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RESEARCH ARTICLE

Additivity of Jordan elementary maps on standard rings

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Abstract. We prove that Jordan elementary surjective maps on standard rings are additive.

1. Standard rings and Jordan elementary map

Throughout this paper the word ring will mean a not necessarily associative or commutative ring. Let $\mathfrak{R}$ be a ring. For $x, y, z \in \mathfrak{R}$, we denote the *associator* by $(x, y, z) = (xy)z - x(yz)$ and the *commutator* by $[x, y] = xy - yx$. A ring $\mathfrak{R}$ is called *k-torsion free* if $k\,x = 0$ implies $x = 0$, for any $x \in \mathfrak{R}$, where $k \in \mathbb{Z}$, $k > 0$. Let us define the linear application $f : \mathfrak{R} \to \mathfrak{R}$, by $f(x) = kx$ for all $x \in \mathfrak{R}$. Clearly, if $\mathfrak{R}$ is a k-torsion free ring, then f is an injective application. In his case, we denote $x = \frac{1}{k}y$ when $y = kx$. A ring $\mathfrak{R}$ is said *prime* if $\mathfrak{I}\mathfrak{J} \neq 0$ for any two nonzero ideals $\mathfrak{I}, \mathfrak{J} \subseteq \mathfrak{R}$.

Let $\mathfrak{R}$ be a 2-torsion free ring satisfying the following identities:

$$(x, y, z) + (z, x, y) - (x, z, y) = 0, \tag{1}$$

$$(wx, y, z) + (xz, y, w) + (wz, y, x) = 0, \tag{2}$$

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for all $w, x, y, z, \in \mathfrak{R}$. These identities are satisfied by any associative ring and any 2-torsion free Jordan ring. Put $z = x$ in (1). Then

$$(x, y, x) = 0, \tag{3}$$

for all $x, y \in \mathfrak{R}$. Moreover, if $\mathfrak{R}$ is a 3-torsion free ring, then (2) implies

$$(x^2, y, x) = 0, \tag{4}$$

for all $x, y \in \mathfrak{R}$.

We say that $\mathfrak{R}$ is a *standard ring* if cases (1), (2) and (4) are satisfied. The condition (4) is redundant if $\mathfrak{R}$ is 3-torsion free.

So, every standard ring is a noncommutative Jordan ring.

Let us consider $\mathfrak{R}$ a standard ring and let us fix a nontrivial idempotent $e_1 \in \mathfrak{R}$, i.e, $e_1^2 = e_1,\ e_1 \neq 0$ and e_1 is not a unity element. Let $e_2 \colon \mathfrak{R} \to \mathfrak{R}$ and $e_2' \colon \mathfrak{R} \to \mathfrak{R}$ be given by $e_2 a = a - e_1 a$ and $e_2' a = a - a e_1$. We denote $e_2' a$ by $a e_2$. Then, by a process similar to [7], we can show that $\mathfrak{R}$ has a Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{\frac{1}{2}\frac{1}{2}} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, where $\mathfrak{R}_{ij} = \left\{x_{ij} \in \mathfrak{R} \mid e_1 x_{ij} = i x_{ij} \text{ and } x_{ij} e_1 = j x_{ij}\right\}$ $(i, j = 1, 2)$ and $\mathfrak{R}_{\frac{1}{2}\frac{1}{2}} = \left\{x_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R} \mid 2e_1 x_{\frac{1}{2}\frac{1}{2}} = x_{\frac{1}{2}\frac{1}{2}} \text{ and } 2x_{ij}e_1 = x_{\frac{1}{2}\frac{1}{2}}\right\}$, satisfying the multiplicative relations:

(i) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} \subseteq \delta_{jk}\mathfrak{R}_{il}$ $(i, j, k, l = 1, 2)$, where δ_{jk} is the Kronecker delta;

(ii) $\mathfrak{R}_{ii}\mathfrak{R}_{\frac{1}{2}\frac{1}{2}} \subseteq \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$ and $\mathfrak{R}_{\frac{1}{2}\frac{1}{2}}\mathfrak{R}_{ii} \subseteq \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$ $(i = 1, 2)$;

(iii) $\mathfrak{R}_{ij}\mathfrak{R}_{\frac{1}{2}\frac{1}{2}} = 0$ and $\mathfrak{R}_{\frac{1}{2}\frac{1}{2}}\mathfrak{R}_{ij} = 0$ $(i, j = 1, 2; i \neq j)$;

(iv) $\mathfrak{R}_{\frac{1}{2}\frac{1}{2}}\mathfrak{R}_{\frac{1}{2}\frac{1}{2}} \subseteq \mathfrak{R}_{11} \oplus \mathfrak{R}_{22}$;

(v) $[\mathfrak{R}, \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}] = 0$.

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two rings and let

$$M : \mathfrak{R} \longrightarrow \mathfrak{R}' \quad \text{and} \quad M^* : \mathfrak{R}' \longrightarrow \mathfrak{R}$$

be two maps. We call the ordered pair (M, M^*) a *Jordan elementary map of* $\mathfrak{R} \times \mathfrak{R}'$ if

$$\begin{cases} M(aM^*(x) + M^*(x)a) = M(a)x + xM(a) \\ M^*(M(a)x + xM(a)) = aM^*(x) + M^*(x)a, \end{cases}$$

for all $a \in \mathfrak{R}$ and $x \in \mathfrak{R}'$.

We say that the Jordan elementary map (M, M^*) of $\mathfrak{R} \times \mathfrak{R}'$ is *additive* (resp., *injective, surjective, bijective*) if both maps M and M^* are additive (resp., injective, surjective, bijective).

The problem of when a map must be additive has been studied for the case of several rings with nontrivial idempotents. The authors [2], [5] showed that under the condition of surjectivity a Jordan elementary map is additive for the case of associative and alternative rings. Also, the question of when a multiplicative map is additive was investigated by [4] in which it was shown that under the condition of bijectivity a multiplicative map is additive for the case of Jordan rings.

Theorem ([5, W. Jing]). *Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two associative rings. Suppose that $\mathfrak{R}$ is a 2-torsion free ring containing a non-trivial idempotent e_1 satisfying:*

(i) $e_iae_j\mathfrak{R}e_k = 0$ *or* $e_k\mathfrak{R}e_iae_j = 0$ *implies* $e_iae_j = 0$ $(i, j, k = 1, 2)$,

(ii) $(e_2ae_2)(be_2) + (e_2b)(e_2ae_2) = 0$, *for each* $b \in \mathfrak{R}$, *then* $e_2ae_2 = 0$.

Then every surjective Jordan elementary map (M, M^*) of $\mathfrak{R} \times \mathfrak{R}'$ is additive.

Theorem ([2, J. C. M. Ferreira, H. Guzzo Jr.]). *Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two alternative rings. Suppose that $\mathfrak{R}$ is a 2-torsion free ring containing a nontrivial idempotent e_1 satisfying:*

(i) $(e_iae_j)(\mathfrak{R}e_k) = 0$ *or* $((e_iae_j)\mathfrak{R})e_k = 0$ *implies* $e_iae_j = 0$ $(1 \leqslant i, j, k \leqslant 2)$;

(ii) $(e_k\mathfrak{R})(e_iae_j) = 0$ *or* $e_k(\mathfrak{R}(e_iae_j)) = 0$ *implies* $e_iae_j = 0$ $(1 \leqslant i, j, k \leqslant 2)$;

(iii) *if* $(e_2ae_2)(be_2) + (e_2b)(e_2ae_2) = 0$ *for each* $b \in \mathfrak{R}$, *then* $e_2ae_2 = 0$.

Then every surjective Jordan elementary map (M, M^*) of $\mathfrak{R} \times \mathfrak{R}'$ is additive.

