $\Omega(M)$ is connected if and only if M contains a unique minimal submodule.*

Proof. Suppose that M is an Artinian R-module, then M has at least one minimal submodule. Moreover, every nonzero submodule of M contains a minimal submodule. Therefore, if M possesses a unique minimal submodule, say L, then L contained in every nonzero submodule of M. Assume that A and B are two distinct vertices of $\Omega(M)$. Then $L \subseteq A$ and $L \subseteq B$, hence $L + A = A \neq M$ and $L + B = B \neq M$. Then there is A, B-path, to form $A \leftrightarrow L \leftrightarrow B$. Therefore, $\Omega(M)$ is connected.

Conversely, suppose that $\Omega(M)$ is connected. Let N_1 and N_2 be two distinct minimal submodules of M. Since $(0) \subseteq N_1 \cap N_2 \subseteq N_i \subsetneqq M$, for $i = 1, 2$, by minimality of N_1 and N_2, if $N_1 \cap N_2 \neq (0)$, then $N_1 \cap N_2 = N_1 = N_2$, a contradiction. If $N_1 \cap N_2 = (0)$, then the only two possible cases are $N_1 + N_2 = M$ or $N_1 + N_2 \neq M$. If $N_1 + N_2 = M$, then $M = N_1 \oplus N_2$ such that N_1 and N_2 are two simple R-modules. Then by Theorem 2.1, $\Omega(M)$ is not connected, a contradiction. But, if $N_1 + N_2 \neq M$, $N_1 = N_1/(N_1 \cap N_2) \cong (N_1 + N_2)/N_2$ and N_1 is simple, then N_2 is maximal submodule of M. Also, similarly N_1 is maximal submodule of M. Since, $(0) \subsetneqq N_i \subseteq N_1 + N_2 \subsetneqq M$, for $i = 1, 2$, by maximality of N_1 and N_2, we have $N_1 + N_2 = N_1 = N_2$, which again this is a contradiction. Consequently, M contains a unique minimal submodule, and the proof is complete. $\square$

Proposition 2.11. *Let M be an R-module, with the graph $\Omega(M)$. Then M is a hollow if and only if $\Omega(M)$ is a complete graph.*

Proof. Suppose that K_1 and K_2 are two distinct vertices of $\Omega(M)$. Since M is a hollow R-module, then $K_1 \ll M$ and $K_2 \ll M$. Then by [3, Proposition 5.17(2)] $K_1 + K_2 \ll M$. Thus, $K_1 + K_2 \neq M$. Therefore, $\Omega(M)$ is a complete graph.

Conversely, assume that $\Omega(M)$ is a complete graph. Let N is a non-trivial submodule of M. Since $\Omega(M)$ is complete, N is adjacent to every other vertex of $\Omega(M)$. Then $N + X \neq M$, for every proper submodule X of M, thus $N \ll M$. Hence, M is a hollow R-module. $\square$

Corollary 2.12. *Let M be an R-module and N be a non-trivial submodule of M. If $|\Omega(M)| = n$, then N is a non-trivial small submodule of M if and only if* $\deg(N) = n - 1$, $n \in \mathbb{N}$.

Proof. It is clear. □

Example 2.13. We consider $\mathbb{Z}_{12}$ as $\mathbb{Z}_{12}$- module. The non-trivial submodules of $\mathbb{Z}_{12}$ are $M_1 = \{0,6\}$, $M_2 = \{0,4,8\}$, $M_3 = \{0,3,6,9\}$, $M_4 = \{0,2,4,6,8,10\}$ such that $M_1 = \{0,6\}$ is the only non-trivial small submodule of $\mathbb{Z}_{12}$ and $|\Omega(\mathbb{Z}_{12})| = 4$. Then, by Corollary 2.12, $\deg(M_1) = 3$ (see Fig. 2).

