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RESEARCH ARTICLE

A geometrical interpretation of infinite wreath powers

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Communicated by R. I. Grigorchuk

To Ani Mikaelian, my daughter

Abstract. A geometrical construction based on an infinite tree graph is suggested to illustrate the concept of infinite wreath powers of P. Hall. We use techniques based on infinite wreath powers and on this geometrical constriction to build a 2-generator group which is not soluble, but in which the normal closure of one of the generators is locally soluble.

1. Introduction

Wreath powers (especially, infinite wreath powers) are one of the most elegant structures, developed by P. Hall, and used by him to construct characteristically simple groups, verbally complete groups, groups containing an isomorphic copy of any finite group, non-strictly simple groups and other interesting group types (see for example [5, 7, 8]). Although wreath powers are flexible constructions, they are not used as widely as many of other constructions of Hall due to their complicated structure and lengthy definitions needed for their usage.

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The main aim of this paper is to suggest a geometrical construction illustrating the concept of wreath powers. Some of the analytical proofs from the mentioned works of Hall can in simpler manner be explained by properties of some geometrical objects. In order to present our geometrical approach, we consider the task of a construction of a group with certain properties, and solve it in two ways: first using the infinite wreath powers, and then by a geometrical method. The result is the group already announced in [2, Remark 5.6] to answer the question:

> *Is there a 2-generator group $G = \langle x, y \rangle$ such that G is not soluble (and, thus, even not locally soluble), but the normal closure $\langle x \rangle^G$ of x in G is locally soluble?*

Here is the context in which existence of such a group was considered. In [2] Guralnick, Plotkin and Shalev consider some "local-global" conditions by which properties of the group can be determined by its 2-generator subgroups. This is motivated by earlier result of Thompson [27], who proved that a finite group is soluble if and only if any of its 2-generator subgroup is soluble, and by later generalisation of Thompson's result in [1]: Let G be a finite group, and let $R(G)$ be the soluble radical of G (the maximal soluble normal subgroup of G). Then $R(G)$ coincides with the set of all radical elements of G, that is, elements $x \in G$ with the following property: for any $y \in G$ the 2-generator subgroup $\langle x, y \rangle$ is soluble. Clearly, $x \in G$ is radical if and only if the normal closure $\langle x \rangle^{\langle y \rangle} = \langle x \rangle^{\langle x,y \rangle}$ of $\langle x \rangle$ in $\langle x, y \rangle$ is soluble.

Generalising this and also a few other concepts, [2] offers the notion of $\mathfrak{X}$-radical elements for the given class of groups $\mathfrak{X}$. Let G be a group. An element $x \in G$ is called *locally $\mathfrak{X}$-radical* if the normal closure $\langle x \rangle^{\langle x,y \rangle}$ belongs to $\mathfrak{X}$ for every $y \in G$. And an element $x \in G$ is called *globally $\mathfrak{X}$-radical* if the normal closure $\langle x \rangle^G$ belongs to $\mathfrak{X}$.

If $\mathfrak{X}$ is the class of locally soluble groups we get the notion of a locally radical element x (if $\langle x \rangle^{\langle x,y \rangle}$ is locally soluble for any y) and the notion of a globally radical element x (if $\langle x \rangle^G$ is locally soluble). For such elements Problem 5.5 in [2] asks: which are the groups in which every locally radical element is globally radical; which are the groups in which every radical element is globally radical; and which are the groups in which every locally radical element is a radical?

In this context the 2-generator group built in the current paper is mentioned in [2, Remark 5.6]. For, it is in some sense a "minimal" example of a group G which is not locally soluble, but contains an element which is both locally radical and globally radical. This example shows that, unlike the case of finite groups, in the case of infinite groups the properties of locally radical and globally radical elements can be "very far" from the properties of the whole group.

In fact, the group G we construct has a few additional properties stressing the difference between G and $\langle x\rangle^G$:

Theorem 1. *There exists a two generator group $G = \langle x, y\rangle$ such that G is not soluble (and, thus, even not locally soluble), but the normal closure $\langle x\rangle^G$ is a locally soluble subgroup. Moreover, the group G can be constructed so that,*

1) *G is not finite (and, thus, even not locally finite), but $\langle x\rangle^G$ is locally finite;*
2) *G is not a p-group, but $\langle x\rangle^G$ is a p-group (p is a any pre-given prime number);*
3) *G is torsion free (and so is $\langle x\rangle^G$).*

We give the first proof of this theorem based on infinite wreath powers in Section 2. We omit the definition and basic properties of the infinite wreath powers, and refer for background information to the articles of Hall mentioned above.

In Section 3 we present the geometrical construction. The group G is built as an automorphism group of a graph Γ generated by some basic permutations of the vertex set of the graph. The graph Γ is an extended version of the graph used for the Grigorchuk group.

In Section 4 we compare these two constructions and point out a few other results in [4, 5, 7] the proofs of which could be realized without infinite wreath products by means of the suggested geometrical method.

The groups G we build in Section 3 and in Section 4 by far are not the only groups with the property mentioned in Theorem 1. When this work was in progress, we had opportunities to discuss the topic, and we got a couple of other examples of groups with that property. We present them in the closing Section 5.

2. The proof based on infinite wreath powers

Before we start using wreath products, let us state that all wreath products used here are permutational (associative) wreath products. These days under "wreath product" one usually understands the direct or cartesian standard wreath products, which are non-associative operations.

Permutational wreath products were more popular in earlier years of the development of the wreath product. Their definition and basic properties can be found in [5] or in [26]. And a description of infinite wreath powers can be found in [5, 7, 8].

Let C be any non-trivial transitive permutation group, and Λ be any infinite ordered set. Index the copies of C by elements $\lambda \in \Lambda$ and denote by C_λ the λ'th copy of it. Form the general (permutational, associative) wreath product

$$W = \mathrm{Wr}_{\lambda\in\Lambda} C_\lambda = \mathrm{Wr} C^\Lambda$$

of the copies C_λ, $\lambda \in \Lambda$. For the group A of order-preserving permutations of Λ build the split extension $G = W \cdot A$ with elements of A "shifting" the copies of the wreath factors C_λ as:

$$C_\lambda^f = C_{f(\lambda)} \text{ for any } f \in A.$$

If A is *irreducible*, in the sense that for any $\lambda_1, \lambda_2 \in \Lambda$ there exists a $f \in A$ such that $\lambda_2 < f(\lambda_1)$, then the commutator subgroup W' is a minimal normal subgroup of G [7, Theorem D].