Theorem ([4, P. Ji]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Jordan algebras over a field $\mathfrak{F}$ of characteristic not two and p a non trivial idempotent in $\mathfrak{A}$. Let $\mathfrak{A} = \mathfrak{A}_1 \oplus \mathfrak{A}_{\frac{1}{2}} \oplus \mathfrak{A}_2$ be the Peirce decomposition of $\mathfrak{A}$ with respect to p. If $\mathfrak{A}$ satisfies the following conditions:*

(i) *Let* $a_i \in \mathfrak{A}_i$ $(i = 1, 2)$. *If* $a_it_{\frac{1}{2}} = 0$ *for all* $t_{\frac{1}{2}} \in \mathfrak{A}_{\frac{1}{2}}$, *then* $a_i = 0$*;*

(ii) *Let* $a_0 \in \mathfrak{A}_0$. *If* $a_0t_0 = 0$ *for all* $t_0 \in \mathfrak{A}_0$, *then* $a_0 = 0$*;*

(iii) *Let* $a_{\frac{1}{2}} \in \mathfrak{A}_{\frac{1}{2}}$. *If* $a_{\frac{1}{2}} t_0 = 0$ *for all* $t_0 \in \mathfrak{A}_0$, *then* $a_{\frac{1}{2}} = 0$;

Then every map ϕ *from* $\mathfrak{A}$ *onto* $\mathfrak{B}$ *that is bijective and satisfies*

$$\phi(ab) = \phi(a)\phi(b),$$

for all $a, b \in \mathfrak{A}$ *is additive.*

It is clear that for 2-torsion free rings an additive map is a Jordan elementary map if, and only if, is multiplicative. But in general, we do not know whether they are still equivalent without the additivity assumption. Thus, in this paper we consider the question and give affirmative answer for the case of a Jordan elementary maps on standard rings.

2. The main theorem

We will prove the following result:

Theorem 1. *Let* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *be two standard rings such that* $\mathfrak{R}$ *is a 2-torsion free ring containing a non-trivial idempotent* e_1 *and* $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{\frac{1}{2}\frac{1}{2}} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, *the Peirce Decomposition of* $\mathfrak{R}$, *relative to* e_1, *satisfying at least one of the two sets of conditions:*

(i) $a_{ij}(\mathfrak{R}e_k) = 0$ *or* $(e_k\mathfrak{R})a_{ij} = 0$ *implies* $a_{ij} = 0$ $(i, j, k = 1, 2; i \neq j)$,

(ii) *If* $(e_i a e_i)t_{ij} = 0$ *for every* $t_{ij} \in \mathfrak{R}_{ij}$, *then* $a_{ii} = 0$ $(i, j = 1, 2; i \neq j)$,

(iii) *If* $t_{ij}(e_j a e_j) = 0$ *for every* $t_{ij} \in \mathfrak{R}_{ij}$, *then* $a_{jj} = 0$ $(i, j = 1, 2; i \neq j)$,

(iv) *If* $a_{ii}t_{22} = 0$ *(res.,* $t_{22}a_{ii} = 0$*)* $(i = \frac{1}{2}, 2)$ *for every* $t_{22} \in \mathfrak{R}_{22}$, *then* $a_{ii} = 0$,

(v) *If* $a_{22}t_{22} + t_{22}a_{22} = 0$ *for every* $t_{22} \in \mathfrak{R}_{22}$, *then* $a_{22} = 0$,

or

(i′) $a_{ij}(\mathfrak{R}e_j) = 0$ *or* $(e_i\mathfrak{R})a_{ij} = 0$ *implies* $a_{ij} = 0$ $(i, j = 1, 2; i \neq j)$,

(ii′) *If* $a_{ii}t_{\frac{1}{2}\frac{1}{2}} = 0$ $(i = 1, 2)$ *for every* $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$, *then* $a_{ii} = 0$,

(iii′) *If* $a_{ii}t_{22} = 0$ *(resp.,* $t_{22}a_{ii} = 0$*)* $(i = \frac{1}{2}, 2)$ *for every* $t_{22} \in \mathfrak{R}_{22}$, *then* $a_{ii} = 0$,

(iv′) *If* $a_{22}t_{22} + t_{22}a_{22} = 0$ *for every* $t_{22} \in \mathfrak{R}_{22}$, *then* $a_{22} = 0$,

Then every surjective Jordan elementary map (M, M^*) *of* $\mathfrak{R} \times \mathfrak{R}'$ *is additive.*

The proof of theorem is the same for both sets of axioms and is organized as a series of lemmas. When necessary in the proves of the lemmas, we shall present both conditions or separately, of the first and second set of axioms of the Theorem, or when derived from them, to the conclusion of each desired result.

Henceforth, where necessary, we shall use the components of the decomposition of Peirce of any element of ring without making any mention.

We begin with the following lemma that your proof is very simple.

Lemma 1. $M(0) = 0$ *and* $M^*(0) = 0$.

Lemma 2. *Let* $a = a_{11} + a_{12} + a_{\frac{1}{2}\frac{1}{2}} + a_{21} + a_{22} \in \mathfrak{R}$.

If $\mathfrak{R}$ *satisfies the conditions (i)–(v), of Theorem, then:*

(i) *If* $a_{ij}t_{jk} = 0$ *for each* $t_{jk} \in \mathfrak{R}_{jk}$ $(i, j, k = 1, 2)$, *then* $a_{ij} = 0$. *Dually, if* $t_{ki}a_{ij} = 0$ *for each* $t_{ki} \in \mathfrak{R}_{ki}$ $(i, j, k = 1, 2)$, *then* $a_{ij} = 0$;

(ii) *If* $t_{ij}a+at_{ij} \in \mathfrak{R}_{ij}$ *for every* $t_{ij} \in \mathfrak{R}_{ij}(i, j = 1, 2; i \neq j)$, *then* $a_{ji} = 0$;

(iii) *If* $a_{ii}t_{ii} + t_{ii}a_{ii} = 0$ *for every* $t_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, *then* $a_{ii} = 0$;

(iv) *If* $t_{jj}a + at_{jj} \in \mathfrak{R}_{ij}$ *for every* $t_{jj} \in \mathfrak{R}_{jj}$ $(i, j = 1, 2; i \neq j)$, *then* $a_{ji} = 0$, $a_{jj} = 0$ *and* $a_{\frac{1}{2}\frac{1}{2}} = 0$. *Dually, if* $t_{jj}a + at_{jj} \in \mathfrak{R}_{ji}$ *for every* $t_{jj} \in \mathfrak{R}_{jj}$ $(i, j = 1, 2; i \neq j)$, *then* $a_{ij} = 0$, $a_{jj} = 0$ *and* $a_{\frac{1}{2}\frac{1}{2}} = 0$.

If $\mathfrak{R}$ *satisfies the conditions (i′)–(iv′), of Theorem, then:*

(i′) *If* $a_{ij}t_{jj} = 0$ *for each* $t_{jj} \in \mathfrak{R}_{jj}$ $(i, j = 1, 2; i \neq j)$, *then* $a_{ij} = 0$. *Dually, if* $t_{ii}a_{ij} = 0$ *for each* $t_{ii} \in \mathfrak{R}_{ii}$ $(i, j = 1, 2; i \neq j)$, *then* $a_{ij} = 0$;

(ii′) *If* $a_{ii}t_{ii} + t_{ii}a_{ii} = 0$ *for every* $t_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, *then* $a_{ii} = 0$;

(iii′) *If* $t_{jj}a + at_{jj} \in \mathfrak{R}_{ij}$ *for every* $t_{jj} \in \mathfrak{R}_{jj}$ $(i, j = 1, 2; i \neq j)$, *then* $a_{ji} = 0$, $a_{jj} = 0$ *and* $a_{\frac{1}{2}\frac{1}{2}} = 0$. *Dually, if* $t_{jj}a + at_{jj} \in \mathfrak{R}_{ji}$ *for every* $t_{jj} \in \mathfrak{R}_{jj}$ $(i, j = 1, 2; i \neq j)$, *then* $a_{ij} = 0$, $a_{jj} = 0$ *and* $a_{\frac{1}{2}\frac{1}{2}} = 0$.

