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Morita equivalence for partially ordered monoids and po-Γ-semigroups with unities

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Abstract. We prove that operator pomonoids of a po-Γ-semigroup with unities are Morita equivalent pomonoids. Conversely, we show that if L and R are Morita equivalent pomonoids then a po-Γ-semigroup A with unities can be constructed such that left and right operator pomonoids of A are Pos-isomorphic to L and R respectively. Using this nice connection between po-Γ-semigroups and Morita equivalence for pomonoids we, in one hand, obtain some Morita invariants of pomonoids using the results of po-Γ-semigroups and on the other hand, some recent results of Morita theory of pomonoids are used to obtain some results of po-Γ-semigroups.

Introduction and notations

A monoid S together with a partially ordered relation $\leqslant$ on it is called a *partially ordered monoid* (in short pomonoid) if $s \leqslant t$ implies $us \leqslant ut$ and $su \leqslant tu$ for all $s, t, u \in S$. Let S be a pomonoid with identity 1. A poset $(A, \leqslant)$ together with a mapping $A \times S \to A$, denoted $(a, s) \mapsto as$, is called a *right S-poset* if for all $a, b \in A$ and $s, t \in S$ we have (1) $a(st) = (as)t$, (2) $a1 = a$, (3) $a \leqslant b$ implies $as \leqslant bs$, and (4) $s \leqslant t$ implies $as \leqslant at$. *Left S-posets* can be defined analogously. We shall distinguish left and right S-posets by writing ${}_SA$ and A_S respectively. If A is simultaneously a left

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T-poset and a right S-poset such that $(ta)s = t(as)$ for all $a \in A$, $t \in T$ and $s \in S$ then we call A an *T-S-biposet* and denote it by ${}_TA_S$. Let A and B be two right S-posets. Then a mapping $f : A \to B$ is called a *right S-poset morphism* if for all $a, b \in A$ and $s \in S$ we have (1) $f(as) = f(a)s$ and (2) $a \leqslant b$ implies $f(a) \leqslant f(b)$. *Left S-morphisms* can be defined analogously. If A and B be two T-S-biposets then a mapping $f : A \to B$ is called a *T-S-biposet morphism* if f is simultaneously a left T-poset morphism and a right S-poset morphism. The category formed by right S-posets together with the right S-morphisms is denoted by Pos_S. Its left analogue is denoted by ${}_SPos$. Also the category of T-S-biposets is denoted by ${}_TPos_S$. In the category Pos_S, ${}_SPos$ and ${}_TPos_S$ the morphism sets are denoted by $Pos_S(A, B)$, ${}_SPos(A, B)$ and ${}_TPos_S(A, B)$ respectively. These categories are enriched over the category Pos of posets (with order preserving maps as morphisms), i.e., the morphism sets are posets with respect to pointwise order. Again a *Pos-functor* between such categories is a functor that preserves the order of morphisms.

Two pomonoids S and T are said to be *Morita equivalent* if the categories Pos_S and Pos_T are two Pos-equivalent categories. Let A be a right S-poset and B be a left S-poset. Denote by $A \times B$ the Cartesian product of sets A and B. Then $A \times B$ is a partially ordered set in the Cartesian order $(a, b) \leqslant (c, d)$ if and only if $a \leqslant c$ and $b \leqslant d$. Put $H = \{((as, b), (a, sb)) \mid a \in A,\ b \in B,\ s \in S\}$ and let $\rho = \rho(H)$ be the congruence generated by H over $A \times B$. The quotient $(A \times B)/\rho$ is a partially ordered set, called the *tensor product* of A and B over S, denoted by $A \otimes_S B$. As usual we denote the equivalence class of (a, b) in $A \otimes_S B$ by $a \otimes b$. The order relation on $A \otimes_S B$ is defined as follows: $a \otimes b \leqslant a' \otimes b'$ in $A \otimes_S B$ if and only if there exist $a_1, a_2, ..., a_n \in A$, $b_2, ..., b_n \in B$, $s_1, t_1, ..., s_n, t_n \in S$ such that

$$\begin{array}{ll} a \leqslant a_1 s_1, & \\ a_1 t_1 \leqslant a_2 s_2, & s_1 b \leqslant t_1 b_2, \\ a_2 t_2 \leqslant a_3 s_3, & s_2 b_2 \leqslant t_2 b_3, \\ \vdots & \vdots \\ a_n t_n \leqslant a', & s_n b_n \leqslant t_n b'. \end{array}$$

A six-tuple $\langle S, T, {}_SP_T, {}_TQ_S, \tau, \mu\rangle$ is said to be a *Morita context* where S and T are pomonoids, ${}_SP_T \in {}_SPos_T$, ${}_TQ_S \in {}_TPos_S$, and $\tau : {}_S(P \otimes_T Q)_S \to {}_SS_S$ and $\mu : {}_T(Q \otimes_S P)_T \to {}_TT_T$ are biposet morphisms such that for all $p, p' \in P$ and $q, q' \in Q$ we have $\tau(p \otimes q)p' = p\mu(q \otimes p')$ and $q\tau(p \otimes q') = \mu(q \otimes p)q'$. For more notions of Morita equivalent pomonoids reader is referred to [11, 12, 20–23].