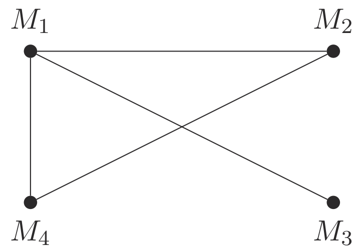

FIGURE 2. $\Omega(\mathbb{Z}_{12})$.

Example 2.14. For every prime number p and for all $n \in \mathbb{N}$ with $n \geqslant 2$, the co-intersection graph of $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$, is a complete graph. Because, $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is local, then it is hollow. Hence, by Proposition 2.11, $\Omega(\mathbb{Z}_{p^n})$ is complete. Also, since the number of non-trivial submodules of $\mathbb{Z}$-module $\mathbb{Z}_{p^n}$ is equal $n-1$. Therefore, $\Omega(\mathbb{Z}_{p^n})$ is a complete graph with $n-1$ vertices, i.e., $\Omega(\mathbb{Z}_{p^n})$=K_{n-1} (see Fig. 3 for $p = 2$ and $n = 5$).

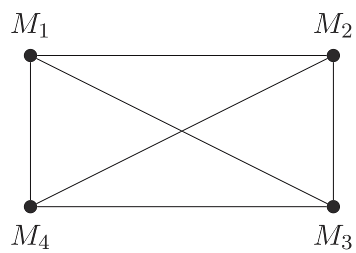

FIGURE 3. $\Omega(\mathbb{Z}_{32})$.

Example 2.15. For every prime number p, the co-intersection graph of $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$, is a complete graph. Because, by [10, 41.23, Exercise (6)], for every prime number p, the $\mathbb{Z}$-module $\mathbb{Z}_{p^\infty}$ is hollow. Therefore, by Proposition 2.11, $\Omega(\mathbb{Z}_{p^\infty})$ is complete.

Corollary 2.16. *Let M be an R-module. Then $\Omega(M)$ is complete, if one of the following holds:*

(1) *if M is an indecomposable R-module, such that every pair of non-trivial submodules of M, have zero intersection;*

(2) *if M is a local R-module;*

(3) *if M is a self-(or quasi-) projective R-module and $\mathrm{End}_R(M)$ is a local ring.*

Proof. (1) It is clear by definition.

(2) Since local R-modules are hollow, it follows from Proposition 2.11.

(3) Since, M is a self- (or quasi-) projective R-module and $\mathrm{End}_R(M)$ is a local ring, M is hollow by [9, Proposition 2.6]. Then it follows from Proposition 2.11. □

3. Clique number, chromatic number and some finiteness conditions

Let M be an R-module. In this section, we obtain some results on the clique and the chromatic number of $\Omega(M)$. We also study the condition under which the chromatic number of $\Omega(M)$ is finite. Finally, it is proved that $\chi(\Omega(M))$ is finite, provided $\omega(\Omega(M))$ is finite.

Lemma 3.1. *Let M be an R-module and $\omega(\Omega(M)) < \infty$. Then the following hold:*

(1) $l_R(M) < \infty$;

(2) $\omega(\Omega(M)) = 1$ *if and only if either* $|\Omega(M)| = 1$ *or* $|\Omega(M)| \geqslant 2$ *and M is a direct sum of two simple R-modules(i.e., $\Omega(M)$ is null);*

(3) *if $\omega(\Omega(M)) > 1$, then the number of minimal submodules of M is finite.*

Proof. (1) Let $M_0 \subset M_1 \subset \cdots \subset M_i \subset M_{i+1} \subset \ldots$, be an infinite strictly increasing sequence of submodules of M. For $i < j$, $M_i + M_j = M_j \neq M$, so similarly for infinite strictly decreasing sequence of submodules of M. Hence, any infinite strictly increasing or decreasing sequence of submodules of M induces a clique in $\Omega(M)$ which contradicts the finiteness $\omega(\Omega)$. This implies that for infinite strictly (increasing and decreasing) sequence of submodules of M, $M_n = M_{n+i}$ for $i = 1, 2, 3, \ldots$. Thus, M should be Noetherian and Artinian. Therefore, $l_R(M) < \infty$.