Proof. If $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then:

(i) For the case $(i = 1, j = 2)$. If $k = 1$, then $2a_{12}(te_1) = 2a_{12}t_{21} = 0$, for all $t \in \mathfrak{R}$. This implies $a_{12}(\mathfrak{R}e_1) = 0$. It follows from condition (i), of Theorem, that $a_{12} = 0$. If $k = 2$, then $2a_{12}(te_2) = 2a_{12}t_{22} = 0$. This implies $a_{12}(\mathfrak{R}e_2) = 0$. Again, it follows from condition (i), of Theorem, that $a_{12} = 0$. For the case $(i = j = 1)$. If $k = 1$, then $a_{11}t_{11} = 0$ implies $a_{11} = 0$, because $e_1 \in \mathfrak{R}_{11}$. If $k = 2$, then $4(e_1ae_1)t_{12} = 4a_{11}t_{12} = 0$. Hence,

$(e_1ae_1)t_{12} = 0$. It follows from condition (ii), of Theorem, that $a_{11} = 0$. Now, for the case $(i = 2, j = 1)$. If $k = 1$, then $2a_{21}(te_1) = 2a_{21}t_{11} = 0$, for all $t \in \mathfrak{R}$. So $a_{21}(\mathfrak{R}e_1) = 0$. It follows from condition (i), of Theorem, that $a_{21} = 0$. If $k = 2$, then $2a_{21}(te_2) = 2a_{21}t_{12} = 0$. Therefore, $a_{21}(\mathfrak{R}e_2) = 0$. Again, it follows from condition (i), of Theorem, that $a_{21} = 0$. For the case $(i = j = 2)$. If $k = 1$, then $4(e_2ae_2)t_{21} = 4a_{22}t_{21} = 0$, that is, $a_{22}t_{21} = 0$. It follows from condition (ii), of Theorem, that $a_{22} = 0$. If $k = 2$, then $a_{22}t_{22} = 0$ implies $a_{22} = 0$, by condition (iv), of Theorem. Similarly, we prove the dual cases.

(ii) Since $t_{ij}a + at_{ij} \in \mathfrak{R}_{ij}$, we have $(t_{ij}a + at_{ij})e_i = 0$ which implies $t_{ij}a_{ji} = 0$. Hence, $t_{ij}a_{ji} = 0$. Thus, $a_{ji} = 0$, by (i), of Lemma.

(iii) For the case $i = 1$, in particular, we have $0 = a_{11}e_1 + e_1a_{11} = 2a_{11}$ and so $a_{11} = 0$, since $\mathfrak{R}$ is 2-torsion free. The case $i = 2$ follows direct of (v), of Theorem.

(iv) If $j = 1$, then $t_{11}a + at_{11} \in \mathfrak{R}_{21}$. Hence, $e_1(t_{11}a + at_{11}) = 0$ which implies $t_{11}a_{11} + a_{11}t_{11} = 0$, $t_{11}a_{12} = 0$ and $t_{11}a_{\frac{1}{2}\frac{1}{2}} = 0$. So $a_{11} = 0$, $a_{12} = 0$ and $a_{\frac{1}{2}\frac{1}{2}} = 0$, because $e_1 \in \mathfrak{R}_{11}$. Now, if $j = 2$, then $t_{22}a + at_{22} \in \mathfrak{R}_{12}$. Hence, $e_2(t_{22}a + at_{22}) = 0$ which implies $t_{22}a_{\frac{1}{2}\frac{1}{2}} = 0$, $t_{22}a_{21} = 0$ and $t_{22}a_{22} + a_{22}t_{22} = 0$. So $a_{\frac{1}{2}\frac{1}{2}} = 0$, $a_{21} = 0$ and $a_{22} = 0$, by (iv) of Theorem, (i) and (iii) of Lemma, respectively. Similarly, we prove the dual cases.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then the demonstrations, of the cases (i′)–(iv′), of Lemma, are made identically to the preceding cases. For this, it is sufficient in the case (i′), of Lemma, to take the case (i), just to $i \neq j$, replacing in the proof, the condition (i), of the theorem, by condition (i′). In the case (ii′), of Lemma, to take the case (iii), replacing in the proof, the condition (v), of the theorem, by the condition (iv′), and in the case (iii′), of Lemma, to take the case (iv), replacing in the proof, the conditions (iv), (i) and (iii), of the theorem, by the conditions (iii′), (i′) and (iii′), respectively. □

Lemma 3. *M and M^* are injective.*

Proof. Let $a, b \in \mathfrak{R}$ be two elements such that $M(a) = M(b)$. For every $t_{jj} \in \mathfrak{R}_{jj}$ $(i = 1, 2)$, there exists $x(j, j) \in \mathfrak{R}'$ such that $M^*(x(j, j)) = t_{jj}$, by hypothesis of the surjectivity of M^*. Hence,

$$\begin{aligned} t_{jj}a + at_{jj} &= M^*(x(j,j))a + aM^*(x(j,j)) \\ &= M^*(x(j,j)M(a) + M(a)x(j,j)) \\ &= M^*(x(j,j)M(b) + M(b)x(j,j)) \\ &= M^*(x(j,j))b + bM^*(x(j,j)) = t_{jj}b + bt_{jj}. \end{aligned}$$

This implies
$$t_{jj}(a-b)+(a-b)t_{jj}=0.$$
By (iv) (resp., (iii′)) in Lemma 2, we have $a_{ij}-b_{ij}=0$ $(i,j=\frac{1}{2},1,2)$, that is $a=b$. Thus, M is injective. Now, let $x,y\in\mathfrak{R}'$ such that $M^*(x)=M^*(y)$. Since M is a bijection, there are $a,b\in\mathfrak{R}$ such that $a=M^{-1}(x)$ and $b=M^{-1}(y)$. Hence, for every $t_{jj}\in\mathfrak{R}_{jj}$ $(j=1,2)$, there is a $c(j,j)\in\mathfrak{R}$ such that $M^*M(c(j,j))=t_{jj}$, by the surjectivity of M^*M. This implies
$$\begin{aligned}
t_{jj}a+at_{jj} &= t_{jj}M^{-1}(x)+M^{-1}(x)t_{jj}\\
&= M^*M(c(j,j))M^{-1}(x)+M^{-1}(x)M^*M(c(j,j))\\
&= M^*\big(M(c(j,j))MM^{-1}(x)+MM^{-1}(x)M(c(j,j))\big)\\
&= M^*\big(M(c(j,j))x+xM(c(j,j))\big)\\
&= c(j,j)M^*(x)+M^*(x)c(j,j)=c(j,j)M^*(y)+M^*(y)c(j,j)\\
&= M^*\big(M(c(j,j))y+yM(c(j,j))\big)\\
&= M^*\big(M(c(j,j))MM^{-1}(y)+MM^{-1}(y)M(c(j,j))\big)\\
&= M^*M(c(j,j))M^{-1}(y)+M^{-1}(y)M^*M(c(j,j))\\
&= t_{jj}M^{-1}(y)+M^{-1}(y)t_{jj}\\
&= t_{jj}b+bt_{jj}.
\end{aligned}$$
Thus,
$$t_{jj}(a-b)+(a-b)t_{jj}=0.$$
Again, by (iv) (resp., (iii′)) in Lemma 2, we have $a_{ij}-b_{ij}=0$ $(i,j=\frac{1}{2},1,2)$ and so $a=b$. Consequently, we have $x=y$, by bijectivity of M. Therefore, we can also infer that M^* is injective. □

The three lemmas that follow, have identical proofs, as in [5]. Thus, they will be omitted.

Lemma 4. *The pair* (M^{*-1},M^{-1}) *is a Jordan elementary map on* $\mathfrak{R}\times\mathfrak{R}'$.

Lemma 5. *Let* $a,b,c\in\mathfrak{R}$ *such that* $M(c)=M(a)+M(b)$. *Then*
$$M^{*-1}(tc+ct)=M^{*-1}(ta+at)+M^{*-1}(tb+bt)$$
for all $t\in\mathfrak{R}$.

Lemma 6. *Let* $x,y,z\in\mathfrak{R}'$ *such that* $M^*(z)=M^*(x)+M^*(y)$. *Then*
$$M^{-1}(wz+zw)=M^{-1}(wx+xw)+M^{-1}(wy+yw)$$
for all $w\in\mathfrak{R}'$.

Corollary 1. *Let $a, b, c \in \mathfrak{R}$ such that $M^{*-1}(c) = M^{*-1}(a) + M^{*-1}(b)$. Then*

$$M(tc + ct) = M(ta + at) + M(tb + bt)$$

for all $t \in \mathfrak{R}$.