If A and Γ be two non-empty sets then A is said to be a Γ-*semigroup* if there exist mappings $A \times \Gamma \times A \to A$, denoted by $(a, \gamma, b) \mapsto a\gamma b$, and $\Gamma \times A \times \Gamma \to \Gamma$, denoted by $(\alpha, a, \beta) \mapsto \alpha a\beta$, satisfying $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ for all $a, b, c \in A$ and $\alpha, \beta \in \Gamma$. For a natural example of a Γ-semigroup let $A := Maps(M, N)$ and $\Gamma := Maps(N, M)$ where M and N are two non-empty sets. Then A is a Γ-semigroup where $a\gamma b$ and $\alpha a\beta$ are usual mapping compositions. If A is a Γ-semigroup and ρ is the equivalence relation on $\Gamma \times A$ defined by $(\alpha, a)\rho(\beta, b)$ if and only if $x\alpha a = x\beta b$ and $\alpha a\gamma = \beta b\gamma$ for all $x \in A$ and $\gamma \in \Gamma$, then we denote the equivalence class of (α, a) by $[\alpha, a]$. Now the *right operator semigroup* of the Γ-semigroup A is defined to be $R = (\Gamma \times A)/\rho = \{[\alpha, a] \mid a \in A, \alpha \in \Gamma\}$, where the composition is defined by $[\alpha, a][\beta, b] = [\alpha, a\beta b]$. The *left operator semigroup* L is defined analogously. We note here that A is a $L - R$-biset. Now if there exists an element $[\gamma, f]$ in R such that $x\gamma f = x$ for all $x \in A$, then it is called the *right unity* of A. We note that $[\gamma, f]$ becomes the identity of R. Similarly the *left unity* of A is defined. A Γ-semigroup is said to be a Γ-*semigroup with unities* if it has both left and right unities. A Γ-semigroup A is said to be a *partially ordered* Γ-*semigroup* (in short po-Γ-semigroup) if (1) A and Γ are posets, (2) $a \leqslant b$ in A implies that $a\alpha c \leqslant b\alpha c$, $c\alpha a \leqslant c\alpha b$ in A and $\gamma a\alpha \leqslant \gamma b\alpha$ in Γ for all $a, b, c \in A$ and $\alpha, \gamma \in \Gamma$, (3) $\alpha \leqslant \beta$ in Γ implies that $\alpha a\gamma \leqslant \beta a\gamma$, $\gamma a\alpha \leqslant \gamma a\beta$ in Γ and $a\alpha b \leqslant a\beta b$ in A for all $a, b \in A$ and $\alpha, \beta, \gamma \in \Gamma$. It is well known that operator semigroups of a (po)Γ-semigroup with unities are (po)monoids. For more details of Γ-semigroups, operator semigroups and po-Γ-semigroups we refer respectively to [14, 15], [4] and [8].

The notion of Γ-semigroup was introduced by M.K. Sen [18, 19] in 1981 as a generalization of semigroup. Subsequently Dutta and Adhikari generalized partially ordered semigroups to po-Γ-semigroups with unities in [8] in the year 2004. They also introduced the notion of operator pomonoids associated with a po-Γ-semigroup. Among others, Knauer [10] and Banaschewski [2], contributed a lot to develop the theory of Morita equivalence of monoids. Recently Laan [12] developed a theory of Morita equivalent partially ordered monoids or shortly pomonoids and obtained Morita I, Morita II and Morita III. The Morita theory of posemigroups and pomonoids are further investigated by Tart [22, 23] and Targla and Laan [21].

Using mainly the works of Knauer [10] and Banaschewski [2], in [16] we could find a close connection between the Morita equivalence of monoids and Γ-semigroups with unities which was used to enrich each other's study. As a sequel to this and motivated by the works of Laan [12], Tart [22, 23], Targla et al. [21] on Morita theory of pomonoids we make an attempt here to find connection between po-Γ-semigroups and Morita equivalent pomonoids. In this venture we show that operator pomonoids of a po-Γ-semigroup with unities are Morita equivalent and conversely if L and R are two Morita equivalent pomonoids then we can construct a po-Γ-semigroup with unities whose left and right operator pomonoids are Pos-isomorphic [12] to L and R respectively. As an application of these results we have been able to apply the existing results of po-Γ-semigroups to extend the Morita theory of pomonoids in the form of obtaining some properties of pomonoids which remain invariant under Morita equivalence. In the opposite direction we have used some recent results of Morita theory of pomonoids due to Tart [22, 23] and Targla and Laan [21] to obtain some results of po-Γ-semigroups.