W' clearly is not an abelian group because W contains the subgroup

$$C \,\mathrm{wr}\, C \,\mathrm{wr}\, C,$$

the commutator of which already is non-abelian. In particular G is not a locally soluble group because in such a group, and, in general, in every SI-group every minimal normal subgroup (as a chief factor) need to be abelian [26, vol. II].

To make G a 2-generator group it is sufficient to take $C = \langle c \rangle$ to be a non-trivial cyclic group, and to take Λ to be the set of integers $\mathbb{Z}$ ordered naturally. Then take $x = c_0$ to be the generator element of the 0'th copy C_0 of C; and take y to be the "shifting" automorphism of W corresponding to the generator 1 of $\mathbb{Z}$. Then $\langle x \rangle^G = W$, and W is a locally soluble group because each of its finitely generated subgroups is a subgroup in a finite wreath product $C \,\mathrm{wr}\, C \,\mathrm{wr} \cdots \mathrm{wr}\, C$, which is soluble. Further, if C is a non-trivial finite cyclic group, then $\langle x \rangle^G$ is locally finite because of the reason above. If, additionally, C is of prime order p, then $\langle x \rangle^G$ is a p-group. In both cases G is not locally finite because y is of infinite order. Finally, if we take C to be an infinite cyclic group, then the group G will be torsion free. This completes the proof.

3. The geometrical proof

Let us construct a graph Γ, which is an infinite tree with vertices $A_{n,k}$ with $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, where $\mathbb{Z}$ is the set of all integers and $\mathbb{N}$ is the set of all positive integers (see Figure 1 below). The vertices of Γ form rows:

$$A_{n,1}, A_{n,2}, \ldots, A_{n,k}, \ldots \quad \text{for any } n \in \mathbb{Z}.$$

Each of the vertices $A_{n,k}$ is connected by two edges to two vertices of the lower row $A_{n-1,2(k-1)+1}$ and $A_{n-1,2k}$; and is connected by one edge to the vertex $A_{n+1,\frac{k}{2}}$ (if k is even) or to $A_{n+1,\frac{k+1}{2}}$ (if k is odd) of the upper row. For any pair (n,k), where $n \in \mathbb{Z}$ and $k \in \mathbb{N}$ denote by $\Gamma_{n,k}$ and call a *peak* the subgraph of Γ that includes the vertex $A_{n,k}$ and *"lies below it"*, that is, the subgraph of Γ consisting of the set of the following vertices of Γ together with all the edges of Γ connecting them:

$$\begin{aligned}
&A_{n,k};\\
&A_{n-1,2(k-1)+1}, A_{n-1,2k};\\
&A_{n-2,2^2(k-1)+1}, A_{n-2,2^2(k-1)+2}, A_{n-2,2^2(k-1)+3}, A_{n-2,2^2\cdot k};\\
&\cdots\cdots\cdot\\
&A_{n-s,2^s(k-1)+1}, A_{n-s,2^s(k-1)+2}, \ldots, A_{n-s,2^s(k-1)+2^s-1}, A_{n-s,2^s\cdot k};\\
&\cdots\cdots\cdot
\end{aligned} \tag{1}$$

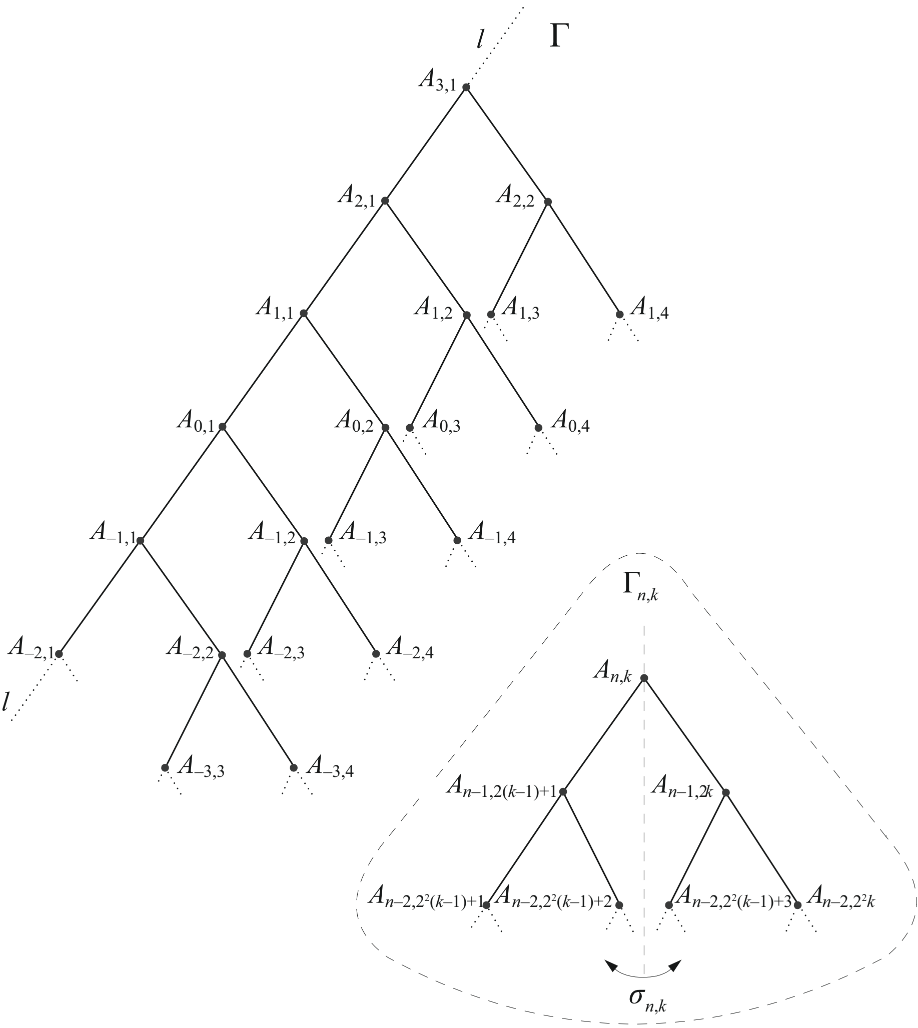
Let us call these: "0'th row of $\Gamma_{n,k}$", "1'st row of $\Gamma_{n,k}$", "2'nd row of $\Gamma_{n,k}$", "s'th row of $\Gamma_{n,k}$", etc.

