

On one class of algebras

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ABSTRACT. In this paper a g -dimonoid which is isomorphic to the free g -dimonoid is given and a free n -nilpotent g -dimonoid is constructed. We also present the least n -nilpotent congruence on a free g -dimonoid and give numerous examples of g -dimonoids.

1. Introduction

Recall that a dialgebra (dimonoid) [1] is a vector space (set) with two binary operations \dashv and \vdash satisfying the axioms $(x \dashv y) \dashv z = x \dashv (y \dashv z)$ ($D1$), $(x \dashv y) \dashv z = x \dashv (y \vdash z)$ ($D2$), $(x \vdash y) \dashv z = x \vdash (y \dashv z)$ ($D3$), $(x \dashv y) \vdash z = x \vdash (y \vdash z)$ ($D4$), $(x \vdash y) \vdash z = x \vdash (y \vdash z)$ ($D5$). In our time dimonoids are standard tool in the theory of Leibniz algebras. So, for example, free dimonoids were used for constructing free dialgebras and for studying a cohomology of dialgebras. There exist papers devoted to studying structural properties of dimonoids (see, e.g., [2 – 4]). If in the definition of a dialgebra delete the axioms ($D1$), ($D3$), ($D5$), then we obtain a 0-dialgebra which was considered in [5]. Algebras obtained from the definition of a dimonoid by deleting the axioms ($D2$) and ($D4$) were considered in [6]. In the last paper the free object in the corresponding variety was constructed. Observe that dimonoids are closely connected with restrictive bisemigroups considered by B.M. Schein [7]. In [8–11] the notions of interassociativity, respectively,

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strong interassociativity, related semigroups and doppelalgebras which are naturally connected with dimonoids were considered. Another reason for interest in dimonoids is their connection with n -tuple semigroups which were used in [12] for studying properties of n -tuple algebras of associative type. If in the definition of a dimonoid delete the axiom (D3), then we obtain an algebraic system which is called a g -dimonoid (see [13, 14]).

In this paper g -dimonoids are studied. In Section 2 we give numerous examples of g -dimonoids. In Section 3 we suggest a new concrete representation of a free g -dimonoid using the construction of a free g -dimonoid from [14]. The main result of this section was announced in [13]. In Section 4 the construction of a free n -nilpotent g -dimonoid is given. Moreover, here we characterize the least n -nilpotent congruence on a free g -dimonoid.

2. Examples of g -dimonoids

In this section we give different examples of g -dimonoids.

a) Obviously, any dimonoid is a g -dimonoid.

b) Let X be an arbitrary nonempty set, $|X| > 1$ and let X^* be the set of all finite nonempty words in the alphabet X . Denoting the first (respectively, the last) letter of a word $w \in X^*$ by $w^{(0)}$ (respectively, by $w^{(1)}$), define operations \dashv and \vdash on X^* by $w \dashv u = w^{(0)}$, $w \vdash u = u^{(1)}$ for all $w, u \in X^*$. From the proof of Theorem 2 [2] it follows that (X^*, \dashv, \vdash) is a g -dimonoid but not a dimonoid.

c) Let $\{D_i\}_{i \in I}$ be a family of arbitrary g -dimonoids D_i , $i \in I$, and let $\overline{\prod}_{i \in I} D_i$ be a set of all functions $f : I \rightarrow \bigcup_{i \in I} D_i$ such that $if \in D_i$ for any $i \in I$.

It is easy to prove the following lemma.

Lemma 1. $\overline{\prod}_{i \in I} D_i$ with multiplications defined by

$$i(f_1 \dashv f_2) = if_1 \dashv if_2, \quad i(f_1 \vdash f_2) = if_1 \vdash if_2, \quad (1)$$

where $i \in I$, $f_1, f_2 \in \overline{\prod}_{i \in I} D_i$, is a g -dimonoid.

The obtained algebra is called the Cartesian product of g -dimonoids D_i , $i \in I$. If I is finite, then the Cartesian product and the direct product coincide. The Cartesian product of a finite number of g -dimonoids D_1, D_2, \dots, D_n is denoted by $D_1 \times D_2 \times \dots \times D_n$. In particular, the Cartesian power of a g -dimonoid can be defined as follows. Let V be an arbitrary g -dimonoid and X be any nonempty set. Denote by $Map(X; V)$ the set of

all maps $X \rightarrow V$. Define operations \dashv and \vdash on $Map(X; V)$ by (1) for all $f_1, f_2 \in Map(X; V)$ and $i \in X$. Then $(Map(X; V), \dashv, \vdash)$ is a g -dimonoid which is called the Cartesian power of V .

d) As usual, \mathbb{N} denotes the set of all positive integers.

Let $F[X]$ be the free semigroup in an alphabet X . We denote the length of a word $w \in F[X]$ by $l(w)$. Fix $n \in \mathbb{N}$ and define operations \dashv and \vdash on $F[X] \times \mathbb{N}$ by

$$(w_1, m_1) \dashv (w_2, m_2) = (w_1 w_2, n),$$

$$(w_1, m_1) \vdash (w_2, m_2) = (w_1 w_2, l(w_1) + m_2)$$

for all $(w_1, m_1), (w_2, m_2) \in F[X] \times \mathbb{N}$. Denote the algebra $(F[X] \times \mathbb{N}, \dashv, \vdash)$ by $X\mathbb{N}_n$.