1. Relating po-Γ-semigroup with Morita equivalence

In this section we are going to explore the relationship between po-Γ-semigroup with unities and Morita equivalence for pomonoids. Let us first obtain the following lemma.

Lemma 1. *Let A be a po-Γ-semigroup with unities and its left and right operator pomonoids be respectively L and R. Then ${}_LA_R$ and ${}_R\Gamma_L$ are biposets.*

Proof. By definition of po-Γ-semigroup, A and Γ both are posets. We define

$$L \times A \to A \quad \text{and} \quad A \times R \to A \quad \text{by}$$

$$[a, \alpha]c := a\alpha c \quad \text{and} \quad c[\beta, b] := c\beta b \quad \text{respectively.}$$

Again, L and R are pomonoids with respect to the ordering defined below:

(i) On L, $[a, \alpha] \leqslant [b, \beta]$ if and only if $a\alpha c \leqslant b\beta c$ and $\gamma a\alpha \leqslant \gamma b\beta$ for all $c \in A$ and $\gamma \in \Gamma$;

(ii) On R, $[\alpha, a] \leqslant [\beta, b]$ if and only if $c\alpha a \leqslant c\beta b$ and $\alpha a\gamma \leqslant \beta b\gamma$ for all $c \in A$ and $\gamma \in \Gamma$.

Now using the fact that A is a po-Γ-semigroup and L and R are partially ordered monoids we deduce the following:

(1) For all $a \in A$ and $[\alpha, b], [\beta, c] \in R$, $a([\alpha, b][\beta, c]) = a[\alpha, b\beta c] = a\alpha(b\beta c) = (a\alpha b)\beta c = (a[\alpha, b])[\beta, c]$.

(2) If $[\gamma, f]$ is the right unity of A then for all $a \in A$ we have $a[\gamma, f] = a$.

(3) For all $[\alpha, c] \in R$, $a \leqslant b$ gives $a\alpha c \leqslant b\alpha c$ whence $a[\alpha, c] \leqslant b[\alpha, c]$.

(4) For all $a \in A$, $[\alpha, b] \leqslant [\beta, c]$ gives $a\alpha b \leqslant a\beta c$ whence $a[\alpha, b] \leqslant a[\beta, c]$.

Hence A is a right R-poset. Similarly, A is a left L-poset. That A is a biposet follows from the associative property of the po-Γ-semigroup A. Similarly we can prove that ${}_R\Gamma_L$ is a biposet. □

In the following theorem we obtain the Morita equivalence of the left and right operator pomonoids of a po-Γ-semigroup with unities.

Theorem 1. *Let A be a po-Γ-semigroup with unities and its left and right operator pomonoids be respectively L and R. Then the following hold:*

(1) *L and R are Morita equivalent,*

(2) *$A \otimes_R \Gamma \cong L$ and $\Gamma \otimes_L A \cong R$,*

(3) *${}_LA$, A_R, ${}_R\Gamma$ and Γ_L are cyclic projective generators of their respective categories,*

(4) *$L \cong End(A_R) \cong End({}_R\Gamma)$ and $R \cong End(\Gamma_L) \cong End({}_LA)$.*

Proof. By Lemma 1, L and R are pomonoids, ${}_LA_R$ and ${}_R\Gamma_L$ are biposets. So we define

$$\tau : A \otimes \Gamma \to L \quad \text{and} \quad \mu : \Gamma \otimes A \to R$$

as follows

$$\tau(a \otimes \alpha) = [a, \alpha] \quad \text{and} \quad \mu(\alpha \otimes a) = [\alpha, a].$$

Let $a \otimes \alpha = b \otimes \beta$, for $a, b \in A$ and $\alpha, \beta \in B$. Then by the Theorem 5.2 of [20], there exist $a_1, a_2, ..., a_n, b_1, b_2, ..., b_m \in A, \alpha_2, ..., \alpha_n, \beta_2, ..., \beta_m \in B$, $s_1, t_1, ..., s_n, t_n, u_1, v_1, ..., u_m, v_m \in R$ such that,

$$\begin{array}{cccc}
a \leqslant a_1 s_1, & & b \leqslant b_1 u_1, & \\
a_1 t_1 \leqslant a_2 s_2, & s_1\alpha \leqslant t_1\alpha_2, & b_1 v_1 \leqslant b_2 u_2, & u_1\beta \leqslant v_1\beta_2, \\
a_2 t_2 \leqslant a_3 s_3, & s_2\alpha_2 \leqslant t_2\alpha_3, & b_2 v_2 \leqslant b_3 u_3, & u_2\beta_2 \leqslant v_2\beta_3, \\
\vdots & \vdots & \vdots & \vdots \\
a_n t_n \leqslant b, & s_n\alpha_n \leqslant t_n\beta; & b_m v_m \leqslant a, & u_m\beta_m \leqslant v_m\alpha.
\end{array}$$