In Figure 1 below the peak $\Gamma_{n,k}$ is highlighted by a dashed line around it. The geometrical form of the subgraph $\Gamma_{n,k}$ well explains the term "peak". The s'th row inside a peak contains 2^s vertices, from each of which a new peak starts. Two peaks $\Gamma_{n,k}$ and $\Gamma_{n',k'}$ are said to be of the *same level*, if $n = n'$; and the peak $\Gamma_{n,k}$ is of *lower level* then $\Gamma_{n',k'}$ if $n < n'$. The peak $\Gamma_{n,k}$ is *inside* the peak $\Gamma_{n',k'}$ if $n < n'$ and $\Gamma_{n,k}$ is a subgraph of $\Gamma_{n',k'}$. It is easy to see that any two peaks:

— either have no common vertices;

— or one of them is inside the other;

— or they coincide.



l
Γ
A3,1
A2,1
A2,2
A1,1
A1,2
A1,3
A1,4
A0,1
A0,2
A0,3
A0,4
A−1,1
A−1,2
A−1,3
A−1,4
A−2,1
A−2,2
A−2,3
A−2,4
l
A−3,3
A−3,4
Γn,k
An,k
An−1,2(k−1)+1
An−1,2k
An−2,2²(k−1)+1
An−2,2²(k−1)+2
An−2,2²(k−1)+3
An−2,2²k
σn,k

Figure 1.

For each peak $\Gamma_{n,k}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$, define the *basic automorphism (of the first type)* $\sigma_{n,k}$ of Γ, which roughly speaking "turns" $\Gamma_{n,k}$ around its vertical axis and leaves all the remaining vertices and edges in $\Gamma \backslash \Gamma_{n,k}$ unaffected.

The exact definition is:

$$
\begin{aligned}
&\sigma_{n,k} : A_{n,k} \mapsto A_{n,k};\\
&\sigma_{n,k} : A_{n-1,2(k-1)+1} \mapsto A_{n-1,2k},\\
&\qquad\qquad \sigma_{n,k} : A_{n-1,2k} \mapsto A_{n-1,2(k-1)+1};\\
&\sigma_{n,k} : A_{n-2,2^2(k-1)+1} \mapsto A_{n-2,2^2\cdot k},\\
&\qquad\qquad \sigma_{n,k} : A_{n-2,2^2(k-1)+2} \mapsto A_{n-2,2^2(k-1)+3},\\
&\qquad\qquad \sigma_{n,k} : A_{n-2,2^2(k-1)+3} \mapsto A_{n-2,2^2(k-1)+2},\\
&\qquad\qquad \sigma_{n,k} : A_{n-2,2^2\cdot k} \mapsto A_{n-2,2^2(k-1)+1};\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\sigma_{n,k} : A_{n-s,2^s(k-1)+1} \mapsto A_{n-s,2^s\cdot k},\\
&\qquad\qquad \sigma_{n,k} : A_{n-s,2^s(k-1)+2} \mapsto A_{n-s,2^s(k-1)+2^s-1},\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\\
&\qquad\qquad \sigma_{n,k} : A_{n-s,2^s(k-1)+2^s-1} \mapsto A_{n-s,2^s(k-1)+2},\\
&\qquad\qquad \sigma_{n,k} : A_{n-s,2^s\cdot k} \mapsto A_{n-s,2^s(k-1)+1};\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,
\end{aligned}
\tag{2}
$$

where $n \in \mathbb{Z}$ and $k \in \mathbb{N}$; and $\sigma_{n,k} : A_{n',k'} \mapsto A_{n',k'}$ for any $A_{n',k'} \in \Gamma \backslash \Gamma_{n,k}$. We would strongly recommend the reader in considerations below to follow the "geometrical logic" of this definition, not the calculation routine.

In analogy with the terminology above we say that the basic automorphism $\sigma_{n,k}$ is of the same level as $\sigma_{n',k'}$, or is of lower level or is inside it, if the corresponding peaks $\Gamma_{n,k}$ and $\Gamma_{n',k'}$ have those properties. As a first illustration of the properties of the basic automorphisms let us observe that:

Lemma 1. *For any $n, n' \in \mathbb{Z}$ and $k, k' \in \mathbb{N}$ the commutator $[\sigma_{n,k}, \sigma_{n',k'}]$ of $\sigma_{n,k}$ and $\sigma_{n',k'}$ is:*

1) *trivial if none of these automorphisms is inside the other;*
2) *non-trivial, if one of them, say $\sigma_{n,k}$, lies in the other. In this case $[\sigma_{n,k}, \sigma_{n',k'}] = \sigma_{n,k} \cdot \sigma_{n,k^*}$, where $k^* = (2k'-1)2^{n'-n} + 2 - k$.*

Proof. The first point is evident: if the basic automorphisms coincide or if they intersect trivially, then their commutator is trivial either because these automorphisms either are equal or because the intersection of the sets on which they act non-trivially is empty.

Assume $\sigma_{n,k}$ lies inside $\sigma_{n',k'}$. Without loss of generality we can assume that $\Gamma_{n,k}$ is "closer to the left edge" of $\Gamma_{n',k'}$, that is, if $\Gamma_{n,k}$ is inside $\Gamma_{n'-1,(k'-1)2+1}$ or coincides with it. This, of course, means that $k - l < l + 2^{n'-n} - k$ (equivalently: $k < 2^{n'-n-1} + l$), where $l = (k'-1)2^{n'-n} + 1$ is the number from which the $(n'-n)$'th row of the peak $\Gamma_{n',k'}$ starts (see Figure 2). Then the peak Γ_{n,k^*} with

$$k^* = (l + 2^{n'-n}) - (k - l) = 2^{n'-n} - k + 2l = (2k'-1)2^{n'-n} + 2 - k$$

is nothing else but the peak symmetrical to $\Gamma_{n,k}$ "around the central vertical axis" of $\Gamma_{n',k'}$ (see Figure 2).