Lemma 2. *The algebra $X\mathbb{N}_n$ is a g -dimonoid but not a dimonoid.*

Proof. One can directly verify that $X\mathbb{N}_n$ is a g -dimonoid. Show that it is not a dimonoid. For all $(w_1, m_1), (w_2, m_2), (w_3, m_3) \in X\mathbb{N}_n$ obtain

$$\begin{aligned} ((w_1, m_1) \vdash (w_2, m_2)) \dashv (w_3, m_3) &= (w_1 w_2, l(w_1) + m_2) \dashv (w_3, m_3) = \\ &= (w_1 w_2 w_3, n) \neq (w_1 w_2 w_3, l(w_1) + n) = (w_1, m_1) \vdash (w_2 w_3, n) = \\ &= (w_1, m_1) \vdash ((w_2, m_2) \dashv (w_3, m_3)). \quad \square \end{aligned}$$

e) Let S be an arbitrary semigroup, $a, b \in S$. By E_S denote the set of all idempotents of S . Define operations \dashv and \vdash on S by

$$x \dashv y = ax, \quad x \vdash y = by$$

for all $x, y \in S$. Denote the algebra (S, \dashv, \vdash) by $S(a, b)$.

Lemma 3. *Let S be an arbitrary right cancellative semigroup, $a, b \in E_S$.*

(i) *If a and b are non-commuting, then $S(a, b)$ is a g -dimonoid but not a dimonoid.*

(ii) *If a and b are commuting, then $S(a, b)$ is a dimonoid.*

Proof. (i) The axioms (D1), (D2), (D4), (D5) are checked directly. Besides,

$$(x \vdash y) \dashv z = by \dashv z = aby, \quad x \vdash (y \dashv z) = x \vdash ay = bay$$

for all $x, y, z \in S$. Suppose that $aby = bay$. Then, using the right cancellability, obtain $ab = ba$. Thus, we arrive at a contradiction, i.e., the

assumption that $aby = bay$ does not hold. Consequently, $S(a, b)$ is not a dimonoid.

(ii) If a and b are commuting, then, obviously, all axioms of a dimonoid hold. \square

f) Let S be an arbitrary semigroup, $a, b \in S$. Define operations \dashv and \vdash on S by

$$x \dashv y = xa, \quad x \vdash y = yb$$

for all $x, y \in S$. Denote the algebra (S, \dashv, \vdash) by $S[a, b]$.

Similarly to Lemma 3, the following lemma can be proved.

Lemma 4. *Let S be an arbitrary left cancellative semigroup, $a, b \in E_S$.*

(i) *If a and b are non-commuting, then $S[a, b]$ is a g -dimonoid but not a dimonoid.*

(ii) *If a and b are commuting, then $S[a, b]$ is a dimonoid.*

g) Let S be an arbitrary semigroup, $a, b \in S$. Define operations \dashv and \vdash on S by

$$x \dashv y = axb, \quad x \vdash y = ayb$$

for all $x, y \in S$. Denote the algebra (S, \dashv, \vdash) by $S(a, b]$.

The following lemma could be proved immediately.

Lemma 5. *If $a, b \in E_S$, then $S(a, b]$ is a dimonoid.*

h) Let Y be an arbitrary nonempty set, $S = S_Y$ be some monoid defined on the set of finite words in the alphabet Y and $\theta \in S$ be an empty word which is a unit of S . Denote the operation on S by $*$ and the length of a word $w \in S$ by $l(w)$. By definition $l(\theta) = 0$, $u^0 = \theta$ for all $u \in S$. Fix elements $a, b \in Y$, $k \in \mathbb{N} \cup \{0\}$ and define operations \dashv and \vdash on S , assuming

$$u_1 \dashv u_2 = u_1 * a^{l(u_2)+k}, \quad u_1 \vdash u_2 = u_2 * b^{l(u_1)+k}$$

for all $u_1, u_2 \in S$. The obtained algebra will be denoted by $S_a^b(k)$.

Lemma 6. *Let T be the free monoid in the alphabet Y . Then for any $a, b \in Y$, $k \in \mathbb{N} \cup \{0\}$ the algebra $T_a^b(k)$ is a g -dimonoid. If $a \neq b$, then it is not a dimonoid.*

Proof. Let $u_1, u_2, u_3 \in T_a^b(k)$. In order to prove that $T_a^b(k)$ is a g -dimonoid we consider the following cases.

Case 1. Let $u_1 \neq \theta$, $u_2 \neq \theta$, $u_3 \neq \theta$. Then

$$\begin{aligned}
 u_1 \dashv (u_2 \dashv u_3) &= u_1 \dashv (u_2 * a^{l(u_3)+k}) = \\
 &= u_1 * a^{l(u_2 a^{l(u_3)+k})+k} = u_1 * a^{l(u_2)+l(u_3)+2k} = \\
 &= u_1 * a^{l(u_2)+k} * a^{l(u_3)+k} = (u_1 * a^{l(u_2)+k}) \dashv u_3 = (u_1 \dashv u_2) \dashv u_3, \\
 u_1 \dashv (u_2 \vdash u_3) &= u_1 \dashv (u_3 * b^{l(u_2)+k}) = \\
 &= u_1 * a^{l(u_3 b^{l(u_2)+k})+k} = u_1 * a^{l(u_3)+l(u_2)+2k}, \\
 u_1 \vdash (u_2 \dashv u_3) &= u_1 \vdash (u_3 * b^{l(u_2)+k}) = \\
 &= u_3 * b^{l(u_2)+k} * b^{l(u_1)+k} = u_3 * b^{l(u_2)+l(u_1)+2k} = \\
 &= u_3 * b^{l(u_2 b^{l(u_1)+k})+k} = (u_2 * b^{l(u_1)+k}) \vdash u_3 = (u_1 \vdash u_2) \vdash u_3, \\
 (u_1 \dashv u_2) \vdash u_3 &= (u_1 * a^{l(u_2)+k}) \vdash u_3 = \\
 &= u_3 * b^{l(u_1 a^{l(u_2)+k})+k} = u_3 * b^{l(u_1)+l(u_2)+2k}.
 \end{aligned}$$