Now we again use the Lemma 1 and obtain

$$\begin{aligned} a\alpha c &\leqslant (a_1 s_1)\alpha c = a_1(s_1\alpha)c \\ &\leqslant a_1(t_1\alpha_2)c = (a_1t_1)\alpha_2 c \\ &\leqslant (a_2s_2)\alpha c = a_2(s_2\alpha)c \\ &\cdots\cdots \\ &\leqslant a_n(t_n\beta)c = (a_nt_n)\beta c \\ &\leqslant b\beta c. \end{aligned} \tag{1}$$

Similarly we have $b\beta c \leqslant a\alpha c$. Since by definition A has partial ordering, so $a\alpha c \leqslant b\beta c$ and $b\beta c \leqslant a\alpha c$ together implies $a\alpha c = b\beta c$ for any $c \in A$. Similarly we can show that $\gamma a\alpha = \gamma b\beta$ for any $\gamma \in \Gamma$. Hence $[a,\alpha] = [b,\beta]$. Thus we see that the mapping τ is well-defined. Also it is clear that τ is onto.

Again, let $a \in A$, $\alpha \in \Gamma$ and $[b,\beta] \in L$. Then we see that

$$\begin{aligned} \tau([b,\beta](a\otimes\alpha)) &= \tau([b,\beta]a\otimes\alpha) \\ &= \tau(b\beta a\otimes\alpha) \\ &= [b\beta a,\alpha] \\ &= [b,\beta][a,\alpha] \\ &= [b,\beta]\tau(a\otimes\alpha). \end{aligned}$$

Similarly $\tau((a\otimes\alpha)[b,\beta]) = \tau(a\otimes\alpha)[b,\beta]$. So τ preserves the action from both sides.

Let $a\otimes\alpha \leqslant b\otimes\beta$, for $a,b \in A$ and $\alpha,\beta \in B$. Then using Lemma 1 along with the definition of $\leqslant$, we apply a similar argument, as given in equation (1), and prove that $a\alpha c \leqslant b\beta c$ for all $c \in A$. Similarly we deduce that $\gamma a\alpha \leqslant \gamma b\beta$ for all $\gamma \in \Gamma$. Hence we obtain $[a,\alpha] \leqslant [b,\beta]$.

Then τ preserves the ordering. Hence τ becomes a surjective biposet morphism. By a similar argument we see that μ is a surjective biposet morphism.

Now we see that for all $a,b \in A,\ \alpha,\beta \in \Gamma$,

$$\begin{aligned} \tau(a\otimes\alpha)b = [a,\alpha]b = a\alpha b = a[\alpha,b] = a\mu(\alpha\otimes b) \text{ and} \\ \alpha\tau(a\otimes\beta) = \alpha[a,\beta] = \alpha a\beta = [\alpha,a]\beta = \mu(\alpha\otimes a)\beta. \end{aligned}$$

Hence $\langle L, R,_L A_R,_R \Gamma_L, \tau, \mu\rangle$ is a Morita context with τ and μ surjective. Hence in view of Theorem 6 and Proposition 7 of [12] we obtain the desired results namely (1)–(4). $\square$

Remark 1. Since we have deduced that operator pomonoids of a po-Γ-semigroup A with unities are Morita equivalent so the examples of non-isomorphic operator pomonoids will also become examples of non-isomorphic Morita equivalent pomonoids.

To prove the converse part of the above theorem we first obtain some characterization of two Morita equivalent pomonoids. In this regard we prove the following lemma.

Lemma 2. *Let M_S be a right S-poset. Then $M_S \cong Pos_S(S, M_S)$.*

Proof. For each $m \in M_S$, if we define

$$\rho_m : \ {}_SS_S \to M_S \quad \text{by} \quad \rho_m(s) := ms$$

then $\rho_m \in Pos_S(S, M_S)$. Now we define

$$\rho : M_S \to Pos_S(S, M_S) \quad \text{by} \quad \rho(m) := \rho_m.$$

Then it will take more than a glance to see that ρ is an isomorphism. □

Now we obtain the following characterization for Morita equivalent pomonoids.