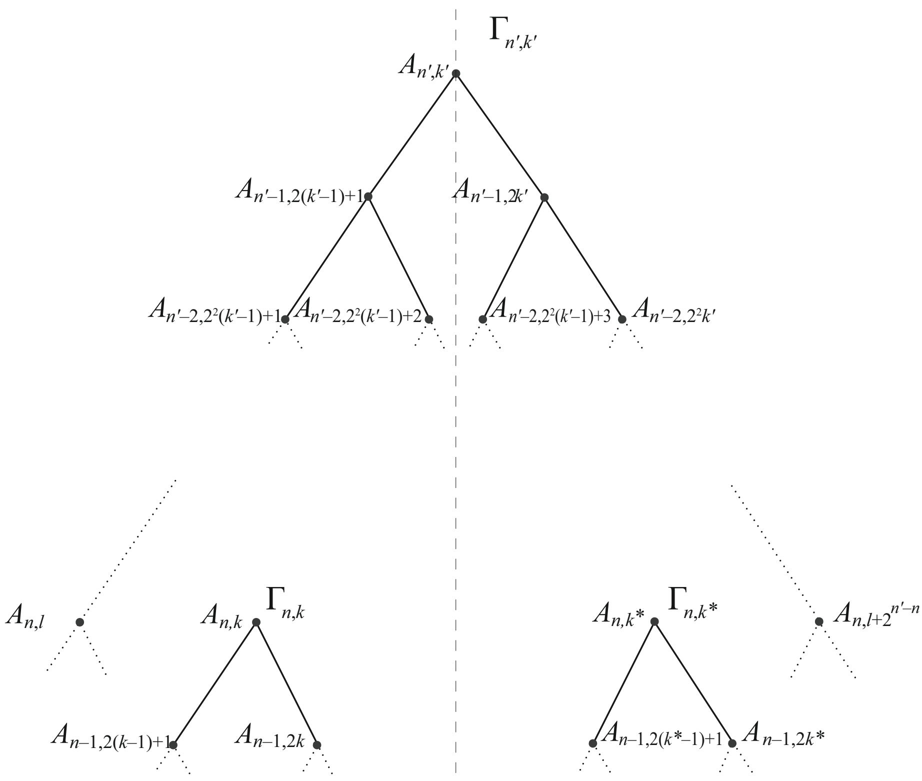


FIGURE 2.

It is clear that $[\sigma_{n,k}, \sigma_{n',k'}]$ does not move the vertices of $\Gamma \backslash \Gamma_{n',k'}$. Also, this commutator acts trivially on the subgraph $\Gamma_{n',k'} \setminus (\Gamma_{n,k} \cup \Gamma_{n,k^*})$ because it acts on its vertices like the square of the automorphism $\sigma_{n',k'}$, which is of order two.

Further, $[\sigma_{n,k}, \sigma_{n',k'}]$ acts on $\Gamma_{n,k}$ like $\sigma_{n,k}$. For, $\sigma_{n,k}^{-1}$ "turns" $\Gamma_{n,k}$; then $\sigma_{n',k'}^{-1}$ maps the "turned" $\Gamma_{n,k}$ onto Γ_{n,k^*}; then comes $\sigma_{n,k}$, which acts trivially on Γ_{n,k^*}; and finally the "turned" Γ_{n,k^*} "comes back" to $\Gamma_{n,k}$ by $\sigma_{n',k'}$. For the same reason $[\sigma_{n,k}, \sigma_{n',k'}]$ acts on Γ_{n,k^*} like σ_{n,k^*}. Thus, $[\sigma_{n,k}, \sigma_{n',k'}] = \sigma_{n,k} \cdot \sigma_{n,k^*}$. □

Our next key element is the *basic automorphism (of the second) type* θ of Γ, which roughly speaking shifts Γ one level higher by axis l. More precisely:

$$\theta : A_{n,k} \mapsto A_{n+1,k}.$$

θ maps a peak $\Gamma_{n,k}$ onto $\Gamma_{n+1,k}$. The main property of θ needed for our construction is:

Lemma 2. *For each basic automorphism $\sigma_{n,k}$, $n \in \mathbb{Z}$ and $k \in \mathbb{N}$:*

$$\sigma_{n,k}^{\theta} = \sigma_{n+1,k}.$$

Proof. Consider the peak $\Gamma_{n+1,k}$ and the peak $\Gamma_{n,k}$, which is inside it, or is to the left from it. The main idea of this proof is that θ^{-1} maps $\Gamma_{n+1,k}$ on $\Gamma_{n,k}$, then $\sigma_{n,k}$ "turns" $\Gamma_{n,k}$, and then θ maps the latter onto the "turned" $\Gamma_{n+1,k}$. So the effect of $\theta^{-1}\sigma_{n,k}\theta$ is $\sigma_{n+1,k}$. More precisely:

$$\begin{aligned}
&A_{n+1,k} \overset{\theta^{-1}}{\longmapsto} A_{n,k} \overset{\sigma_{n,k}}{\longmapsto} A_{n,k} \overset{\theta}{\longmapsto} A_{n+1,k};\\
&A_{n,(k-1)2+1} \overset{\theta^{-1}}{\longmapsto} A_{n-1,(k-1)2+1} \overset{\sigma_{n,k}}{\longmapsto} A_{n-1,2k} \overset{\theta}{\longmapsto} A_{n,2k};\\
&A_{n,2k} \overset{\theta^{-1}}{\longmapsto} A_{n-1,2k} \overset{\sigma_{n,k}}{\longmapsto} A_{n-1,(k-1)2+1} \overset{\theta}{\longmapsto} A_{n,(k-1)2+1};\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&A_{n+1-s,(k-1)2^s+1} \overset{\theta^{-1}}{\longmapsto} A_{n-s,(k-1)2^s+1} \overset{\sigma_{n,k}}{\longmapsto}\\
&\qquad\qquad A_{n-s,k2^s} \overset{\theta}{\longmapsto} A_{n+1-s,k2^s};\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&A_{n+1-s,k2^s} \overset{\theta^{-1}}{\longmapsto} A_{n-s,k2^s} \overset{\sigma_{n,k}}{\longmapsto}\\
&\qquad\qquad A_{n-s,(k-1)2^s+1} \overset{\theta}{\longmapsto} A_{n+1-s,(k-1)2^s+1};\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{3}$$