Case 2. Let $u_1 = u_2 = u_3 = \theta$. Then

$$\begin{aligned}
 \theta \dashv (\theta \dashv \theta) &= \theta \dashv (\theta * a^{l(\theta)+k}) = \theta \dashv a^k = \theta * a^{l(a^k)+k} = a^{2k} = \\
 &= a^k * a^{l(\theta)+k} = a^k \dashv \theta = (\theta * a^{l(\theta)+k}) \dashv \theta = (\theta \dashv \theta) \dashv \theta, \\
 \theta \dashv (\theta \vdash \theta) &= \theta \dashv (\theta * b^{l(\theta)+k}) = \theta \dashv b^k = \theta * a^{l(b^k)+k} = a^{2k}, \\
 \theta \vdash (\theta \dashv \theta) &= \theta \vdash (\theta * b^{l(\theta)+k}) = \theta \vdash b^k = b^k * b^{l(\theta)+k} = b^{2k} = \\
 &= \theta * b^{l(b^k)+k} = b^k \vdash \theta = (\theta * b^{l(\theta)+k}) \vdash \theta = (\theta \vdash \theta) \vdash \theta, \\
 (\theta \dashv \theta) \vdash \theta &= (\theta * a^{l(\theta)+k}) \vdash \theta = a^k \vdash \theta = \theta * b^{l(a^k)+k} = b^{2k}.
 \end{aligned}$$

Case 3. Let $u_1 = \theta$, $u_2 \neq \theta$, $u_3 \neq \theta$. Then

$$\begin{aligned}
 \theta \dashv (u_2 \dashv u_3) &= \theta \dashv (u_2 * a^{l(u_3)+k}) = \theta * a^{l(u_2 a^{l(u_3)+k})+k} = a^{l(u_2)+l(u_3)+2k} = \\
 &= a^{l(u_2)+k} * a^{l(u_3)+k} = (\theta * a^{l(u_2)+k}) \dashv u_3 = (\theta \dashv u_2) \dashv u_3, \\
 \theta \dashv (u_2 \vdash u_3) &= \theta \dashv (u_3 * b^{l(u_2)+k}) = \theta * a^{l(u_3 b^{l(u_2)+k})+k} = a^{l(u_2)+l(u_3)+2k}, \\
 \theta \vdash (u_2 \dashv u_3) &= \theta \vdash (u_3 * b^{l(u_2)+k}) = u_3 * b^{l(u_2)+k} * b^{l(\theta)+k} = u_3 * b^{l(u_2)+2k} = \\
 &= u_3 * b^{l(u_2 * b^{l(\theta)+k})+k} = (u_2 * b^{l(\theta)+k}) \vdash u_3 = (\theta \vdash u_2) \vdash u_3, \\
 (\theta \dashv u_2) \vdash u_3 &= (\theta * a^{l(u_2)+k}) \vdash u_3 = u_3 * b^{l(a^{l(u_2)+k})+k} = u_3 * b^{l(u_2)+2k}.
 \end{aligned}$$

Case 4. Let $u_1 = \theta$, $u_2 \neq \theta$, $u_3 = \theta$. Then

$$\begin{aligned} \theta \dashv (u_2 \dashv \theta) &= \theta \dashv (u_2 * a^{l(\theta)+k}) = \theta \dashv (u_2 * a^k) = \theta * a^{l(u_2 * a^k)+k} = a^{l(u_2)+2k} = \\ &= a^{l(u_2)+k} * a^{l(\theta)+k} = (\theta * a^{l(u_2)+k}) \dashv \theta = (\theta \dashv u_2) \dashv \theta, \\ \theta \dashv (u_2 \vdash \theta) &= \theta \dashv (\theta * b^{l(u_2)+k}) = \theta * a^{l(b^{l(u_2)+k})+k} = a^{l(u_2)+2k}, \\ \theta \vdash (u_2 \dashv \theta) &= \theta \vdash (\theta * b^{l(u_2)+k}) = b^{l(u_2)+k} * b^{l(\theta)+k} = b^{l(u_2)+2k} = \\ &= \theta * b^{l(u_2 * b^k)+k} = (u_2 * b^{l(\theta)+k}) \vdash \theta = (\theta \dashv u_2) \vdash \theta, \\ (\theta \dashv u_2) \vdash \theta &= (\theta * a^{l(u_2)+k}) \vdash \theta = \theta * b^{l(a^{l(u_2)+k})+k} = b^{l(u_2)+2k}. \end{aligned}$$

The cases $u_1 \neq \theta$, $u_2 = \theta$, $u_3 \neq \theta$; $u_1 \neq \theta$, $u_2 \neq \theta$, $u_3 = \theta$; $u_1 = u_2 = \theta$, $u_3 \neq \theta$; $u_1 \neq \theta$, $u_2 = u_3 = \theta$ are considered in a similar way.