Theorem 2. *Let S and T be two Morita equivalent pomonoids via inverse Pos-equivalences $F : Pos_S \to Pos_T$ and $G : Pos_T \to Pos_S$. Set $P = F(S)$ and $Q = G(T)$. Then P and Q are biposets ${}_SP_T$ and ${}_TQ_S$ such that*

1. *P_T, Q_S are respectively cyclic projective generators for Pos_T and Pos_S;*
2. *$T \cong End(Q_S)$ and $S \cong End(P_T)$;*
3. *$F \cong Pos_S(Q_S, \_)$ and $G \cong Pos_T(P_T, \_)$;*
4. *${}_SP_T \cong Pos_S(Q, S)$ and ${}_TQ_S \cong Pos_T(P, T)$.*

Proof. By Theorem 4 of [12], P and Q are biposets ${}_SP_T$ and ${}_TQ_S$. Also in view of Theorem 3 of [12], (1) and (2) follow.

Now by Lemma 2, for any right S-poset M_S, $M_S \cong Pos_S(S, M_S)$. Then

$$G(M_T) \ \cong Pos_S(S, G(M_T)) \ \cong Pos_T(F(S), M_T) \ \cong Pos_T(P_T, M_T).$$

So we obtain $G \cong Pos_T(P_T, \_)$. Using this on T we deduce that

$$Q \ = \ G(T) \ \cong \ Pos_T(P_T, T).$$

Similarly we can prove $F \cong Pos_S(Q_S, \_)$ and ${}_SP_T \cong Pos_S(Q, S)$. Hence the theorem. □

Now we can obtain the following theorem which gives the converse of Theorem 1(1).

Theorem 3. *Let L and R be two Morita equivalent pomonoids. Then there exists a po-Γ-semigroup with unities whose left and right operator pomonoids are Pos-isomorphic to L and R respectively.*

Proof. Since L and R are Morita equivalent pomonoids, the categories Pos_L and Pos_R are Pos-equivalent via inverse Pos-equivalences, say,

$$F : Pos_L \to Pos_R \quad \text{and} \quad G : Pos_R \to Pos_L.$$

Let $A = F(L)$ and $\Gamma = G(R)$. Then by Theorem 2 we see that

(1) ${}_LA_R$ and ${}_R\Gamma_L$ are biposets;

(2) A_R and Γ_L are respectively cyclic projective generators for Pos_R and Pos_L;

(3) $R \cong End(\Gamma_L)$ and $L \cong End(A_R)$;

(4) $F \cong Pos_L(\Gamma_L, \_)$ and $G \cong Pos_R(A_R, \_)$;

(5) ${}_LA_R \cong Pos_L(\Gamma, L)$ and ${}_R\Gamma_L \cong Pos_R(A, R)$.

Now considering Γ as $Pos_R(A, R)$ we define the mappings

$$A \times \Gamma \times A \to A \quad \text{and} \quad \Gamma \times A \times \Gamma \to \Gamma$$

such that for $a, b, x \in A$ and $\alpha, \beta, \gamma \in \Gamma$

$$(a, \gamma, b) \mapsto a(\gamma(b)) \quad \text{and} \quad (\alpha, x, \beta) \mapsto (\alpha(x))\beta.$$

As a consequence, we deduce the following equalities:

$(a\alpha b)\beta c = (a(\alpha(b)))(\beta(c)) = a(\alpha(b)\beta(c))$, since A is a right R-poset;

$a\alpha(b\beta c) = a(\alpha(b(\beta(c)))) = a(\alpha(b)\beta(c))$, since α is a right R-poset morphism;

$a(\alpha b\beta)c = a(((\alpha(b))\beta)(c)) = a(\alpha(b)\beta(c))$, since $Pos_R(A, R)$ is a left R-poset.

Consequently, A becomes a Γ-semigroup. Let L' and R' be the left and right operator semigroups of the Γ-semigroup A. Then in order to complete the proof we consider the case of R and R' as the case of L and L' will follow in a similar fashion. We define

$$f : R' \to R \quad \text{by} \quad f([\alpha, a]) = \alpha(a).$$

Then

$$\begin{aligned}[\alpha, a] = [\beta, b] \quad & \text{implies that} \quad \alpha a\gamma = \beta b\gamma \text{ for all } \gamma \in \Gamma \\ & \text{i.e.,} \quad (\alpha(a))\gamma = (\beta(b))\gamma \text{ for all } \gamma \in \Gamma \\ & \text{i.e.,} \quad \alpha(a) = \beta(b).\end{aligned}$$