Finally, the effect of $\sigma_{n,k}^{\theta}$ on $\Gamma \backslash \Gamma_{n+1,k}$ is trivial. □

Now we have the necessary items to define the group G we are looking for by automorphisms of Γ as follows:

$$G = \langle \sigma_{n,k}, \theta \mid n \in \mathbb{Z},\ k \in \mathbb{N} \rangle.$$

Lemma 3. *The group G constructed above is a 2-generator group:*

$$G = \langle \sigma_{0,1}, \theta \rangle.$$

Proof. Since each peak $\Gamma_{n,k}$ is inside a peak of type $\Gamma_{n',1}$ (where $n' \geqslant n$), and since by Lemma 2 for each n' holds:

$$\sigma_{n',1} = \sigma_{0,1}^{\theta^{n'}} \in G,$$

it is sufficient to prove that for each $\sigma_{n',1}$ all basic automorphisms $\sigma_{n,k}$ inside it belong to G. Prove this by the row number of $\Gamma_{n',1}$. For the first two rows the situation is simple:

$$\begin{aligned}
&\sigma_{n',1} \in G;\\
&\sigma_{n'-1,1} = \sigma_{n',1}^{\theta^{-1}} \in G;\\
&\sigma_{n'-1,2} \in G \text{ because } [\sigma_{n',1}, \sigma_{n'-1,1}]\\
&\qquad\qquad = \sigma_{n'-1,1}\sigma_{n'-1,2}, \text{ and } \sigma_{n',1}, \sigma_{n'-1,1} \in G
\end{aligned}$$

(we used Lemma 1 on $[\sigma_{n',1}, \sigma_{n'-1,1}]$). Assume lemma is proved for all rows of $\Gamma_{n',1}$ until the s'th row included. Let us consider the $(s+1)$'st row using the idea above:

$$\begin{aligned}
&\sigma_{n'-s-1,1} = \sigma_{n'-s,1}^{\theta^{-1}} \in G;\\
&\sigma_{n'-s-1,2} \in G \text{ because } [\sigma_{n'-s-1,1}, \sigma_{n'-s,1}]\\
&\qquad\qquad = \sigma_{n'-s-1,1}\,\sigma_{n'-s-1,2}, \text{ and } \sigma_{n'-s-1,1}, \sigma_{n'-s,1} \in G;\\
&\sigma_{n'-s-1,3} \in G \text{ because } [\sigma_{n'-s-1,2}, \sigma_{n'-s+1,1}]\\
&\qquad\qquad = \sigma_{n'-s-1,2}\,\sigma_{n'-s-1,3}, \text{ and } \sigma_{n'-s-1,2}, \sigma_{n'-s+1,1} \in G;\\
&\sigma_{n'-s-1,4} \in G \text{ because } [\sigma_{n'-s-1,3}, \sigma_{n'-s,2}]\\
&\qquad\qquad = \sigma_{n'-s-1,3}\,\sigma_{n'-s-1,4} \text{ and } \sigma_{n'-s-1,3}, \sigma_{n'-s,2} \in G;
\end{aligned}$$

....

The Lemma is proved. □

The formal calculations above may seem confusing, but the geometrical meaning is not complicated: for each row of automorphisms $\sigma_{n'-s-1,1}$,

$\sigma_{n'-s-1,2}$, $\sigma_{n'-s-1,3}$,... the first $\sigma_{n'-s-1,1}$ is in G because by Lemma 2 it is a shifted image of a basic automorphism that is located "to the North" from $\sigma_{n'-s-1,1}$. And the rest are in G because by Lemma 1 we can obtain each $\sigma_{n'-s-1,j}$ $(j > 1)$ as a product of basic automorphisms (and of commutators of basic automorphisms) that all are located "to the North-West" from $\sigma_{n'-s-1,j}$.

It remains to prove the desired two properties of the group G.

Lemma 4. *The group G constructed above is not a soluble group (and, thus, is not a locally soluble group).*

Proof. For any pre-given positive integer m we will construct elements

$$\tau_1, \tau_2, \ldots, \tau_{2^m} \in G,$$

such that the solubility identity $w_m(x_1, x_2, \ldots, x_{2^m})$ [20] of length m is falsified on them. These elements could be pointed out directly, and the proof could be completed by computations. However, we deduce the proof from two geometrical observations to illustrate why we introduced the graph Γ and its basic automorphisms.

Firstly, assume we have an automorphism $\lambda = \sigma_{n_1,k_1} \cdots \sigma_{n_s,k_s}$, which, clearly, can act non-trivially only on the following subgraph of Γ:

$$\Gamma_\lambda = \Gamma_{n_1,k_1} \cup \cdots \cup \Gamma_{n_s,k_s} \subset \Gamma.$$

Denote $\bar{n} = \max_{i=1,\ldots,s}\{n_i\}$. It always is possible to choose $v > \bar{n}$ large enough so that all peaks Γ_{n_i,k_i}, $i = 1, \ldots, s$, "lie in the left half" of the peak $\Gamma_{v,1}$, that is, Γ_{n_i,k_i} is inside $\Gamma_{v-1,1}$ for all $i = 1, \ldots, s$ (see Figure 3). Applying the idea of Lemma 1 we get that the commutator $[\lambda, \sigma_{v,1}]$ is a product of λ and of the automorphism $\sigma_{n_1,k_1^*} \cdots \sigma_{n_s,k_s^*}$. We omit the calculations because the geometrical meaning is clear: $\sigma_{v,1}$ just "turns" around the vertical axis of $\Gamma_{v,1}$ all the peaks Γ_{n_i,k_i}, $i = 1, \ldots, s$.