Thus, $T_a^b(k)$ is a g -dimonoid.

Finally, show that $T_a^b(k)$ is not a dimonoid when $a \neq b$. For $u_1 \neq \theta$, $u_2 \neq \theta$ and $u_3 \neq \theta$ we have

$$\begin{aligned} (u_1 \vdash u_2) \dashv u_3 &= (u_2 * b^{l(u_1)+k}) \dashv u_3 = u_2 * b^{l(u_1)+k} * a^{l(u_3)+k} = \\ &= u_2 b^{l(u_1)+k} a^{l(u_3)+k} \neq u_2 a^{l(u_3)+k} b^{l(u_1)+k} = \\ &= u_2 * a^{l(u_3)+k} * b^{l(u_1)+k} = u_1 \vdash (u_2 * a^{l(u_3)+k}) = u_1 \vdash (u_2 \dashv u_3) \end{aligned}$$

and so, the axiom (D3) of a dimonoid does not hold. \square

The following lemma gives an answer on the question when $S_a^b(k)$ is a dimonoid.

Lemma 7. *Let M be the free commutative monoid in the alphabet Y . For any $a, b \in Y$, $k \in \mathbb{N} \cup \{0\}$ algebras $M_a^b(k)$ and $T_a^a(k)$ are dimonoids.*

Proof. From Lemma 6 it follows that $M_a^b(k)$ satisfies the axioms (D1), (D2), (D4), (D5). Show that the axiom (D3) also holds.

Let $u_1, u_2, u_3 \in M_a^b(k)$. Consider the following eight cases.

Case 1. Let $u_1 \neq \theta$, $u_2 \neq \theta$, $u_3 \neq \theta$. Then

$$\begin{aligned} (u_1 \vdash u_2) \dashv u_3 &= (u_2 * b^{l(u_1)+k}) \dashv u_3 = u_2 * b^{l(u_1)+k} * a^{l(u_3)+k} = \\ &= u_2 * a^{l(u_3)+k} * b^{l(u_1)+k} = u_1 \vdash (u_2 * a^{l(u_3)+k}) = u_1 \vdash (u_2 \dashv u_3). \end{aligned}$$

Case 2. Let $u_1 = u_2 = u_3 = \theta$. Then

$$(\theta \vdash \theta) \dashv \theta = (\theta * b^{l(\theta)+k}) \dashv \theta = b^k \dashv \theta = b^k * a^{l(\theta)+k} = b^k * a^k =$$

$$= a^k * b^k = a^k * b^{l(\theta)+k} = \theta \vdash a^k = \theta \vdash (\theta * a^{l(\theta)+k}) = \theta \vdash (\theta \dashv \theta).$$

Case 3. Let $u_1 = \theta$, $u_2 \neq \theta$, $u_3 \neq \theta$. Then

$$\begin{aligned} (\theta \vdash u_2) \dashv u_3 &= (u_2 * b^{l(\theta)+k}) \dashv u_3 = u_2 * b^{l(\theta)+k} * a^{l(u_3)+k} = u_2 * b^k * a^{l(u_3)+k} = \\ &= u_2 * a^{l(u_3)+k} * b^k = u_2 * a^{l(u_3)+k} * b^{l(\theta)+k} = \theta \vdash (u_2 * a^{l(u_3)+k}) = \theta \vdash (u_2 \dashv u_3). \end{aligned}$$

Case 4. Let $u_1 \neq \theta$, $u_2 = \theta$, $u_3 \neq \theta$. Then

$$\begin{aligned} (u_1 \vdash \theta) \dashv u_3 &= (\theta * b^{l(u_1)+k}) \dashv u_3 = b^{l(u_1)+k} * a^{l(u_3)+k} = \\ &= a^{l(u_3)+k} * b^{l(u_1)+k} = u_1 \vdash (\theta * a^{l(u_3)+k}) = u_1 \vdash (\theta \dashv u_3). \end{aligned}$$

Case 5. Let $u_1 \neq \theta$, $u_2 \neq \theta$, $u_3 = \theta$. Then

$$\begin{aligned} (u_1 \vdash u_2) \dashv \theta &= (u_2 * b^{l(u_1)+k}) \dashv \theta = u_2 * b^{l(u_1)+k} * a^{l(\theta)+k} = u_2 * b^{l(u_1)+k} * a^k = \\ &= u_2 * a^k * b^{l(u_1)+k} = u_1 \vdash (u_2 * a^{l(\theta)+k}) = u_1 \vdash (u_2 \dashv \theta). \end{aligned}$$

Case 6. Let $u_1 = u_2 = \theta$, $u_3 \neq \theta$. Then

$$\begin{aligned} (\theta \vdash \theta) \dashv u_3 &= (\theta * b^{l(\theta)+k}) \dashv u_3 = b^k \dashv u_3 = b^k * a^{l(u_3)+k} = \\ &= a^{l(u_3)+k} * b^k = a^{l(u_3)+k} * b^{l(\theta)+k} = \theta \vdash (\theta * a^{l(u_3)+k}) = \theta \vdash (\theta \dashv u_3). \end{aligned}$$

Case 7. Let $u_1 \neq \theta$, $u_2 = u_3 = \theta$. Then

$$\begin{aligned} (u_1 \vdash \theta) \dashv \theta &= (\theta * b^{l(u_1)+k}) \dashv \theta = b^{l(u_1)+k} * a^{l(\theta)+k} = b^{l(u_1)+k} * a^k = \\ &= a^k * b^{l(u_1)+k} = u_1 \vdash a^k = u_1 \vdash (\theta * a^{l(\theta)+k}) = u_1 \vdash (\theta \dashv \theta). \end{aligned}$$