(2) Suppose that $\omega(\Omega) = 1$ and $|\Omega(M)| \geqslant 2$. This implies that $\Omega(M)$ is not connected. Hence, by Theorem 2.1, M is a direct sum of two simple R-modules.

Conversely, it is clear by Theorem 2.1.

(3) Since $\omega(\Omega) > 1$, by Part (2), M is not a direct sum of two simple R-modules. Then, by Theorem 2.1, $\Omega(M)$ is not connected. Therefore, by Corollary 2.3(2), every pair of minimal submodules of M, have non-trivial sum. Suppose that $\Omega^\star(M)$ is a subgraph of $\Omega(M)$ with the vertex set $V^\star$=$\{L \leqslant M | L$ is *minimal* submodule *of* $M\}$. Then $\Omega^\star(M)$ is a clique in M, and $\mathrm{Card}(V^\star) = \omega(\Omega^\star(M)) \leqslant \omega(\Omega(M)) < \infty$. Hence, then the number of minimal submodules of M is finite. □

Remark 3.2. Let M be an R-module with the length $l_R(M)$ and N be a submodule of M and $\triangle(\Omega) = \max\{\deg(v_i) | v_i \in V(\Omega)\}$, then:

(1) Clearly, $\omega(\Omega(N)) \leqslant \omega(\Omega(M))$ and $\omega(\Omega(M/N)) \leqslant \omega(\Omega(M))$. Hence, $\omega(\Omega(M)) < \infty$, implies that $\omega(\Omega(N)) < \infty$ and $\omega(\Omega(M/N)) < \infty$.

(2) Clearly, $l_R(M) \leqslant \omega(\Omega(M)) + 1$. Also if $\Omega(M)$ is a connected graph, then $\omega(\Omega(M)) \leqslant \chi(\Omega(M)) \leqslant \triangle(\Omega) + 1$ by Theorem 10.3(1) of [6, p. 289]. Hence, $\triangle(\Omega) < \infty$, implies that $\chi(\Omega(M)) < \infty$, $\omega(\Omega(M)) < \infty$, and $l_R(M) < \infty$.

Theorem 3.3. *Let M be an R-module and $|\Omega(M)| \geqslant 2$. Then the following conditions are equivalent:*

(1) *$\Omega(M)$ is a star graph;*
(2) *$\Omega(M)$ is a tree;*
(3) *$\chi(\Omega(M) = 2$;*
(4) *$l_R(M) = 3$, M has a unique minimal submodule L such that every non-trivial submodule contains L is maximal submodule of M.*

Proof. (1) ⇒ (2) ⇒ (3) It follows from definitions.

(3) ⇒ (4), Let $\chi(\Omega(M) = 2$. Then $\Omega(M)$ is not null and by Corollary 2.4(1), $\Omega(M)$ is connected. By Remark 3.2(2), $\omega(\Omega(M)) \leqslant \chi(\Omega(M))$, hence $\omega(\Omega(M)) < \infty$ and by Lemma 3.1(1), $l_R(M) < \infty$. Then M is Artinian. Hence M contains a minimal submodule L. We show that L is unique. Let there exist two minimal submodules L_1 and L_2 of M. Then by Corollary 2.3(2), $L_1 + L_2 \neq M$. Since $(L_1 + L_2) + L_1 = L_1 + L_2 \neq M$ and $(L_1 + L_2) + L_2 = L_1 + L_2 \neq M$, then $(L_1, L_1 + L_2, L_2)$ is a cycle of length 3 in $\Omega(M)$, which contradicts $\chi(\Omega(M) = 2$. Hence, L is a unique minimal submodule of M. Suppose that L contained in every non-trivial submodule of M. If K is a non-trivial submodule of M such that $L \subsetneqq K$, we show that K is a maximal submodule of M. Let $L \subsetneqq K \subsetneqq X \subsetneqq M$, since $L + K = K$, $K + X = X$ and $L + X = X$, (L, K, X) is a cycle of

length 3, which is a contradiction. Consequently, K is a maximal submodule contains L, and $(0) \subsetneqq L \subsetneqq K \subsetneqq M$, is a composition series of M with length 3. Therefore, $l_R(M) = 3$.