Corollary 2. *Let $x, y, z \in \mathfrak{R}'$ such that $M^{-1}(z) = M^{-1}(x) + M^{-1}(y)$. Then*

$$M^*(wz + zw) = M^*(wx + xw) + M^*(wy + yw)$$

for all $w \in \mathfrak{R}'$.

Lemma 7. *Let $a_{11} \in \mathfrak{R}_{11}$ and $b_{22} \in \mathfrak{R}_{22}$. Then*

(i) $M(a_{11} + a_{22}) = M(a_{11}) + M(a_{22})$;

(ii) $M^{*-1}(a_{11} + a_{22}) = M^{*-1}(a_{11}) + M^{*-1}(a_{22})$.

Proof. (i) Suppose $M(c) = M(a_{11}) + M(a_{22})$, for some $c \in \mathfrak{R}$. For arbitrary $t_{11} \in \mathfrak{R}_{11}$, by Lemma 5, we have

$$\begin{aligned} M^{*-1}(t_{11}c + ct_{11}) &= M^{*-1}(t_{11}a_{11} + a_{11}t_{11}) + M^{*-1}(t_{11}a_{22} + a_{22}t_{11}) \\ &= M^{*-1}(t_{11}a_{11} + a_{11}t_{11}). \end{aligned}$$

Hence, $t_{11}c + ct_{11} = t_{11}a_{11} + a_{11}t_{11}$ which implies $c_{11} = a_{11}$, $c_{12} = 0$, $c_{\frac{1}{2}\frac{1}{2}} = 0$ and $c_{21} = 0$, because $e_1 \in \mathfrak{R}_{11}$. Now, for arbitrary $t_{22} \in \mathfrak{R}_{22}$, by Lemma 5, we have

$$\begin{aligned} M^{*-1}(t_{22}c + ct_{22}) &= M^{*-1}(t_{22}a_{11} + a_{11}t_{22}) + M^{*-1}(t_{22}a_{22} + a_{22}t_{22}) \\ &= M^{*-1}(t_{22}a_{22} + a_{22}t_{22}). \end{aligned}$$

Hence, $t_{22}c + ct_{22} = t_{22}a_{22} + a_{22}t_{22}$ which implies $c_{22} = a_{22}$, by (iv) (resp., (iii′)) in Lemma 2. So $c = a_{11} + a_{22}$.

(ii) This proof is similar to case (i). □

Lemma 8. *Let $a_{ii} \in \mathfrak{R}_{ii}$ and $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$. Then*

(i) $M(a_{ii} + a_{ij}) = M(a_{ii}) + M(a_{ij})$;

(ii) $M^{*-1}(a_{ii} + a_{ij}) = M^{*-1}(a_{ii}) + M^{*-1}(a_{ij})$.

Proof. (i) For the case $(i = 1, j = 2)$, suppose that $M(c) = M(a_{11}) + M(a_{12})$ for some $c \in \mathfrak{R}$. For arbitrary $t_{22} \in \mathfrak{R}_{22}$, using Lemma 5, we have

$$\begin{aligned} M^{*-1}(t_{22}c + ct_{22}) &= M^{*-1}(t_{22}a_{11} + a_{11}t_{22}) + M^{*-1}(t_{22}a_{12} + a_{12}t_{22}) \\ &= M^{*-1}(a_{12}t_{22}) \end{aligned}$$

which yields $t_{22}c + ct_{22} = a_{12}t_{22} \in \mathfrak{R}_{12}$. It follows from (iv) (resp., (iii′)), in Lemma 2, that $c_{\frac{1}{2}\frac{1}{2}} = 0$, $c_{21} = 0$ and $c_{22} = 0$.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, by Lemma 5, we have

$$\begin{aligned} M^{*-1}(t_{12}c + ct_{12}) &= M^{*-1}(t_{12}a_{11} + a_{11}t_{12}) + M^{*-1}(t_{12}a_{12} + a_{12}t_{12}) \\ &= M^{*-1}(a_{11}t_{12}). \end{aligned}$$

It follows that $t_{12}c + ct_{12} = a_{11}t_{12}$ which implies $c_{11}t_{12} = a_{11}t_{12}$. Thus, $c_{11} = a_{11}$, by (i) in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$ we have

$$M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}a_{11} + a_{11}t_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}),$$

by Lemma 5. It follows that $t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}} = 2a_{11}t_{\frac{1}{2}\frac{1}{2}}$ which implies $c_{11}t_{\frac{1}{2}\frac{1}{2}} = a_{11}t_{\frac{1}{2}\frac{1}{2}}$. So $c_{11} = a_{11}$, by condition (ii′) of Theorem.

In both cases we have $c_{11} = a_{11}$.

Since $t_{22}c + ct_{22} = a_{12}t_{22} \in \mathfrak{R}_{12}$, then we can conclude that $c_{12}t_{22} = a_{12}t_{22}$. Using (i) (resp., (i′)), in Lemma 2 again, we see that $c_{12} = a_{12}$. Thus, $c = a_{11}+a_{12}$. Therefore, $M(a_{11}+a_{12}) = M(a_{11})+M(a_{12})$.

Similarly, we prove the case $(i = 2, j = 1)$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Similarly, we can get the following result.

Lemma 9. *Let* $a_{ii} \in \mathfrak{R}_{ii}$ $(i, j = 1, 2; i \neq j)$ *and* $a_{ji} \in \mathfrak{R}_{ji}$. *Then*

(i) $M(a_{ii} + a_{ji}) = M(a_{ii}) + M(a_{ji})$;

(ii) $M^{*-1}(a_{ii} + a_{ji}) = M^{*-1}(a_{ii}) + M^{*-1}(a_{ji})$.

Lemma 10. *Let* $a_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$ *and* $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$. *Then*

(i) $M(a_{11} + a_{ij} + a_{22}) = M(a_{11}) + M(a_{ij}) + M(a_{22})$;

(ii) $M^{*-1}(a_{11} + a_{ij} + a_{22}) = M^{*-1}(a_{11}) + M^{*-1}(a_{ij}) + M^{*-1}(a_{22})$.

Proof. (i) For the case $(i = 1, j = 2)$, suppose that $M(c) = M(a_{11}) + M(a_{12}) + M(a_{22})$ for some $c \in \mathfrak{R}$. From Lemma 5, we have

$$M^{*-1}(e_1c + ce_1) = M^{*-1}(2a_{11}) + M^{*-1}(a_{12}) = M^{*-1}(2a_{11} + a_{12}),$$

by Lemma 8, which yields $e_1c + ce_1 = 2a_{11} + a_{12}$. Hence, $c_{11} = a_{11}$, $c_{12} = a_{12}$, $c_{\frac{1}{2}\frac{1}{2}} = 0$ and $c_{21} = 0$. Now, for arbitrary $t_{22} \in \mathfrak{R}_{22}$, by Lemma 5 again, we have

$$\begin{aligned} M^{*-1}(t_{22}c + ct_{22}) &= M^{*-1}(a_{12}t_{22}) + M^{*-1}(t_{22}a_{22} + a_{22}t_{22}) \\ &= M^{*-1}(a_{12}t_{22} + t_{22}a_{22} + a_{22}t_{22}), \end{aligned}$$

by Lemma 9. It follows that $t_{22}(c_{22} - a_{22}) + (c_{22} - a_{22})t_{22} = 0$, which implies $c_{22} = a_{22}$, by (v) (resp., (iv$'$)) of Theorem. Therefore, $c = a_{11} + a_{12} + a_{22}$. Hence, $M(a_{11} + a_{12} + a_{22}) = M(a_{11}) + M(a_{12}) + M(a_{22})$.

Similarly, we prove the case $(i = 2, j = 1)$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 11. *Let* $t_{22} \in \mathfrak{R}_{22}$ *and* $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$*. Then*

(i) $M(a_{12}t_{22} + t_{22}a_{21}) = M(a_{12}t_{22}) + M(t_{22}a_{21})$;

(ii) $M^{*-1}(a_{12}t_{22} + t_{22}a_{21}) = M^{*-1}(a_{12}t_{22}) + M^{*-1}(t_{22}a_{21})$.