Hence the mapping f is well-defined. Again,

$$\begin{aligned}\alpha(a) = \beta(b) \quad & \text{implies that} \quad (\alpha(a))\gamma = (\beta(b))\gamma \text{ and } x(\alpha(a)) = x(\beta(b)) \\ & \qquad\qquad\quad \text{for all } x \in A,\ \gamma \in \Gamma \\ & \text{i.e.,} \quad \alpha a\gamma = \beta b\gamma \text{ and } x\alpha a = x\beta b \\ & \qquad\quad \text{for all } x \in A,\ \gamma \in \Gamma \\ & \text{i.e.,} \quad [\alpha, a] = [\beta, b].\end{aligned}$$

Hence f is injective. Again, since A_R is a generator of Pos_R, there exists a right R-poset epimorphism $\psi : A \to R$ (see [11]) such that for any $r \in R$, there exists $a \in A$ with $\psi(a) = r$. Then we have $f([\psi, a]) = r$. Hence the mapping f is surjective.

Also for all $a, b \in A$ and $\alpha, \beta \in \Gamma$,

$$\begin{aligned}f([\alpha, a][\beta, b]) &= f([\alpha, a\beta b]) = \alpha(a(\beta(b))) \\ &= \alpha(a)\beta(b) = f([\alpha, a])f([\beta, b]).\end{aligned}$$

Hence f is a semigroup morphism. Consequently, R' and R are isomorphic as monoids.

Hence A is a Γ-semigroup with unities whose left and right operator semigroups are isomorphic to L and R respectively. Now in view of Lemma 1 and the fact that $\Gamma \cong Pos_R(A, R)$ has pointwise ordering, we deduce that A is a po-Γ-semigroup.

Now in order to complete the proof we show that R' is Pos-isomorphic to R which we accomplish by proving that the isomorphism, defined above, preserves the orders. Now, if the right unity of A is $[\delta, e]$ then for $[\alpha, a], [\beta, b] \in R'$, $[\alpha, a] \leqslant [\beta, b]$ implies that $\alpha a\gamma \leqslant \beta b\gamma$ for all $\gamma \in \Gamma$. So, in particular,

$$\begin{aligned} & \alpha a\delta \leqslant \beta b\delta \\ \text{i.e.,} \quad & (\alpha(a))\delta \leqslant (\beta(b))\delta \\ \text{i.e.,} \quad & ((\alpha(a))\delta)(e) \leqslant ((\beta(b))\delta)(e) \text{ as } {}_R\Gamma_L \cong Pos_R(A, R) \\ \text{i.e.,} \quad & (\alpha(a))(\delta(e)) \leqslant (\beta(b))(\delta(e)) \text{ as } {}_R\Gamma_L \cong Pos_R(A, R) \\ \text{i.e.,} \quad & \alpha(a) \leqslant \beta(b) \text{ as } \delta(e) \text{ becomes the identity of } R.\end{aligned}$$

Again,

$$\begin{aligned} f([\alpha,a]) \leqslant f([\beta,b]) \quad & \text{implies that } \alpha(a) \leqslant \beta(b) \\ & \text{i.e., } \ c(\alpha(a)) \leqslant c(\beta(b)) \text{ for all } c \in A \\ & \qquad \text{and } (\alpha(a))\gamma \leqslant (\beta(b))\gamma \text{ for all } \gamma \in \Gamma \\ & \text{i.e., } \ c\alpha a \leqslant c\beta b \text{ for all } c \in A \\ & \qquad \text{and } \alpha a\gamma \leqslant \beta b\gamma \text{ for all } \gamma \in \Gamma \\ & \text{i.e., } \ [\alpha,a] \leqslant [\beta,b]. \end{aligned}$$

Consequently, f, f^{-1} preserve the ordering. This completes the proof. □

We conclude this section by combining Theorems 1 and 3 into the following theorem.

Theorem 4. *Two pomonoids L and R are Morita equivalent if and only if there exists a po-Γ-semigroup with unities whose operator pomonoids are isomorphic to L and R.*

2. Applications

In this section we find some applications of Theorem 4. In fact by this theorem and some well-known results of po-Γ-semigroups we obtain some properties of pomonoids which remain invariant under Morita equivalence.

Remark 2. In this section, what we use as the results of po-Γ-semigroup, they are actually taken without proof from their counterparts in Γ-semigroup [1, 4–7] because their proofs can be accomplished by slight modification of the proofs of their analogues.