(4) $\Rightarrow$ (1) Suppose that $l_R(M) = 3$ and M has a unique minimal submodule L, such that every non-trivial submodule L_i, $(i \in I)$ of M contains L, is a maximal submodule of M. Then, $(0) \subsetneqq L \subsetneqq L_i \subsetneqq M$, for all $i \in I$, are composition series of M with length 3, such that $L_i + L = L_i \neq M$ and $L_i + L_j = M$ for $i \neq j$, Therefore, $\Omega(M)$ is a star graph. The proof is complete. □

Lemma 3.4. *Let M be an R-module and N a vertex of the graph $\Omega(M)$. If $\deg(N) < \infty$, then $l_R(M) < \infty$.*

Proof. Suppose that N contains an infinite strictly increasing sequence of submodules $N_0 \subset N_1 \subset N_2 \subset \cdots$. Then $N_i + N = N \neq M$, for all $i \in I$, which contradicts $\deg(N) < \infty$. Similarly, if N contains an infinite strictly decreasing sequence of submodules, which again yields a contradiction. Also assume that M/N contains an infinite strictly increasing sequence of submodules $M_0/N \subset M_1/N \subset M_2/N \subset \cdots$. Since $N \subset M_0 \subset M_1 \subset M_2 \subset \cdots$. Then $M_i + N = M_i \neq M$, for all $i \in I$, a contradiction. Similarly, if M/N contains an infinite strictly decreasing sequence of submodules, which again yields a contradiction. Hence, N and M/N can not contain an infinite strictly increasing or decreasing sequence of submodules. Thus, they are Noetherian R-module as well as Artinian R-module. Hence, M is Noetherian R-module as well as Artinian R-module. Therefore, $l_R(M) < \infty$. □

Lemma 3.5. *Let M be an R-module and N a minimal submodule of M. Assume that L is a non-trivial submodule of M such that $L + N = M$. Then, L is a maximal submodule of M.*

Proof. Let U be a submodule of M such that $(0) \neq L \subseteq U \subsetneqq M$. Then $L + U = U$ and $(0) \subseteq L \cap N \subseteq U \cap N \subseteq N$. Since N is a minimal submodule of M, $L \cap N = N$ or $U \cap N = (0)$. If $L \cap N = N$ then $N \subseteq L$ thus $N + U \subseteq L + U = U$ and $M = L + N \subseteq N + U$, implies that $M = U$, which is a contradiction. Hence $U \cap N = (0)$ and $U = U \cap (L + N) = L + U \cap N = L$ by Modularity condition. Therefore, L is a maximal submodule of M. □

Theorem 3.6. *Let M be an R-module with the graph $\Omega(M)$ and N is a minimal submodule of M, such that $\deg(N) < \infty$. If $\Omega(M)$ is connected, then the following hold:*

(1) *the number of minimal submodules of M is finite;*
(2) $\chi(\Omega(M)) < \infty$*;*
(3) *if* $\mathrm{Rad}(M) \neq (0)$*, then* $\Omega(M)$ *has a vertex which is adjacent to every other vertex.*

Proof. (1) Let $\Sigma = \{K \leqslant M |$ K *be* a *minimal* submodule *of* $M\}$. Clearly, $\Sigma \neq \varnothing$. Since $\Omega(M)$ is connected, then by Corollary 2.3(2), for all $K \in \Sigma$, $K + N \neq M$, for N and every $K \in \Sigma$ are minimal submodules of M and adjacent vertices of $\Omega(M)$ with $\deg(N) < \infty$. Hence, $\mathrm{Card}(\Sigma) < \infty$, thus the number of minimal submodules of M is finite.