Proof. First at all, let us note that

$$2e_1 + a_{12}a_{21} + a_{12} + a_{21} + a_{21}a_{12} = (e_1 + a_{12})(e_1 + a_{21}) + (e_1 + a_{21})(e_1 + a_{12}).$$

Hence,

$$\begin{aligned} &M(2e_1 + a_{12}a_{21} + a_{12} + a_{21} + a_{21}a_{12}) \\ &\quad = M((e_1 + a_{12})(e_1 + a_{21}) + (e_1 + a_{21})(e_1 + a_{12})) \\ &\quad = M((e_1 + a_{12})M^*M^{*-1}(e_1 + a_{21}) + M^*M^{*-1}(e_1 + a_{21})(e_1 + a_{12})) \\ &\quad = M(e_1 + a_{12})M^{*-1}(e_1 + a_{21}) + M^{*-1}(e_1 + a_{21})M(e_1 + a_{12}) \\ &\quad = M(e_1 + a_{12})M^{*-1}(e_1) + M(e_1 + a_{12})M^{*-1}(a_{21}) \\ &\qquad + M^{*-1}(e_1)M(e_1 + a_{12}) + M^{*-1}(a_{21})M(e_1 + a_{12}) \\ &\quad = M((e_1 + a_{12})e_1 + e_1(e_1 + a_{12})) + M((e_1 + a_{12})a_{21} + a_{21}(e_1 + a_{12})) \\ &\quad = M(2e_1 + a_{12}) + M(a_{12}a_{21} + a_{21} + a_{21}a_{12}) \\ &\quad = M(2e_1) + M(a_{12}) + M(a_{12}a_{21}) + M(a_{21}) + M(a_{21}a_{12}), \end{aligned}$$

by (i), in Lemma 8, (i), in Lemma 9 and (i), in Lemma 10. So

$$\begin{aligned} &M(2e_1 + a_{12}a_{21} + a_{12} + a_{21} + a_{21}a_{12}) \\ &\qquad = M(2e_1) + M(a_{12}) + M(a_{12}a_{21}) + M(a_{21}) + M(a_{21}a_{12}). \end{aligned}$$

Now,

$$M^{*-1}(2^2e_1 + 2a_{12}a_{21} + a_{12} + a_{21}) \\ = M^{*-1}(2^2e_1) + M^{*-1}(a_{12}) + M^{*-1}(2a_{12}a_{21}) + M^{*-1}(a_{21}),$$

by Lemma 5, which implies $M(a_{12}t_{22} + t_{22}a_{21}) = M(a_{12}t_{22}) + M(t_{22}a_{21})$, by Corollary 1.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 12. *Let* $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$. *Then*

(i) $M(a_{12} + a_{21}) = M(a_{12}) + M(a_{21})$;

(ii) $M^{*-1}(a_{12} + a_{21}) = M^{*-1}(a_{12}) + M^{*-1}(a_{21})$.

Proof. (i) Suppose that $M(c) = M(a_{12}) + M(a_{21})$ for some $c \in \mathfrak{R}$. By Lemma 5, we have

$$M^{*-1}(e_1c + ce_1) = M^{*-1}(e_1a_{12} + a_{12}e_1) + M^{*-1}(e_1a_{21} + a_{21}e_1)$$

which implies

$$M^{*-1}(2c_{11} + c_{12} + c_{\frac{1}{2}\frac{1}{2}} + c_{21}) = M^{*-1}(a_{12}) + M^{*-1}(a_{21}). \qquad (5)$$

Next, for arbitrary $t_{22} \in \mathfrak{R}_{22}$, by Corollary 1 and Lemma 11, we have

$$M(c_{12}t_{22}+2c_{\frac{1}{2}\frac{1}{2}}t_{22}+t_{22}c_{21}) = M(a_{12}t_{22})+M(t_{22}a_{21}) = M(a_{12}t_{22}+t_{22}a_{21}).$$

It follows that $c_{12}t_{22} = a_{12}t_{22}$, $c_{\frac{1}{2}\frac{1}{2}}t_{22} = 0$ and $t_{22}c_{21} = t_{22}a_{21}$. So $c_{12} = a_{12}$, $c_{\frac{1}{2}\frac{1}{2}} = 0$ and $c_{21} = a_{21}$, by (i) (resp., (i′)) in Lemma 2 and (iv) (resp., (iii′)) in Theorem.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, by (5) and Lemma 7, we have

$$M(2c_{11}t_{12} + t_{12}c_{21} + c_{21}t_{12}) = M(t_{12}a_{21} + a_{21}t_{12})$$

which implies $c_{11}t_{12} = 0$. So $c_{11} = 0$, by (i) in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$, by (5) and Lemma 7, we have $M(2^2c_{11}t_{\frac{1}{2}\frac{1}{2}}) = 0$ which implies $c_{11}t_{\frac{1}{2}\frac{1}{2}} = 0$. So $c_{11} = 0$, by (ii′) in Theorem.

In both cases we have $c_{11} = a_{11}$.

Finally, for arbitrary $t_{22} \in \mathfrak{R}_{22}$, we have

$$M^{*-1}(t_{22}c+ct_{22}) = M^{*-1}(a_{12}t_{22})+M^{*-1}(t_{22}a_{21}) = M^{*-1}(a_{12}t_{22}+t_{22}a_{21}),$$

by Lemma 11. It follows that $t_{22}c + ct_{22} = a_{12}t_{22} + t_{22}a_{21}$ which implies $c_{12}t_{22} = a_{12}t_{22}$, $t_{22}c_{21} = t_{22}a_{21}$ and $c_{22}t_{22} + t_{22}t_{22} = 0$. So $c_{22} = 0$, by (v) (resp., (iv′)) of Theorem.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 13. *Let* $a_{11} \in \mathfrak{R}_{11}$ *and* $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$. *Then*

(i) $M(a_{11} + a_{12} + a_{21}) = M(a_{11}) + M(a_{12}) + M(a_{21})$;

(ii) $M^{*-1}(a_{11} + a_{12} + a_{21}) = M^{*-1}(a_{11}) + M^{*-1}(a_{12}) + M^{*-1}(a_{21})$.

Proof. (i) Suppose that $M(c) = M(a_{11}) + M(a_{12}) + M(a_{21})$ for some $c \in \mathfrak{R}$. For arbitrary $t_{22} \in \mathfrak{R}_{22}$, we have

$$M^{*-1}(t_{22}c+ct_{22})=M^{*-1}(a_{12}t_{22})+M^{*-1}(t_{22}a_{21})=M^{*-1}(a_{12}t_{22}+t_{22}a_{21}),$$

by Lemmas 5 and 12. Hence, $t_{22}c + ct_{22} = a_{12}t_{22} + t_{22}a_{21}$ which implies $c_{12}t_{22} = a_{12}t_{22}$, $c_{\frac{1}{2}\frac{1}{2}}t_{22} = 0$, $t_{22}c_{21} = t_{22}a_{21}$ and $c_{22}t_{22} + t_{22}t_{22} = 0$. So $c_{12} = a_{12}$, $c_{\frac{1}{2}\frac{1}{2}} = 0$, $c_{21} = a_{21}$ and $c_{22} = 0$, by (i) (resp., (i′)) in Lemma 2, (iv) (resp., (iii′)) and (v) (resp., (iv′)) in Theorem, respectively.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, by Lemmas 5 and 10, we have

$$\begin{aligned} M^{*-1}(t_{12}c + ct_{12}) &= M^{*-1}(a_{11}t_{12}) + M^{*-1}(t_{12}a_{21} + a_{21}t_{12}) \\ &= M^{*-1}(a_{11}t_{12} + t_{12}a_{21} + a_{21}t_{12}). \end{aligned}$$

It follows that $t_{12}c + ct_{12} = a_{11}t_{12} + t_{12}a_{21} + a_{21}t_{12}$ which implies $c_{11}t_{12} = a_{11}t_{12}$. Thus, $c_{11} = a_{11}$, by (i) in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$, we have

$$M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}).$$

It follows that $t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}} = 2a_{11}t_{\frac{1}{2}\frac{1}{2}}$ which implies $c_{11}t_{\frac{1}{2}\frac{1}{2}} = a_{11}t_{\frac{1}{2}\frac{1}{2}}$. Thus, $c_{11} = a_{11}$, by (ii′) in Theorem.