Case 8. Let $u_1 = \theta$, $u_2 \neq \theta$, $u_3 = \theta$. Then

$$\begin{aligned} (\theta \vdash u_2) \dashv \theta &= (u_2 * b^{l(\theta)+k}) \dashv \theta = u_2 * b^k * a^{l(\theta)+k} = u_2 * b^k * a^k = \\ &= u_2 * a^k * b^k = u_2 * a^k * b^{l(\theta)+k} = \theta \vdash (u_2 * a^{l(\theta)+k}) = \theta \vdash (u_2 \dashv \theta). \end{aligned}$$

Thus, $M_a^b(k)$ is a dimonoid.

A proof is the same for $T_a^a(k)$. □

Note that independence of axioms of a g -dimonoid follows from independence of axioms of a dimonoid (see [2], Theorem 2).

3. Free g -dimonoids

In this section we construct a g -dimonoid which is isomorphic to the free g -dimonoid of an arbitrary rank and consider separately free g -dimonoids of rank 1.

A nonempty subset A of a g -dimonoid (D, \dashv, \vdash) is called a g -subdimonoid, if for any $a, b \in D$, $a, b \in A$ implies $a \dashv b$, $a \vdash b \in A$.

Note that the class of all g -dimonoids is a variety as it is closed under taking of homomorphic images, g -subdimonoids and Cartesian products. A g -dimonoid which is free in the variety of all g -dimonoids is called a free g -dimonoid.

In order to prove the main result of this section we need the construction of a free g -dimonoid from [14].

Let e be an arbitrary symbol. Consider the following sets:

$$I^1 = \{e\}, \quad I^n = \{(\varepsilon_1, \dots, \varepsilon_{n-1}) \mid \varepsilon_k \in \{0, 1\}, 1 \leq k \leq n-1\}, \quad n > 1,$$

$$I = \bigcup_{n \geq 1} I^n.$$

If $l = 0$, we will regard the sequence $\varepsilon_1, \dots, \varepsilon_l$ without brackets as empty, and the sequence $(\varepsilon_1, \dots, \varepsilon_l)$ with brackets as e . Define operations \dashv and \vdash on I by

$$(\varepsilon_1, \dots, \varepsilon_{n-1}) \dashv (\theta_1, \dots, \theta_{m-1}) = (\varepsilon_1, \dots, \varepsilon_{n-1}, \underbrace{1, 1, \dots, 1}_m),$$

$$(\varepsilon_1, \dots, \varepsilon_{n-1}) \vdash (\theta_1, \dots, \theta_{m-1}) = (\theta_1, \dots, \theta_{m-1}, \underbrace{0, 0, \dots, 0}_n).$$

By Lemma 3 from [14] (I, \dashv, \vdash) is a g -dimonoid. Observe that $e \dashv e = (1)$, $e \vdash e = (0)$ and (I, \dashv, \vdash) is not a dimonoid.

Let X be an arbitrary nonempty set and $F[X]$ be the free semigroup in the alphabet X . Define operations \dashv and \vdash on $FG = \{(w, \varepsilon) \mid w \in F[X], \varepsilon \in I^{l(w)}\}$ by

$$(w_1, \varepsilon) \dashv (w_2, \xi) = (w_1 w_2, \varepsilon \dashv \xi),$$

$$(w_1, \varepsilon) \vdash (w_2, \xi) = (w_1 w_2, \varepsilon \vdash \xi)$$

for all $(w_1, \varepsilon), (w_2, \xi) \in FG$. The algebra (FG, \dashv, \vdash) is denoted by $FG[X]$. By Theorem 4 from [14] $FG[X]$ is the free g -dimonoid.

Using notations from Section 2, introduce the set

$$XT_a^b(k) = \{(w, u) \in F[X] \times T_a^b(k) \mid l(w) - l(u) = 1\}.$$

If $s = 1$, we will regard the sequence $y_1y_2\dots y_{s-1} \in T_a^b(k)$ as θ .

The main result of this section is the following.

Theorem 1. *The g -dimonoid $XT_a^b(1)$ is free if $|Y| = 2$ and $a \neq b$.*

Proof. By Lemma 1 $F[X] \times T_a^b(k)$ is a g -dimonoid. It is not difficult to check that $XT_a^b(1)$ is a g -subdimonoid of $F[X] \times T_a^b(1)$.

Let $|Y| = 2$ and $a \neq b$. Let us show that $XT_a^b(1)$ is free. Take $(x_1x_2\dots x_s, y_1y_2\dots y_{s-1}) \in XT_a^b(1)$, where $x_i \in X, 1 \leq i \leq s, y_j \in Y, 1 \leq j \leq s - 1$, and define a map

$$\pi : XT_a^b(1) \rightarrow FG[X] :$$

$$(x_1x_2\dots x_s, y_1y_2\dots y_{s-1}) \mapsto (x_1x_2\dots x_s, y_1y_2\dots y_{s-1})\pi,$$

assuming

$$(x_1x_2\dots x_s, y_1y_2\dots y_{s-1})\pi = (x_1x_2\dots x_s, (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{s-1})),$$

where

$$\tilde{y}_i = \begin{cases} 1, & y_i = a, \\ 0, & y_i = b \end{cases}$$

for all $1 \leq i \leq s - 1, s \neq 1$, and $(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{s-1})$ is e for $s = 1$. Show that π is an isomorphism.