(2) Let $\{U_i\}_{i\in I}$ be the family of non-trivial submodules which are not adjacent to N. Thus, $U_i + N = M$, for all $i \in I$. Hence by Lemma 3.5, U_i is maximal submodule of M, for all $i \in I$. Since $U_i + U_j = M$, for $i \neq j$, distinct vertices U_i and U_j are not two adjacent vertices of $\Omega(M)$. Hence, one can color all $\{U_i\}_{i\in I}$ by a color, and other vertices, which are a finite number of adjacent vertices N, by a new color to obtain a proper vertex coloring of $\Omega(M)$. Therefore, $\chi(\Omega(M)) < \infty$.

(3) In order to establish this part, consider $\mathrm{Rad}(M)$. Since N is a vertex of $\Omega(M)$ and $\deg(N) < \infty$, by Lemma 3.4, $l_R(M) < \infty$ and thus M is Noetherian. Then by [3, Proposition 10.9], M is finitely generated and by [3, Theorem 10.4(1)], $\mathrm{Rad}(M) \ll M$. Now, we know that $\mathrm{Rad}(M) \neq M$, otherwise, $M = (0)$, which is a contradiction. Hence, $\mathrm{Rad}(M)$ is a non-trivial submodule of M and for each non-trivial submodule K of M, we have $K + \mathrm{Rad}(M) \neq M$. Consequently, $\Omega(M)$ has the vertex $\mathrm{Rad}(M)$, which is adjacent to every other vertex. □

Corollary 3.7. *Let M be an R-module with the graph $\Omega(M)$. Then the following hold:*
(1) *if M has no maximal or no minimal submodule, then $\Omega(M)$ is infinite;*
(2) *if M contains a minimal submodule and every minimal submodule of M has finite degree, then $\Omega(M)$ is either null or finite.*

Proof. (1) If M has no maximal submodule, since $(0) \subsetneqq M$ and (0) is not maximal, there exists a submodule M_0 of M, such that $(0) \subsetneqq M_0 \subsetneqq M$, and M_0 is not maximal, there exists a submodule M_1 of M , such that $(0) \subsetneqq M_0 \subsetneqq M_1 \subsetneqq M$. Consequently, there exists $(0) \subsetneqq M_1 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M$, and for $i < j$, $M_i + M_j = M_j \neq M$. Thus M contains an infinite strictly increasing sequence of submodules. Therefore, $\Omega(M)$ is infinite. If M has no minimal submodule, since $M \supsetneqq (0)$ and M is not minimal, there exists a submodule N_0, such that $M \supsetneqq N_0 \supsetneqq (0)$, and N_0 is not minimal, there exists a submodule N_1, such that $M \supsetneqq N_0 \supsetneqq N_1 \supsetneqq (0)$. Consequently, there exists $M \supsetneqq N_0 \supsetneqq N_1 \supsetneqq \cdots \supsetneqq (0)$, and for $i < j$, $N_i + N_j = N_i \neq M$. Thus M contains an infinite strictly decreasing sequence of submodules. Therefore, $\Omega(M)$ is infinite.

(2) Suppose that $\Omega(M)$ is not null and by contrary assume that $\Omega(M)$ is infinite. Since $\Omega(M)$ is not null, by Corollary 2.4(1), $\Omega(M)$ is connected and since every minimal submodule of M has finite degree, by Lemma 3.4, $l_R(M) < \infty$. Hence, M is Artinian and by Theorem 3.6(1), the number of minimal submodules of M is finite. Since $\Omega(M)$ is infinite, and $V(\Omega(M)) = \{N_i | i \in I\}$, there exists a minimal submodule N which $N \subseteq N_i$, for each $i \in I$, then $N + N_i = N_i \neq M$, for each $i \in I$. This contradicts $\deg(N) < \infty$. Hence, $\Omega(M)$ is a finite graph. □

Theorem 3.8. *Let M be an R-module such that $\Omega(M)$ is infinite and $\omega(\Omega(M)) < \infty$. Then the following hold:*

(1) *the number of maximal submodules of M is infinite;*
(2) *the number of non-maximal submodules of M is finite;*
(3) $\chi(\Omega(M)) < \infty$;
(4) $\alpha(\Omega(M)) = \infty$.