In both cases we have $c_{11} = a_{11}$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 14. *Let* $a_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{ii} + a_{ij} + a_{\frac{1}{2}\frac{1}{2}}) = M(a_{ii}) + M(a_{ij} + a_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(a_{ii} + a_{ij} + a_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(a_{ii}) + M^{*-1}(a_{ij} + a_{\frac{1}{2}\frac{1}{2}})$.

Proof. (i) For the case $(i = 1, j = 2)$, suppose that $M(c) = M(a_{11}) + M(a_{12} + a_{\frac{1}{2}\frac{1}{2}})$ for some $c \in \mathfrak{R}$. For arbitrary $t_{22} \in \mathfrak{R}_{22}$, by Lemma 5, we have

$$M^{*-1}(t_{22}c + ct_{22}) = M^{*-1}(t_{22}a_{11} + a_{11}t_{22}) \\ + M^{*-1}(t_{22}(a_{12} + a_{\frac{1}{2}\frac{1}{2}}) + (a_{12} + a_{\frac{1}{2}\frac{1}{2}})t_{22})$$

which implies

$$M^{*-1}(t_{22}c + ct_{22}) = M^{*-1}(a_{12}t_{22} + 2a_{\frac{1}{2}\frac{1}{2}}t_{22}).$$

It follows that $t_{22}c + ct_{22} = a_{11}t_{22} + 2a_{\frac{1}{2}\frac{1}{2}}t_{22}$ which implies $c_{12}t_{22} = a_{12}t_{22}$, $c_{\frac{1}{2}\frac{1}{2}}t_{22} = a_{\frac{1}{2}\frac{1}{2}}t_{22}$, $t_{22}c_{21} = 0$ and $c_{22}t_{22} + t_{22}c_{22} = 0$. By (i) (resp., (i′)), in Lemma 2, (iv) (resp., (iii′)) and (v) (resp., (iv′)), in Theorem, we have $c_{12} = a_{12}$, $c_{\frac{1}{2}\frac{1}{2}} = a_{\frac{1}{2}\frac{1}{2}}$, $c_{21} = 0$ and $c_{22} = 0$.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, by Lemma 5, we have

$$M^{*-1}(t_{12}c + ct_{12}) = M^{*-1}(a_{11}t_{12}).$$

It follows that $t_{12}c + ct_{12} = a_{11}t_{12}$ which implies $c_{11}t_{12} = a_{11}t_{12}$. Thus, $c_{11} = a_{11}$, by (i) in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$ we have

$$M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})$$

and

$$M\big(e_1(t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}}) + (t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}})e_1\big) = M(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M\big(2(a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}\big),$$

by Lemma 5 and Corollary 1, which implies

$$M\big(2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2^2(c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}\big) = M(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M\big(2(a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}\big).$$

Hence, we have

$$M^{*-1}(2^2(c_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}) = M^{*-1}(2^2(a_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}),$$

for arbitrary $u_{22} \in \mathfrak{R}_{22}$, which implies $(c_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22} = (a_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}$. Thus $c_{11} = a_{11}$, by (ii′) and (iii′) of Theorem.

In both cases we have $c_{11} = a_{11}$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Similarly, we can get the following result.

Lemma 15. *Let* $a_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{ii} + a_{\frac{1}{2}\frac{1}{2}} + a_{ji}) = M(a_{ii}) + M(a_{\frac{1}{2}\frac{1}{2}} + a_{ji})$;

(ii) $M^{*-1}(a_{ii} + a_{\frac{1}{2}\frac{1}{2}} + a_{ji}) = M^{*-1}(a_{ii}) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}} + a_{ji})$.

Lemma 16. *Let* $t_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{ij} + 2t_{ii}a_{\frac{1}{2}\frac{1}{2}}) = M(a_{ij}) + M(2t_{ii}a_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(a_{ij} + 2t_{ii}a_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(a_{ij}) + M^{*-1}(2t_{ii}a_{\frac{1}{2}\frac{1}{2}})$.

Proof. For the case $(i = 1, j = 2)$, note that for arbitrary $t_{11} \in \mathfrak{R}_{11}$

$$a_{12} + 2t_{11} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}} = (e_1 + a_{\frac{1}{2}\frac{1}{2}})(a_{12} + t_{11}) + (a_{12} + t_{11})(e_1 + a_{\frac{1}{2}\frac{1}{2}}).$$

Hence,

$$\begin{aligned}
M(2t_{11}) + M(a_{12} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}}) &= M(2t_{11} + a_{12} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}}) \\
&= M((e_1 + a_{\frac{1}{2}\frac{1}{2}})(a_{12} + t_{11}) + (a_{12} + t_{11})(e_1 + a_{\frac{1}{2}\frac{1}{2}})) \\
&= M((e_1 + a_{\frac{1}{2}\frac{1}{2}})M^*M^{*-1}(a_{12} + t_{11}) \\
&\quad + M^*M^{*-1}(a_{12} + t_{11})(e_1 + a_{\frac{1}{2}\frac{1}{2}})) \\
&= M(e_1 + a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(a_{12} + t_{11}) + M^{*-1}(a_{12} + t_{11})M(e_1 + a_{\frac{1}{2}\frac{1}{2}}) \\
&= M(e_1 + a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(a_{12}) + M(e_1 + a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(t_{11}) \\
&\quad + M^{*-1}(a_{12})M(e_1 + a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(t_{11})M(e_1 + a_{\frac{1}{2}\frac{1}{2}}) \\
&= M((e_1 + a_{\frac{1}{2}\frac{1}{2}})a_{12} + a_{12}(e_1 + a_{\frac{1}{2}\frac{1}{2}})) \\
&\quad + M((e_1 + a_{\frac{1}{2}\frac{1}{2}})t_{11} + t_{11}(e_1 + a_{\frac{1}{2}\frac{1}{2}})) \\
&= M(a_{12}) + M(2t_{11} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}}) \\
&= M(a_{12}) + M(2t_{11}) + M(2t_{11}a_{\frac{1}{2}\frac{1}{2}}),
\end{aligned}$$

by (i) in Lemma 14. So $M(a_{12} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}}) = M(a_{12}) + M(2t_{11}a_{\frac{1}{2}\frac{1}{2}})$. Similarly, we prove the case $(i = 2, j = 1)$, from the identity

$$a_{21} + 2t_{22}a_{\frac{1}{2}\frac{1}{2}} = (e_1 + a_{\frac{1}{2}\frac{1}{2}})(a_{21} + t_{22}) + (a_{21} + t_{22})(e_1 + a_{\frac{1}{2}\frac{1}{2}}).$$

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 17. *Let* $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$, $t_{jj} \in \mathfrak{R}_{jj}$ $(j = 1, 2)$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{ij} + 2t_{jj}a_{\frac{1}{2}\frac{1}{2}}) = M(a_{ij}) + M(2t_{jj}a_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(a_{ij} + 2t_{jj}a_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(a_{ij}) + M^{*-1}(2t_{jj}a_{\frac{1}{2}\frac{1}{2}})$.