For all

$$(x_1x_2\dots x_s, y_1y_2\dots y_{s-1}), (a_1a_2\dots a_m, b_1b_2\dots b_{m-1}) \in XT_a^b(1),$$

where $a_i \in X, 1 \leq i \leq m, b_j \in Y, 1 \leq j \leq m - 1$, obtain

$$\begin{aligned} & ((x_1x_2\dots x_s, y_1y_2\dots y_{s-1}) \dashv (a_1a_2\dots a_m, b_1b_2\dots b_{m-1}))\pi = \\ & = (x_1x_2\dots x_s a_1a_2\dots a_m, y_1y_2\dots y_{s-1} * a^m)\pi = \\ & = \left(x_1x_2\dots x_s a_1a_2\dots a_m, (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{s-1}, \underbrace{\tilde{a}, \tilde{a}, \dots, \tilde{a}}_m) \right) = \\ & = \left(x_1x_2\dots x_s a_1a_2\dots a_m, (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{s-1}, \underbrace{1, 1, \dots, 1}_m) \right) = \\ & = (x_1x_2\dots x_s, (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{s-1})) \dashv (a_1a_2\dots a_m, (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_{m-1})) = \\ & = (x_1x_2\dots x_s, y_1y_2\dots y_{s-1})\pi \dashv (a_1a_2\dots a_m, b_1b_2\dots b_{m-1})\pi, \end{aligned}$$

$$\begin{aligned}
 & ((x_1x_2 \dots x_s, y_1y_2 \dots y_{s-1}) \vdash (a_1a_2 \dots a_m, b_1b_2 \dots b_{m-1})) \pi = \\
 & = (x_1x_2 \dots x_s a_1a_2 \dots a_m, b_1b_2 \dots b_{m-1} * b^s) \pi = \\
 & = \left(x_1x_2 \dots x_s a_1a_2 \dots a_m, (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_{m-1}, \underbrace{\tilde{b}, \tilde{b}, \dots, \tilde{b}}_s) \right) = \\
 & = \left(x_1x_2 \dots x_s a_1a_2 \dots a_m, (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_{m-1}, \underbrace{0, 0, \dots, 0}_s) \right) = \\
 & = (x_1x_2 \dots x_s, (\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_{s-1})) \vdash (a_1a_2 \dots a_m, (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_{m-1})) = \\
 & = (x_1x_2 \dots x_s, y_1y_2 \dots y_{s-1}) \pi \vdash (a_1a_2 \dots a_m, b_1b_2 \dots b_{m-1}) \pi.
 \end{aligned}$$

So, π is a homomorphism. Obviously, π is a bijection and thus, π is an isomorphism. Hence we obtain that $XT_a^b(1)$ is the free g -dimonoid. \square

The following lemma gives one property of $S_a^b(k)$.

Lemma 8. *If $S_a^b(k)$ is a dimonoid, then a^k and b^k are commuting in S .*

Proof. Let $S_a^b(k)$ be a dimonoid. Then

$$\begin{aligned}
 (\theta \dashv \theta) \dashv \theta & = (\theta * b^{l(\theta)+k}) \dashv \theta = b^k \dashv \theta = b^k * a^{l(\theta)+k} = b^k * a^k, \\
 \theta \vdash (\theta \dashv \theta) & = \theta \vdash (\theta * a^{l(\theta)+k}) = \theta \vdash a^k = a^k * b^{l(\theta)+k} = a^k * b^k
 \end{aligned}$$

and, using the axiom (D3), obtain $b^k * a^k = a^k * b^k$. \square

Now we construct a g -dimonoid which is isomorphic to the free g -dimonoid of rank 1.

Let $|Y| = 2, a \neq b$. Define operations \dashv and \vdash on

$$\widetilde{NT}_a^b(1) = \{(m, u) \in \mathbb{N} \times T_a^b(1) \mid m - l(u) = 1\}$$

by

$$\begin{aligned}
 (m_1, u_1) \dashv (m_2, u_2) & = (m_1 + m_2, u_1 * a^{l(u_2)+1}), \\
 (m_1, u_1) \vdash (m_2, u_2) & = (m_1 + m_2, u_2 * b^{l(u_1)+1})
 \end{aligned}$$

for all $(m_1, u_1), (m_2, u_2) \in \widetilde{NT}_a^b(1)$. By Lemma 1 $(\mathbb{N}, +) \times T_a^b(1)$ is a g -dimonoid. An immediate verification shows that operations \dashv and \vdash are well-defined. Thus, $(\widetilde{NT}_a^b(1), \dashv, \vdash)$ is a g -subdimonoid of $(\mathbb{N}, +) \times T_a^b(1)$. Denote it by $NT_a^b(1)$.

Lemma 9. *The free g -dimonoid of rank 1 is isomorphic to the g -dimonoid $\mathbb{N}T_a^b(1)$.*

Proof. Let $X = \{r\}$. An easy verification shows that a map

$$\xi : XT_a^b(1) \rightarrow \mathbb{N}T_a^b(1),$$

defined by $\omega\xi = (k, u) \Leftrightarrow \omega = (r^k, u)$, is an isomorphism. □

4. Free n -nilpotent g -dimonoids

In this section we construct a free n -nilpotent g -dimonoid of an arbitrary rank and consider separately free n -nilpotent g -dimonoids of rank 1. We also characterize the least n -nilpotent congruence on a free g -dimonoid.