Theorem 5. *Let L and R be Morita equivalent pomonoids. Then there exists an inclusion preserving bijection between the set of all ideals of R and the set of all ideals of L.*

Proof. By Theorem 3, there exists a po-Γ-semigroup A with unities whose left and right operator pomonoids L_1 and R_1 are isomorphic to L and R respectively. So it is sufficient to prove the result for L_1 and R_1. Now for each $P \subseteq L_1$ and $M \subseteq R_1$, we define (see [4])

$$P^{+} = \{x \in A \mid [x,\alpha] \in P \text{ for all } \alpha \in \Gamma\} \quad \text{and}$$
$$M^{*} = \{x \in A \mid [\alpha,x] \in M, \text{ for all } \alpha \in \Gamma\}.$$

Also for each $Q \subseteq A$, we define (see [4])

$$Q^{+'} = \{[x, \alpha] \in L_1 \mid x\alpha a \in Q, \text{ for all } a \in A\} \quad \text{and}$$
$$Q^{*'} = \{[\alpha, x] \in R_1 \mid a\alpha x \in Q, \text{ for all } a \in A\}.$$

Then there exists an inclusion preserving bijection between the set of all ideals of A and the set of all ideals of L_1 (see [4] and Remark 2) via the mapping

$$Q \mapsto Q^{+'} \quad \text{with the inverse mapping} \quad P \mapsto P^{+}.$$

And also there exists an inclusion preserving bijection between the set of all ideals of A and the set of all ideals of R_1 via the mapping

$$Q \mapsto Q^{*'} \quad \text{with the inverse mapping} \quad M \mapsto M^{*}.$$

Then we obtain the desired inclusion preserving bijection between the set of all ideals of R_1 and the set of all ideals of L_1 via the composition mapping given by

$$J \mapsto J^* \mapsto (J^*)^{+'} \text{ with the corresponding inverse } I \mapsto I^+ \mapsto (I^+)^{*'}. \quad \square$$

Now from [1, 7, 8] and in view of Remark 2, we know that all the mappings $+$, $+'$, $*$ and $*'$ carry prime ideals to prime ideals; weakly prime ideals to weakly prime ideals; semiprime ideals to semiprime ideals; primary ideals to primary ideals; semiprimary ideals to semiprimary ideals; nil ideals to nil ideals and nilpotent ideals to nilpotent ideals. Then by the same argument as applied in the above theorem we see that the composition mappings

$$J \mapsto J^* \mapsto (J^*)^{+'} \quad \text{and its inverse} \quad I \mapsto I^+ \mapsto (I^+)^{*'}.$$

are respectively mappings from R_1 to L_1 and L_1 to R_1 carrying prime ideals to prime ideals; weakly prime ideals to weakly prime ideals; semiprime ideals to semiprime ideals; nil ideals to nil ideals and nilpotent ideals to nilpotent ideals. This gives rise to the following result.

Theorem 6. *Let L and R be Morita equivalent pomonoids. Then there exists an inclusion preserving bijection between the set of all prime (weakly prime, semiprime, primary, semiprimary, nil, nilpotent) ideals of R and the set of all prime (weakly prime, semiprime, primary, semiprimary, nil, nilpotent) ideals of L.*

Also using Remark 2 and combining the respective results of [5, 7] for left operator L and right operator R of a po-Γ-semigroup A we find that different types of radicals viz. prime radical, Schwarz radical, Clifford radical are also preserved by the mappings

$$J \mapsto J^* \mapsto (J^*)^{+'} \quad \text{and its inverse} \quad I \mapsto I^+ \mapsto (I^+)^{*'}.$$

Hence we obtain the following result.

Theorem 7. *Let L and R be Morita equivalent pomonoids. Then the mapping* $J \mapsto J^* \mapsto (J^*)^{+'}$ *(*$I \mapsto I^+ \mapsto (I^+)^{*'}$*) carries prime (Schwarz, Clifford) radical of* R (L) *to prime (Schwarz, Clifford) radical of L (respectively R).*

Theorem 8. *Let L and R be Morita equivalent pomonoids. Then L is Noetherian if and only if R is Noetherian.*

Proof. By Theorem 3, there exists a po-Γ-semigroup A with left and right unities whose left and right operator pomonoids L_1 and R_1 are isomorphic to L and R respectively. According to [6] and Remark 2, L_1 is Noetherian if and only if A is Noetherian if and only if R_1 is Noetherian. Hence the result. □

In a similar way by using the analogous results of [1] and in view of Remark 2, we get the following theorem.