Proof. (1) On the contrary, assume that the number of maximal submodules of M is finite. Since $\Omega(M)$ is infinite, $\Omega(M)$ has an infinite clique which contradicts the finiteness of $\omega(\Omega(M))$.

(2) Suppose that $\omega(\Omega(M)) < \infty$, then by Lemma 3.1(1), $l_R(M) < \infty$. Also for each $U \leqslant M$, $l_R(M/U) \leqslant l_R(M)$, $l_R(M/U) < \infty$. We claim that the number of non-maximal submodules of M is finite. To see this, assume that

$$T_n = \{X \lneqq M | l_R(M/X) = n\} \quad \text{and} \quad n_0 = \max\{n | \operatorname{Card}(T_n) = \infty\}.$$

Since $T_1 = \{X \lneqq M | l_R(M/X) = 1\}$, then M/X is a simple R-module, thus X is a maximal submodule of M. Hence, $T_1 = \{X \lneqq M | X \leqslant^{\max} M\}$. By Part(1), T_1 is infinite, then there exists n_0 and $n_0 \geqslant 1$. Since $l_R(M/X) < l_R(M)$ and by Remark 3.2(2), $l_R(M) \leqslant \omega(\Omega(M)) + 1$. Clearly, $1 \leqslant n_0 \leqslant \omega(\Omega(M))$. However, since $l_R(M) < \infty$, Theorem 5 of [8, p. 19], implies that every proper submodule of length n_0 is contained in a submodule of length $n_0 + 1$. Moreover, by the definition of n_0, the number of submodules of length $n_0 + 1$ is finite. Hence there exists a submodule N of M such that $l_R(M/N) = n_0 + 1$ and N contains an infinite number of submodules $\{N_i\}_{i \in I}$ of M, where $l_R(M/N_i)) = n_0$, for all $i \in I$. Now,

$\omega(\Omega(M)) < \infty$ implies that, there exist submodules K and L of M, with $K, L \subseteq N$ and $l_R(M/K) = l_R(M/L) = n_0$, such that $K + L = M$. Since $K \cap L \subseteq N$ and $M/(K \cap L) \cong M/K \oplus M/L$,

$$\begin{aligned} n_0 + 1 = l_R(M/N) &\geqslant l_R(M/(K \cap L)) = l_R(M/K \oplus M/L) \\ &= l_R(M/K) + l_R(M/L) = 2n_0. \end{aligned}$$

Then $n_0 = 1$. Thus, only T_1 is infinite. Consequently, the number of non-maximal submodules of M is finite.

(3) In order to establish this Part, if $\omega(\Omega(M)) = 1$, there is nothing to prove. Let $\omega(\Omega(M)) > 1$. Since, the sum of two distinct maximal submodules is equal M, they are not two adjacent vertices of $\Omega(M)$. Now, by Part (1), the number of maximal submodules of M is infinite. Hence, one can color all maximal submodules by a color, and other vertices, which are finite number, by a new color, to obtain a proper vertex coloring of $\Omega(M)$. Therefore, $\chi(\Omega(M)) < \infty$.

(4) Suppose that $S = \{N \leqslant M | N \leqslant^{\max} M\}$. Since each two elements of S are not two adjacent vertices of $\Omega(M)$, then S is an independent set of the graph $\Omega(M)$. By Part (1), $\mathrm{Card}(S) = \infty$. Hence, $\alpha(\Omega(M)) = \infty$. $\square$

4. Conclusions and future work

In this work we investigated many fundamental properties of the graph $\Omega(M)$ such as connectivity, the diameter, the girth, the clique number, the chromatic number and obtain some interesting results with finiteness conditions on them. However, in future work, shall search the supplement of this graph and research on deeper properties of them.

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