Secondly, assume we have another automorphism $\mu = \sigma_{n'_1,k'_1} \cdots \sigma_{n'_q,k'_q}$, such that all the peaks $\Gamma_{n'_i,k'_i}$, $i = 1, \ldots, q$, are inside one of the peaks of λ, say, are inside Γ_{n_1,k_1} and, moreover, Γ_{n_1,k_1} has empty intersection with the rest of Γ_{n_2,k_2}, ..., Γ_{n_s,k_s}. Then $[\lambda, \mu] = [\sigma_{n_1,k_1}, \mu]$ because $[\lambda, \mu]$ acts trivially on all vertices of both subgraphs: $\Gamma \setminus \Gamma_\lambda$ and $\Gamma_{n_2,k_2} \cup \cdots \cup \Gamma_{n_s,k_s}$, and there only remains to calculate its action on the peak Γ_{n_1,k_1}.

We can now turn to the construction of the elements $\tau_1, \ldots, \tau_{2^m}$. Take two peaks on "near" levels, for example, $\Gamma_{0,1}$ and $\Gamma_{1,1}$, and set $\tau_1 = \sigma_{0,1}$ and $\tau_2 = \sigma_{1,1}$.

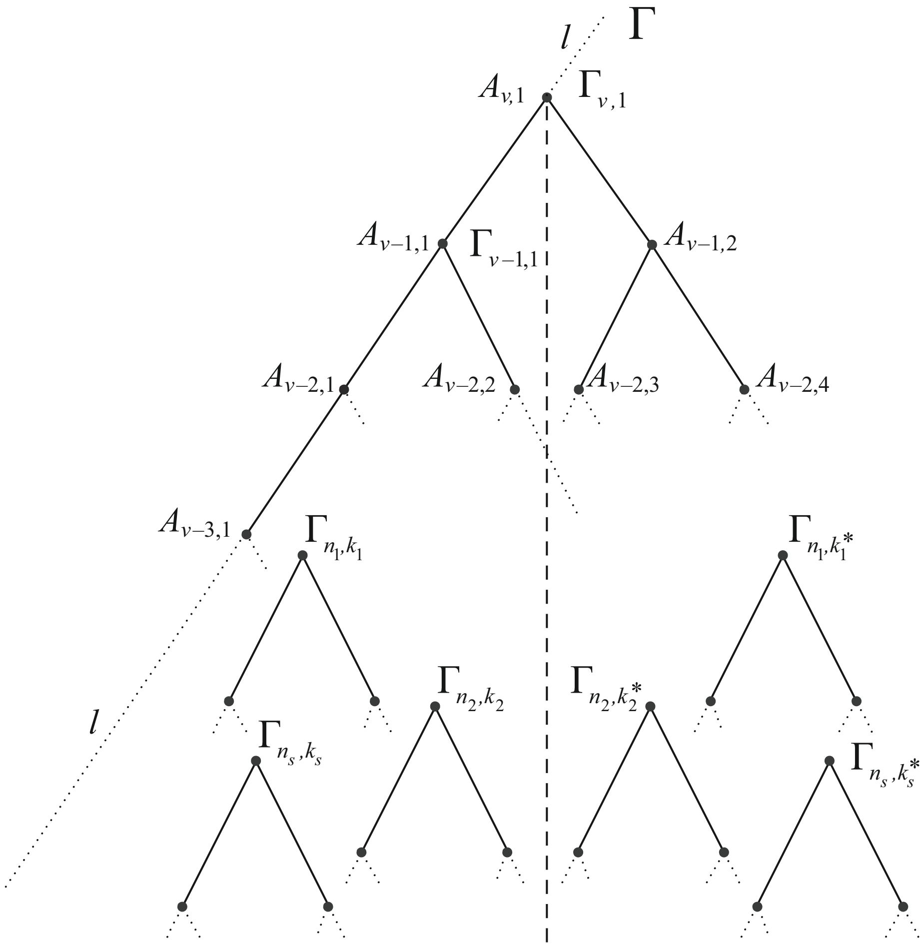


Figure 3.

Then by Lemma 1 (or by the argument above) $[\tau_1, \tau_2] = \sigma_{0,1}\, \sigma_{0,2}$ (in particular, this commutator is not trivial). Take the positive integer v large enough (in this case it is sufficient to take $v = 2$) and consider the peaks:

$$\Gamma_{0,1}^{\theta^2} = \Gamma_{2,1} \quad \text{and} \quad \Gamma_{1,1}^{\theta^2} = \Gamma_{3,1}.$$

Set $\tau_3 = \sigma_{2,1}$ and $\tau_4 = \sigma_{3,1}$. By our definition:

$$[\tau_3, \tau_4] = [\tau_1, \tau_2]^{\theta^2} = \sigma_{2,1}\, \sigma_{2,2}.$$

(in particular, it also is not trivial). We use the geometrical observation above: since $\Gamma_{0,1}$ and $\Gamma_{1,1}$ are inside $\Gamma_{2,1}$, we get that:

$$[[\tau_1, \tau_2], [\tau_3, \tau_4]] = [\sigma_{0,1}\,\sigma_{0,2}\,,\, \sigma_{2,1}\,\sigma_{2,2}] = [\sigma_{0,1}\,\sigma_{0,2}\,,\, \sigma_{2,1}].$$

Since $\Gamma_{0,1}$ and $\Gamma_{0,2}$ are inside the "left half" of $\Gamma_{2,1}$, we by the geometrical observation above get that

$$[\sigma_{0,1}\,\sigma_{0,2}\,,\, \sigma_{2,1}] = \sigma_{0,1}\,\sigma_{0,2}\,\sigma_{0,3}\,\sigma_{0,4} \neq 1.$$

Next take v large enough (in this case it is sufficient to take $v = 4$) and consider the peaks:

$$\Gamma_{0,1}^{\theta^4} = \Gamma_{4,1} \;\;,\;\; \Gamma_{1,1}^{\theta^4} = \Gamma_{5,1} \;\;,\;\; \Gamma_{2,1}^{\theta^4} = \Gamma_{6,1} \;\;,\;\; \Gamma_{3,1}^{\theta^4} = \Gamma_{7,1}.$$