Proof. For the case $(i = 1, j = 2)$, note that for arbitrary $t_{22} \in \mathfrak{R}_{22}$

$$a_{12} + 2t_{22}a_{\frac{1}{2}\frac{1}{2}} = (e_1 + a_{\frac{1}{2}\frac{1}{2}})(a_{12} + t_{22}) + (a_{12} + t_{22})(e_1 + a_{\frac{1}{2}\frac{1}{2}}).$$

Hence,

$$\begin{aligned}
&M(a_{12} + 2t_{22}a_{\frac{1}{2}\frac{1}{2}}) \\
&\quad = M((e_1 + a_{\frac{1}{2}\frac{1}{2}})(a_{12} + t_{22}) + (a_{12} + t_{22})(e_1 + a_{\frac{1}{2}\frac{1}{2}})) \\
&\quad = M((e_1 + a_{\frac{1}{2}\frac{1}{2}})M^*M^{*-1}(a_{12} + t_{22}) \\
&\qquad + M^*M^{*-1}(a_{12} + t_{22})(e_1 + a_{\frac{1}{2}\frac{1}{2}})) \\
&\quad = M(e_1 + a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(a_{12} + t_{22}) + M^{*-1}(a_{12} + t_{22})M(e_1 + a_{\frac{1}{2}\frac{1}{2}}) \\
&\quad = M(e_1 + a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(a_{12}) + M(e_1 + a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(t_{22}) \\
&\qquad + M^{*-1}(a_{12})M(e_1 + a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(t_{22})M(e_1 + a_{\frac{1}{2}\frac{1}{2}}) \\
&\quad = M((e_1 + a_{\frac{1}{2}\frac{1}{2}})a_{12} + a_{12}(e_1 + a_{\frac{1}{2}\frac{1}{2}})) \\
&\qquad + M((e_1 + a_{\frac{1}{2}\frac{1}{2}})t_{22} + t_{22}(e_1 + a_{\frac{1}{2}\frac{1}{2}})) \\
&\quad = M(a_{12}) + M(2t_{22}a_{\frac{1}{2}\frac{1}{2}}),
\end{aligned}$$

by (ii) in Lemma 9. So $M(a_{12} + 2t_{22}a_{\frac{1}{2}\frac{1}{2}}) = M(a_{12}) + M(2t_{22}a_{\frac{1}{2}\frac{1}{2}})$. Similarly, we prove the case $(i = 2, j = 1)$ from the identity

$$a_{21} + 2t_{11} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}} = (e_1 + a_{\frac{1}{2}\frac{1}{2}})(a_{21} + t_{11}) + (a_{21} + t_{11})(e_1 + a_{\frac{1}{2}\frac{1}{2}}).$$

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 18. *Let* $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{ij} + a_{\frac{1}{2}\frac{1}{2}}) = M(a_{ij}) + M(a_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(a_{ij} + a_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(a_{ij}) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}})$.

Proof. For the case $(i=1, j=2)$, suppose that $M(c) = M(a_{12}) + M(a_{\frac{1}{2}\frac{1}{2}})$ for some $c \in \mathfrak{R}$. For arbitrary $t_{11} \in \mathfrak{R}_{11}$, by Lemma 16, we have

$$\begin{aligned} M^{*-1}(t_{11}c + ct_{11}) &= M^{*-1}(t_{11}a_{12} + a_{12}t_{11}) + M^{*-1}(t_{11}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{11}) \\ &= M^{*-1}(t_{11}a_{12}) + M^{*-1}(2t_{11}a_{\frac{1}{2}\frac{1}{2}}) \\ &= M^{*-1}(t_{11}a_{12} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}}). \end{aligned}$$

Hence, $t_{11}c + ct_{11} = t_{11}a_{12} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}}$ which implies $t_{11}c_{11} + c_{11}t_{11} = 0$, $t_{11}c_{12} = t_{11}a_{12}$, $t_{11}c_{\frac{1}{2}\frac{1}{2}} = t_{11}a_{\frac{1}{2}\frac{1}{2}}$ and $c_{21}t_{11} = 0$. So $c_{11} = 0$, $c_{12} = a_{12}$, $c_{\frac{1}{2}\frac{1}{2}} = a_{\frac{1}{2}\frac{1}{2}}$ and $c_{21} = 0$. Finally, for arbitrary $t_{22} \in \mathfrak{R}_{22}$, we have

$$\begin{aligned} M^{*-1}(t_{22}c + ct_{22}) &= M^{*-1}(t_{22}a_{12} + a_{12}t_{22}) + M^{*-1}(t_{22}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{22}) \\ &= M^{*-1}(a_{12}t_{22}) + M^{*-1}(2t_{22}a_{\frac{1}{2}\frac{1}{2}}) \\ &= M^{*-1}(a_{12}t_{22} + 2t_{22}a_{\frac{1}{2}\frac{1}{2}}), \end{aligned}$$

by Lemma 17. Hence, $t_{22}c + ct_{22} = a_{12}t_{22} + 2t_{22}a_{\frac{1}{2}\frac{1}{2}}$ which implies $t_{22}c_{22} + c_{22}t_{22} = 0$. So $c_{22} = 0$, by (v) (resp., (iv′)) of Theorem.

Similarly, we prove the case $(i=2, j=1)$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 19. *Let* $a_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{ii} + a_{\frac{1}{2}\frac{1}{2}}) = M(a_{ii}) + M(a_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(a_{ii} + a_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(a_{ii}) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}})$.

Proof. (i) For the case $(i=1)$, suppose that $M(c) = M(a_{11}) + M(a_{\frac{1}{2}\frac{1}{2}})$ for some $c \in \mathfrak{R}$. For arbitrary $t_{22} \in \mathfrak{R}_{22}$, by Lemma 5, we have

$$\begin{aligned} M^{*-1}(t_{22}c + ct_{22}) &= M^{*-1}(t_{22}a_{11} + a_{11}t_{22}) + M^{*-1}(t_{22}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{22}) \\ &= M^{*-1}(2t_{22}a_{\frac{1}{2}\frac{1}{2}}). \end{aligned}$$

It follows that $t_{22}c + ct_{22} = 2t_{22}a_{\frac{1}{2}\frac{1}{2}}$ which implies $c_{12}t_{22} = 0$, $t_{22}c_{\frac{1}{2}\frac{1}{2}} = t_{22}a_{\frac{1}{2}\frac{1}{2}}$, $t_{22}c_{21} = 0$ and $t_{22}c_{22} + c_{22}t_{22} = 0$. By (i) (resp., (i')) in Lemma 2, and (iv) and (v) (resp., (iii′) and (iv′)) of Theorem, we have $c_{12} = 0$, $c_{\frac{1}{2}\frac{1}{2}} = a_{\frac{1}{2}\frac{1}{2}}$, $c_{21} = 0$ and $c_{22} = 0$.

Next, if $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, by Lemma 5, we have

$$\begin{aligned} M^{*-1}(t_{12}c + ct_{12}) &= M^{*-1}(t_{12}a_{11} + a_{11}t_{12}) + M^{*-1}(t_{12}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{12}) \\ &= M^{*-1}(a_{11}t_{12}). \end{aligned}$$

It follows that $t_{12}c + ct_{12} = a_{11}t_{12}$ which implies $c_{11}t_{12} = a_{11}t_{12}$. So $c_{11} = a_{11}$, by (ii) of Theorem.

If if $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$, we have

$$M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}),$$

by Lemma 5, which implies

$$M^{*-1}(2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}).$$

It follows that

$$M\big(2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2^2(c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}\big) = M(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M\big(2^2(a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}\big),$$

by Corollary 1. Now, for arbitrary $u_{22} \in R_{22}$, we have

$$M^{*-1}\big(2^2(c_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}\big) = M^{*-1}\big(2^2(a_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}\big),$$

by Lemma 5, which implies $\big((c_{11} - a_{11})t_{\frac{1}{2}\frac{1}{2}}\big)u_{22} = 0$. From hypothesis (ii′) and (iii′), of Theorem, we have $c_{11} = a_{11}$.

In both cases we conclude that $c_{11} = a_{11}$. Hence,

$$M(a_{11} + a_{\frac{1}{2}\frac{1}{2}}) = M(a_{11}) + M(a_{\frac{1}{2}\frac{1}{2}}).$$

Similarly, we prove the case $(i = 2)$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 20. *Let* $a_{11} \in \mathfrak{R}_{11}$ *and* $a_{22} \in \mathfrak{R}_{22}$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{11} + a_{\frac{1}{2}\frac{1}{2}} + a_{22}) = M(a_{11}) + M(a_{\frac{1}{2}\frac{1}{2}}) + M(a_{22})$;

(ii) $M^{*-1}(a_{11} + a_{\frac{1}{2}\frac{1}{2}} + a_{22}) = M^{*-1}(a_{11}) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(a_{22})$.