An element 0 of a g -dimonoid (D, \dashv, \vdash) will be called zero, if $x \star 0 = 0 = 0 \star x$ for all $x \in D$ and $\star \in \{\dashv, \vdash\}$.

A g -dimonoid (D, \dashv, \vdash) with zero will be called nilpotent, if for some $n \in \mathbb{N}$ and any $x_i \in D$, $1 \leq i \leq n + 1$, and $\star_j \in \{\dashv, \vdash\}$, $1 \leq j \leq n$, any parenthesizing of

$$x_1 \star_1 x_2 \star_2 \dots \star_n x_{n+1} \tag{2}$$

gives $0 \in D$. The least such n we shall call the nilpotency index of (D, \dashv, \vdash) . For $k \in \mathbb{N}$ a nilpotent g -dimonoid of nilpotency index $\leq k$ is said to be k -nilpotent.

Note that from (2) it follows that operations of any 1-nilpotent g -dimonoid coincide and it is a zero semigroup.

It is not difficult to see that the class of all n -nilpotent g -dimonoids is a subvariety of the variety of all g -dimonoids. A g -dimonoid which is free in the variety of n -nilpotent g -dimonoids will be called a free n -nilpotent g -dimonoid.

Fix $n \in \mathbb{N}$ and, using notations from Section 3, assume

$$G_n = \{(w, u) \in XT_a^b(1) \mid l(w) \leq n\} \cup \{0\} \quad (|Y| = 2, a \neq b).$$

Define operations \prec and \succ on G_n by

$$\begin{aligned} (w_1, u_1) \prec (w_2, u_2) &= \begin{cases} (w_1 w_2, u_1 \star a^{l(u_2)+1}), & l(w_1 w_2) \leq n, \\ 0, & l(w_1 w_2) > n, \end{cases} \\ (w_1, u_1) \succ (w_2, u_2) &= \begin{cases} (w_1 w_2, u_2 \star b^{l(u_1)+1}), & l(w_1 w_2) \leq n, \\ 0, & l(w_1 w_2) > n, \end{cases} \end{aligned}$$

$$(w_1, u_1) \star 0 = 0 \star (w_1, u_1) = 0 \star 0 = 0$$

for all $(w_1, u_1), (w_2, u_2) \in G_n \setminus \{0\}$ and $\star \in \{\prec, \succ\}$. The algebra (G_n, \prec, \succ) will be denoted by $G_n(X)$.

Theorem 2. $G_n(X)$ is the free n -nilpotent g -dimonoid.

Proof. Prove that $G_n(X)$ is a g -dimonoid. Let $(w_1, u_1), (w_2, u_2), (w_3, u_3) \in G_n \setminus \{0\}$. If $l(w_1 w_2) > n$ or $l(w_2 w_3) > n$, then the proof is straightforward. The fact that axioms of a g -dimonoid hold when $l(w_1 w_2 w_3) \leq n$ follows from Theorem 1. In the case $l(w_1 w_2) \leq n$, $l(w_2 w_3) \leq n$ and $l(w_1 w_2 w_3) > n$ we have

$$((w_1, u_1) *_1 (w_2, u_2)) *_2 (w_3, u_3) = 0 = (w_1, u_1) *_1 ((w_2, u_2) *_2 (w_3, u_3))$$

for $*_1, *_2 \in \{\prec, \succ\}$. The proofs of the remaining cases are obvious. Thus, $G_n(X)$ is a g -dimonoid.

For any $(w_i, u_i) \in G_n \setminus \{0\}$, $1 \leq i \leq n+1$, and $*_j \in \{\prec, \succ\}$, $1 \leq j \leq n$, any parenthesizing of

$$(w_1, u_1) *_1 (w_2, u_2) *_2 \dots *_n (w_{n+1}, u_{n+1})$$

gives 0, hence $G_n(X)$ is nilpotent. Moreover, for any $(x_i, \theta) \in G_n \setminus \{0\}$, where $x_i \in X$, $1 \leq i \leq n$,

$$(x_1, \theta) \prec (x_2, \theta) \prec \dots \prec (x_n, \theta) = (x_1 x_2 \dots x_n, a^{n-1}) \neq 0.$$

It means that $G_n(X)$ has nilpotency index n .

Let us show that $G_n(X)$ is free in the variety of n -nilpotent g -dimonoids.

The g -dimonoid $(\mathcal{G}(X), \dashv, \vdash)$ which is isomorphic to $FG[X]$ from Section 3 was constructed in [14]. The corresponding isomorphism $(\mathcal{G}(X), \dashv, \vdash) \rightarrow FG[X]$ is denoted by σ (see [14], Theorem 4). In the last paper for an arbitrary g -dimonoid $(\mathcal{D}, \dashv, \vdash)$ the homomorphism ψ_0 from $(\mathcal{G}(X), \dashv, \vdash)$ to $(\mathcal{D}, \dashv, \vdash)$ was given. We will call ψ_0 as a canonical homomorphism. Observe that ψ_0 sends an arbitrary term with elements x_1, \dots, x_n to the product of some n elements from \mathcal{D} .