Set: $\tau_5 = \sigma_{4,1}$, $\tau_6 = \sigma_{5,1}$, $\tau_7 = \sigma_{6,1}$ and $\tau_8 = \sigma_{7,1}$. By definition and arguments above:

$$\begin{aligned}
&\Big[[[\tau_1, \tau_2], [\tau_3, \tau_4]]\,,[[\tau_5, \tau_6], [\tau_7, \tau_8]]\Big]\\
&\qquad = \Big[[[\tau_1, \tau_2], [\tau_3, \tau_4]]\,, \sigma_{4,1}\Big]\\
&\qquad = \sigma_{0,1}\,\sigma_{0,2}\,\sigma_{0,3}\,\sigma_{0,4} \cdot (\sigma_{0,1}\,\sigma_{0,2}\,\sigma_{0,3}\,\sigma_{0,4})^{\theta^4}\\
&\qquad = \sigma_{0,1}\,\sigma_{0,2} \cdots \sigma_{0,8} \neq 1.
\end{aligned}$$

This process can be continued for $v = 2^3, 2^4, \ldots$, and in each step the automorphism θ^v will lift the already constructed elements "high enough" so that the resulting commutator is of the form $\sigma_{0,1}\,\sigma_{0,2} \cdots \sigma_{0,2^i}$ and is not trivial. □

Lemma 5. *The normal closure $\langle \sigma_{0,1} \rangle^G$ is a locally soluble subgroup in G.*

Proof. Denote $H = \langle \sigma_{n,k} \,|\, n \in \mathbb{Z},\; k \in \mathbb{N} \rangle$. Evidently $\sigma_{0,1}^{\sigma_{n,k}} \in H$ for any $n \in \mathbb{Z},\; k \in \mathbb{N}$; and $\sigma_{0,1}^{\theta} \in H$. Thus, it is sufficient to prove that any set of basic automorphisms (of the first type)

$$\sigma_{n_1,k_1}, \ldots, \sigma_{n_s,k_s} \tag{4}$$

generates a soluble subgroup S in H. Denote $\bar{n} = \max_{i=1,\ldots,s}\{n_{i_j}\}$ and $\underline{n} = \min_{i=1,\ldots,s}\{n_{i_j}\}$. For any non-trivial $\nu \in S$ fix a presentation

$$\nu = \sigma_{n_{i_1},\, k_{i_1}} \cdots\; \sigma_{n_{i_r},\, k_{i_r}} \tag{5}$$

(we may without loss of generality assume that $n_{i_j} \leqslant \bar{n}$, $j = 1, \ldots, r$) and denote $\bar{n}(\nu) = \max_{i=1,\ldots,r}\{n_{i_j}\}$ and $\underline{n}(\nu) = \min_{i=1,\ldots,r}\{n_{i_j}\}$. For any n_1, k_1, n_2, k_2 by Lemma 1 we have that either $[\sigma_{n_1,\,k_1}, \sigma_{n_2,\,k_2}]$ is trivial or:

$$\bar{n}([\sigma_{n_1,\,k_1}, \sigma_{n_2,\,k_2}]) < \max\{\bar{n}(\sigma_{n_1,\,k_1}), \bar{n}(\sigma_{n_2,\,k_2})\} = \max\{n_1, n_2\}.$$

Also, evidently, either $[\sigma_{n_1,k_1}, \sigma_{n_2,k_2}]$ is trivial or:

$$\underline{n}([\sigma_{n_1,k_1}, \sigma_{n_2,k_2}]) \geqslant \min\{\underline{n}(\sigma_{n_1,k_1}), \underline{n}(\sigma_{n_2,k_2})\} = \min\{n_1, n_2\}.$$

By these and by the well known commutator identities [25]:

$$[ab, c] = [a, c]^b [b, c], \quad [a, bc] = [a, c][a, b]^c$$

we have that for any $\nu_1, \nu_2 \in S$ either $[\nu_1, \nu_2]$ is trivial or:

$$\bar{n}([\nu_1, \nu_2]) < \max\{\bar{n}(\nu_1), \bar{n}(\nu_2)\}, \quad \underline{n}([\nu_1, \nu_2]) \geqslant \min\{\bar{n}(\nu_1), \bar{n}(\nu_2)\}$$

because each of ν_1, ν_2 can be presented in the form (5) and then their commutator can be "split" by commutator identities to the commutators of the factors and their conjugates (we omit the routine calculations). Thus, for any $\nu \in S'$ we have that either ν is trivial or $\bar{n}(\nu) \leqslant \bar{n} - 1$ and $\underline{n}(\nu) \geqslant \underline{n}$. For the same reason for any $\nu \in S''$, either ν is trivial or $\bar{n}(\nu) \leqslant \bar{n} - 2$ and $\underline{n}(\nu) \geqslant \underline{n}$, etc. This process cannot repeat more than $\bar{n} - \underline{n}$ times. Thus, the commutator subgroup $S^{(\bar{n}-\underline{n})}$ is trivial. □

4. Comparison of the arguments and other applications of the method

The reader familiar with the notion of infinite wreath products has probably noticed that our geometrical construction imitates some of its aspects. For example, the group $H = \langle \sigma_{n,k} \mid n \in \mathbb{Z},\ k \in \mathbb{N} \rangle$ in the proof of Lemma 5 is the infinite wreath power $W = \mathrm{Wr}_{\lambda \in \Lambda} C_\lambda = \mathrm{Wr} C^\Lambda$ for $\Lambda = \mathbb{Z}$ (ordered in the natural way) and for C a cyclic group of order 2. The automorphism θ of Γ in the second construction has a role similar to the role of the group automorphism f in the first construction.

The group $G = \langle \sigma_{0,1}, \theta \rangle$ also satisfies the second statement of Theorem 1: the normal closure $\langle \sigma_{0,1} \rangle^G$ is a p-group whereas G is not a p-group (here $p = 2$, of course). One could assume that the second construction is less powerful, since it does not cover the cases of $p = 3, 5, \ldots$ However, some modification of Γ can allow to cover the case of of any prime p. Namely, in the graph Γ the number of edges incident to each vertex need not be $3 = 2 + 1$ but $p + 1$ so that each peak has not two but p "branches". However, we built our construction for $p = 2$ because it makes the construction and the calculations simpler (simplicity was our main motivation why we suggest it as an alternative of infinite wreath powers of Hall). In the general case for any p we would have to change our construction as follows:

Firstly, for $p = 2$ with each peak $\Gamma_{n,k}$ connected is just one basic automorphism $\sigma_{n,k}$ (and, of course, the trivial automorphism leaving $\Gamma_{n,k}$ unaffected). In the general case we have p vertices in the second row of $\Gamma_{n,k}$:

$$A_{n-1,(k-1)p^2+1}, A_{n-1,(k-1)p^2+2}, \dots, A_{n-1,kp^2},$$

and we get a cyclic group of order p generated by a basic automorphism of the first type given by "shifting" permutation:

$$\begin{pmatrix} A_{n-1,(k-1)p^2+1} & A_{n-1,(k-1)p^2+2} & \cdots & A_{n-1,kp^2} \\ A_{n-1,(k-1)p^2+2} & A_{n-1,(k-1)p^2+3} & \cdots & A_{n-1,(k-1)p^2+1} \end{pmatrix}$$

The basic automorphism of the second type is defined as before: $\theta : A_{n,k} \mapsto A_{n+1,k}$.