Theorem 9. *Let L and R be Morita equivalent pomonoids. Then L is primary (semiprimary) if and only if R is primary (semiprimary).*

Theorem 10. *Let L and R be Morita equivalent pomonoids. Then there exists an inclusion preserving bijection between the set of all ordered semilattice congruence (ordered fuzzy semilattice congruence) of R and the set of all ordered semilattice congruence (respectively, ordered fuzzy semilattice congruence) of L.*

Proof. By Theorem 3, there exists a po-Γ-semigroup A with unities whose left and right operator pomonoids L_1 and R_1 are isomorphic to L and R respectively. So it is sufficient to prove the result for L_1 and R_1. Now for each relation σ on L_1 and ω on R_1, we define (see [17]) relations σ^+ and ω^* respectively on A as follows:

$$x\sigma^+y \text{ if and only if } [x, \alpha]\sigma[y, \alpha] \text{ for all } \alpha \in \Gamma \quad \text{and}$$

$$x\omega^+y \text{ if and only if } [\alpha, x]\sigma[\alpha, y] \text{ for all } \alpha \in \Gamma.$$

Also for each relation ρ on A, we define (see [17]) relations $\rho^{+'}$ on L_1 and $\rho^{*'}$ on R_1 as follows:

$$[x,\alpha]\rho^{+'}[y,\beta] \text{ if and only if } (x\alpha a)\rho(y\beta a) \text{ for all } a \in A \quad \text{and}$$

$$[\alpha,x]\rho^{*'}[\beta,y] \text{ if and only if } (a\alpha x)\rho(a\beta y) \text{ for all } a \in A.$$

Then there exists an inclusion preserving bijection between the set of all ordered semilattice congruence (ordered fuzzy semilattice congruence) of A and the set of all ordered semilattice congruence (ordered fuzzy semilattice congruence) of L_1 (see [17]) via the mapping

$$\rho \mapsto \rho^{+'} \quad \text{with the inverse} \quad \sigma \mapsto \sigma^{+}.$$

And also there exists an inclusion preserving bijection between the set of all ordered semilattice congruence (ordered fuzzy semilattice congruence) of A and the set of all ordered semilattice congruence (ordered fuzzy semilattice congruence) of R_1 via the mapping

$$\rho \mapsto \rho^{*'} \quad \text{with the inverse} \quad \omega \mapsto \omega^{*}.$$

Then we find an inclusion preserving bijection between the set of all ordered semilattice congruence (ordered fuzzy semilattice congruence) of R_1 and the set of all ordered semilattice congruence (ordered fuzzy semilattice congruence) of L_1 via the composition mapping

$$\omega \mapsto \omega^{*} \mapsto (\omega^{*})^{+'} \quad \text{with the inverse} \quad \sigma \mapsto \sigma^{+} \mapsto (\sigma^{+})^{*'}. \qquad \square$$

So far, in this section, we have used the results of po-Γ-semigroups to obtain results of pomonoids viz. the Morita invariants. Now in the rest of this section we move the other way round i.e., we use results of posemigroups or pomonoids obtained by Tart [22, 23] and Targla and Laan [21] to obtain some results of po-Γ-semigroups.

In what follows A denotes a po-Γ-semigroup with unities and L and R respectively denote the left and right operator pomonoids of A. Then in view of Theorem 1, L and R are Morita equivalent pomonoids. So the following result follows easily from Proposition 6.1 of [22].

Theorem 11. (1) *L is regular if and only if R is regular.*

(2) *Sets of regular $\mathcal{D}$-classes of L and R have same cardinalities.*

Theorem 12. *If the order on L is either total, discrete, directed or a semiorder, then the order on R is also total, discrete, directed or a semiorder and vice-versa.*

Proof. Since L and R are Morita equivalent pomonoids, they are strongly Morita equivalent posemigroups with common local units. Hence the result follows from Proposition 3.2 of [23]. □

Theorem 13. (1) *If L satisfies an inequality, then R also satisfies the same inequality and vice-versa. Moreover if L satisfies an identity, then R also satisfies the same identity and vice-versa.*

(2) *If L is commutative or a band or a semilattice then R is also so and vice-versa.*

(3) *Greatest commutative images of L and R are isomorphic.*

Proof. Since L and R are strongly Morita equivalent posemigroups with common two-sided weak local units, (1), (2) and (3) are just consequences of Theorem 3.1, Corollary 3.2 and Theorem 3.2 of [23] respectively. □

By a similar argument, the following theorem follows from our Theorem 1 and Theorem 4.1 of [23].

Theorem 14. *There is a lattice isomorphism between the lattices of ideals (downwards closed ideals, upwards closed ideals, convex ideals) of L and R which takes finitely generated ideals to finitely generated ideals and principal ideals to principal ideals.*

Theorem 15. (1) *Congruence lattices of L and R are isomorphic.*

(2) *Picard groups of L and R are isomorphic.*

Proof. Since L and R are Morita equivalent pomonoids, (1) and (2) are just consequences of Corollary 1 of [21] and Corollary 5 of [12] respectively. □

Concluding remark

(1) The results of the last section illustrates that Theorem 4, which connects closely Morita equivalence of pomonoids and po-Γ-semigroups, can supplement the study of each other.

(3) It will be nice if the results obtained here for pomonoids still hold in partially ordered semigroups (without identity).

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