Let (P, \dashv', \vdash') be an arbitrary n -nilpotent g -dimonoid, α be the canonical homomorphism from $(\mathcal{G}(X), \dashv, \vdash)$ to (P, \dashv', \vdash') and $\mu = \pi \sigma^{-1} \alpha$ (see Section 3). Obviously, μ is a homomorphism from $XT_a^b(1)$, where $|Y| = 2$, $a \neq b$, to (P, \dashv', \vdash') . Define a map

$$\delta : G_n(X) \rightarrow (P, \dashv', \vdash') : \omega \mapsto \omega \delta,$$

assuming

$$\omega\delta = \begin{cases} \omega\mu, & \omega \in G_n \setminus \{0\}, \\ 0, & \omega = 0. \end{cases}$$

Show that δ is a homomorphism.

Let $\omega_1 = (x_1x_2\dots x_s, y_1y_2\dots y_{s-1})$, $\omega_2 = (a_1a_2\dots a_m, b_1b_2\dots b_{m-1}) \in G_n \setminus \{0\}$, where $x_i \in X$, $1 \leq i \leq s$, $y_j \in Y$, $1 \leq j \leq s-1$, $a_i \in X$, $1 \leq i \leq m$, $b_j \in Y$, $1 \leq j \leq m-1$. Assume $s+m \leq n$. As $\omega_1 \prec \omega_2 \in G_n \setminus \{0\}$, then

$$(\omega_1 \prec \omega_2)\delta = (\omega_1 \prec \omega_2)\mu = (\omega_1 \dashv \omega_2)\mu = \omega_1\mu \dashv' \omega_2\mu = \omega_1\delta \dashv' \omega_2\delta.$$

Analogously, $(\omega_1 \succ \omega_2)\delta = \omega_1\delta \dashv' \omega_2\delta$. Taking into account the previous arguments, in the remaining cases the equalities

$$(\omega_1 \prec \omega_2)\delta = (\omega_1 \succ \omega_2)\delta = 0 = \omega_1\delta \dashv' \omega_2\delta = \omega_1\delta \dashv' \omega_2\delta$$

hold. Thus, δ is a homomorphism.

The proof is complete. □

Now we construct a g -dimonoid which is isomorphic to the free n -nilpotent g -dimonoid of rank 1.

Assume $|Y| = 2$, $a \neq b$. For any $n \in \mathbb{N}$ let

$$\tilde{\mathbb{L}}_n = \{(m, u) \in \mathbb{N} \times T_a^b(1) \mid m - l(u) = 1, m \leq n\} \cup \{0\}.$$

Define operations \dashv and \dashv' on $\tilde{\mathbb{L}}_n$ by the rule

$$\begin{aligned} (m_1, u_1) \dashv (m_2, u_2) &= \begin{cases} (m_1 + m_2, u_1 * a^{l(u_2)+1}), & m_1 + m_2 \leq n, \\ 0, & m_1 + m_2 > n, \end{cases} \\ (m_1, u_1) \dashv' (m_2, u_2) &= \begin{cases} (m_1 + m_2, u_2 * b^{l(u_1)+1}), & m_1 + m_2 \leq n, \\ 0, & m_1 + m_2 > n, \end{cases} \\ (m_1, u_1) * 0 &= 0 * (m_1, u_1) = 0 * 0 = 0 \end{aligned}$$

for all $(m_1, u_1), (m_2, u_2) \in \tilde{\mathbb{L}}_n \setminus \{0\}$ and $*$ \in $\{\dashv, \dashv'\}$. An immediate verification shows that axioms of a g -dimonoid hold concerning operations \dashv and \dashv' . So, $(\tilde{\mathbb{L}}_n, \dashv, \dashv')$ is a g -dimonoid. Denote it by \mathbb{L}_n .

Lemma 10. *If $|X| = 1$, then $G_n(X) \cong \mathbb{L}_n$.*

Proof. Let $X = \{r\}$. An easy verification shows that a map $\varrho : G_n(X) \rightarrow \mathbb{L}_n$, defined by

$$\omega\varrho = \begin{cases} (k, u), & \omega = (r^k, u), \\ 0, & \omega = 0, \end{cases}$$

is an isomorphism. □

We finish this section with the description of the least n -nilpotent congruence on a free g -dimonoid.

If $f : D_1 \rightarrow D_2$ is a homomorphism of g -dimonoids, then the corresponding congruence on D_1 will be denoted by Δ_f . If ρ is a congruence on a g -dimonoid (D, \dashv, \vdash) such that $(D, \dashv, \vdash) / \rho$ is an n -nilpotent g -dimonoid, then we say that ρ is an n -nilpotent congruence.

Let $XT_a^b(1)$ be the free g -dimonoid ($|Y| = 2, a \neq b$) (see Section 3). Fix $n \in \mathbb{N}$ and define a relation $\kappa(n)$ on $XT_a^b(1)$ by

$$(w_1, u_1)\kappa(n)(w_2, u_2) \text{ if and only if} \\ (w_1, u_1) = (w_2, u_2) \text{ or } l(w_1) > n, l(u_1) > n.$$

Theorem 3. *The relation $\kappa(n)$ on the free g -dimonoid $XT_a^b(1)$ is the least n -nilpotent congruence.*

Proof. Define a map $\tau : XT_a^b(1) \rightarrow G_n(X)$ by

$$(w, u)\tau = \begin{cases} (w, u), & l(w) \leq n, \\ 0, & l(w) > n, \end{cases} \quad (w, u) \in XT_a^b(1).$$

Similarly to the proof of Theorem 4 from [4], the facts that τ is a surjective homomorphism and $\Delta_\tau = \kappa(n)$ can be proved. \square

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