The proofs of Lemma 1, Lemma 3, Lemma 4 and Lemma 5 can be modified accordingly. We omit the calculations.

Moreover, our construction can be modified to cover the case of torsion-free groups of the third statement in Theorem 1. In this case countably many edges need be incident to each vertex, and each basic automorphism $\sigma_{n,k}$ will be of infinite order, "shifting" infinitely many vertices of the second row in $\Gamma_{n,k}$.

Closing this paper we would like to stress that the group $G = \langle \sigma_{0,1}, \theta \rangle$ we constructed is useful not only in the context of the problem of Plotkin mentioned above, but also in other problems based on infinite wreath products.

For example, our group $G = \langle \sigma_{0,1}, \theta \rangle$ can be presented as a product of its two locally soluble normal subgroups, but it is not a locally soluble group, is not an SI-group and is not a radical group. A group with such properties was built in [4] to answer Plotkin's question posed in [22]. The construction of [4] also is based on infinite wreath powers, and can be replaced by G.

Another example is the *verbally complete group* built in [5] by Hall. A group G is verbally complete if for any non-trivial word $w(x_1,\dots,x_n)$ and for any element $a \in G$ there exist $a_1, \dots, a_n \in G$ such that $w(a_1, \dots, a_n) = a$. Hall builds this group as infinite wreath product of a series of finite groups, each of which can be realized in our construction as a fragment of the tree Γ and of some automorphisms working on it.

Yet another example is the characteristically simple group built in [7]. That group can be presented as a factor group of $G = \langle \sigma_{0,1}, \theta \rangle$ by some normal subgroup which also has a geometrical meaning.

5. Examples based on other constructions

When this work was in progress, we had an opportunity to discuss this topic with A.Yu. Ol'shanskii, who kindly offered two other ideas of the proof. One is built by Ol'shanskii, Osin and Sapir in [21] in subsection 3.5 for different purposes of lacunary hyperbolic amenable groups.

The other idea is based on the concept of verbal wreath products of groups. Let $G = F_\infty(\mathfrak{V})$ be the relatively free group of countable rank in a variety $\mathfrak{V}$ which is locally soluble but not soluble. Such a variety exists: by a theorem of Razmislov [24] the Burnside variety of exponent 4 has this property. G can be interpreted as a verbal product $\prod_{\mathfrak{V}} C$ of countably many copies of the finite cycle $C = C_4$ of order 4 (see definition in [20]).

We take the infinite cycle $Z = \langle z \rangle$ and index by its elements the generators of copies $C = \langle c_{z^i} \rangle$ of the cycle C in G. An action of $Z = \langle z \rangle$ by automorphisms on G can be defined by the rule $z : c_{z^{i-1}} \mapsto c_{z^i}$, which defines a structure of verbal direct wreath product

$$W = C \operatorname{wr}_{\mathfrak{V}} Z$$

(see [20]). The constructed extension certainly is a 2-generator group $W = \langle c_{z^0}, z \rangle$ and is not soluble. The normal closure of c_{z^0} in W is locally soluble since it is inside the base subgroup $\prod_{\mathfrak{V}} C = G$ of this wreath product.

Razmislov's example is not the only variety we can use. In [15] for different purposes we listed a few other varieties that can also be used here, since their relatively free groups of countable rank also are locally soluble but are not soluble. Let us list them:

Since the Kostrikin variety $\mathfrak{V} = \mathfrak{K}_p$ is a locally nilpotent variety, all relatively free groups $F_n(\mathfrak{K}_p)$, $n = 1, 2, \ldots$, are nilpotent. On the other hand $\mathfrak{K}_p$ is not a soluble variety (see [23] for the case $p \geqslant 5$, and [3] for the case $p = 5$). So we can take $W = C \operatorname{wr}_{\mathfrak{K}_p} Z$.

All 3-Engel groups are locally nilpotent [10]. So put $G_n = F_n(\mathfrak{E}_3)$. There is no bound on the solubility lengths of these groups, for the free group of infinite rank in variety $\mathfrak{K}_5 \cap \mathfrak{E}_3$ is insoluble [3]. We can take $W = C \operatorname{wr}_{\mathfrak{E}_3} Z$.

Varieties $\mathfrak{E}_n$ for $n > 3$ can be another source for examples. Intersections $\mathfrak{K}_p \cap \mathfrak{E}_n$, where $n \geqslant 3$, $p = n+2$, are locally nilpotent, but still not soluble varieties [23]. To get torsion-free groups with the desired properties take relatively free groups of finite ranks in the variety $\mathfrak{A} \cdot (\mathfrak{K}_p \cap \mathfrak{E}_n)$. We can take $W = C \operatorname{wr}_{\mathfrak{K}_p \cap \mathfrak{E}_n} Z$.

The final example was suggested by the first referee. Let U be the group of infinite unitary upper triangular matrices over $\mathbb{Z}$ or over $\mathbb{Z}/p\mathbb{Z}$ with only finitely many non-zero entries outside of the main diagonal; more precisely, U is the inductive limit of UT_n, where UT_n is embedded in the upper left corner of UT_{n+1}. Let t be the injective endomorphism of U that shifts matrix coefficients one position right and then one position down and adds $(1, 0, 0, ...)$ as the first row and the first column (so t acts as a "shift" along the main diagonal). Let x be the matrix with second coefficient in the first row equal to 1 and all other coefficients outside of the main diagonal equal to 0. Then the HNN-extension of U by $\langle t\rangle$ is generated by $\{x, t\}$ and it is fairly easy to see that it satisfies all the required properties.

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