Proof. (i) Suppose that $M(c) = M(a_{11}) + M(a_{\frac{1}{2}\frac{1}{2}}) + M(a_{22})$ for some $c \in \mathfrak{R}$. For arbitrary $t_{11} \in \mathfrak{R}_{11}$, by Lemma 19, we have

$$\begin{aligned} M^{*-1}(t_{11}c + ct_{11}) &= M^{*-1}(t_{11}a_{11} + a_{11}t_{11}) + M^{*-1}(t_{11}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{11}) \\ &\quad + M^{*-1}(t_{11}a_{22} + a_{22}t_{11}) \\ &= M^{*-1}(t_{11}a_{11} + a_{11}t_{11}) + M^{*-1}(2t_{11}a_{\frac{1}{2}\frac{1}{2}}) \\ &= M^{*-1}(t_{11}a_{11} + a_{11}t_{11} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}}). \end{aligned}$$

It follows that $t_{11}c + ct_{11} = t_{11}a_{11} + a_{11}t_{11} + 2t_{11}a_{\frac{1}{2}\frac{1}{2}}$ which implies $t_{11}c_{11}+c_{11}t_{11} = t_{11}a_{11}+a_{11}t_{11}, t_{11}c_{12} = 0, t_{11}c_{\frac{1}{2}\frac{1}{2}} = t_{11}a_{\frac{1}{2}\frac{1}{2}}$ and $c_{21}t_{11} = 0$. Thus, $c_{11} = a_{11}$, $c_{12} = 0$, $c_{\frac{1}{2}\frac{1}{2}} = a_{\frac{1}{2}\frac{1}{2}}$ and $c_{21} = 0$. Next, for arbitrary $t_{22} \in \mathfrak{R}_{22}$, we have

$$\begin{aligned} M^{*-1}(t_{22}c + ct_{22}) &= M^{*-1}(t_{22}a_{11} + a_{11}t_{22}) + M^{*-1}(t_{22}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{22}) \\ &\quad + M^{*-1}(t_{22}a_{22} + a_{22}t_{22}) \\ &= M^{*-1}(2t_{22}a_{\frac{1}{2}\frac{1}{2}} + t_{22}a_{22} + a_{22}t_{22}), \end{aligned}$$

by Lemma 19. It follows that $t_{22}c+ct_{22} = 2t_{22}a_{\frac{1}{2}\frac{1}{2}} + t_{22}a_{22} + a_{22}t_{22}$ which implies $t_{22}c_{22} + c_{22}t_{22} = t_{22}a_{22} + a_{22}t_{22}$. Thus $c_{22} = a_{22}$, by (v) (resp., (iv′)), of the Theorem.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 21. *Let* $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{ii} + a_{ij} + a_{\frac{1}{2}\frac{1}{2}}) = M(a_{ii}) + M(a_{ij}) + M(a_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(a_{ii} + a_{ij} + a_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(a_{ii}) + M^{*-1}(a_{ij}) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}})$.

Proof. (i) For the case $(i = 1, j = 2)$, suppose that $M(c) = M(a_{11}) + M(a_{12}) + M(a_{\frac{1}{2}\frac{1}{2}})$ for some $c \in \mathfrak{R}$. For arbitrary $t_{22} \in \mathfrak{R}_{22}$, by Lemmas 5 and 18, we have

$$\begin{aligned} M^{*-1}(t_{22}c + ct_{22}) &= M^{*-1}(t_{22}a_{11} + a_{11}t_{22}) + M^{*-1}(t_{22}a_{12} + a_{12}t_{22}) \\ &\quad + M^{*-1}(t_{22}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{22}) \\ &= M^{*-1}(a_{12}t_{22}) + M^{*-1}(2t_{22}a_{\frac{1}{2}\frac{1}{2}}) \\ &= M^{*-1}(a_{12}t_{22} + 2t_{22}a_{\frac{1}{2}\frac{1}{2}}). \end{aligned}$$

Hence $t_{22}c+ct_{22} = a_{12}t_{22}+2t_{22}a_{\frac{1}{2}\frac{1}{2}}$ which implies $c_{12}t_{22} = a_{12}t_{22}, t_{22}c_{\frac{1}{2}\frac{1}{2}} = t_{22}a_{\frac{1}{2}\frac{1}{2}}$, $t_{22}c_{21} = 0$ and $t_{22}c_{22} + c_{22}t_{22} = 0$. By (i) (resp., (i′)) in Lemma 2 and (iv) and (v) (resp., (iii′) and (iv′)), of Theorem, we have $c_{12} = a_{12}$, $c_{\frac{1}{2}\frac{1}{2}} = a_{\frac{1}{2}\frac{1}{2}}$, $c_{21} = 0$ and $c_{22} = 0$.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(vi), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, we have

$$M^{*-1}(t_{12}c + ct_{12}) = M^{*-1}(t_{12}a_{11} + a_{11}t_{12}) + M^{*-1}(t_{12}a_{12} + a_{12}t_{12}) + M^{*-1}(t_{12}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{12}) = M^{*-1}(a_{11}t_{12}),$$

by Lemma 5. Hence, $t_{12}c + ct_{12} = a_{11}t_{12}$ which implies $c_{11}t_{12} = a_{11}t_{12}$, So $c_{11} = a_{11}$, by (i) in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then by Lemma 5, we have

$$M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})$$

implying

$$M^{*-1}(2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}).$$

Hence,

$$M(2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2^2(c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}) = M(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M(2^2(a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}),$$

by Corollary 1. Thus, for arbitrary $u_{22} \in R_{22}$, we have

$$M^{*-1}(2^2(c_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}) = M^{*-1}(2^2(a_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}),$$

by Lemma 5, which implies $\big((c_{11} - a_{11})t_{\frac{1}{2}\frac{1}{2}}\big)u_{22} = 0$. From hypothesis (ii′) and (iii′), of Theorem, we have $c_{11} = a_{11}$.

In both cases we conclude that $c_{11} = a_{11}$. Hence,

$$M(a_{11} + a_{12} + a_{\frac{1}{2}\frac{1}{2}}) = M(a_{11}) + M(a_{12}) + M(a_{\frac{1}{2}\frac{1}{2}}).$$

Similarly, we prove the case $(i = 2, j = 1)$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Similarly, we can get the following result.

Lemma 22. *Let* $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$ *and* $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{ii}+a_{ji}+a_{\frac{1}{2}\frac{1}{2}})=M(a_{ii})+M(a_{ji})+M(a_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(a_{ii}+a_{ji}+a_{\frac{1}{2}\frac{1}{2}})=M^{*-1}(a_{ii})+M^{*-1}(a_{ji})+M^{*-1}(a_{\frac{1}{2}\frac{1}{2}})$.

Lemma 23. *Let* $a_{ij}\in\mathfrak{R}_{ij}$ $(i,j=1,2;i\neq j)$ *and* $b_{\frac{1}{2}\frac{1}{2}}\in\mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{11}+a_{ij}+a_{\frac{1}{2}\frac{1}{2}}+a_{22})=M(a_{11})+M(a_{ij})+M(a_{\frac{1}{2}\frac{1}{2}})+M(a_{22})$;

(ii) $M^{*-1}(a_{11}+a_{ij}+a_{\frac{1}{2}\frac{1}{2}}+a_{22})=M^{*-1}(a_{11})+M^{*-1}(a_{ij})+M^{*-1}(a_{\frac{1}{2}\frac{1}{2}})+M^{*-1}(a_{22})$.

Proof. (i) For the case $(i=1,j=2)$, suppose that $M(c)=M(a_{11})+M(a_{12})+M(b_{\frac{1}{2}\frac{1}{2}})+M(a_{22})$ for some $c\in\mathfrak{R}$. For arbitrary $t_{11}\in\mathfrak{R}_{11}$, we have

$$\begin{aligned}M^{*-1}(t_{11}c+ct_{11})&=M^{*-1}(t_{11}a_{11}+a_{11}t_{11})+M^{*-1}(t_{11}a_{12}+a_{12}t_{11})\\&\quad+M^{*-1}(t_{11}b_{\frac{1}{2}\frac{1}{2}}+a_{\frac{1}{2}\frac{1}{2}}t_{11})+M^{*-1}(t_{11}a_{22}+a_{22}t_{11})\\&=M^{*-1}(t_{11}a_{11}+a_{11}t_{11}+t_{11}a_{12}+2t_{11}a_{\frac{1}{2}\frac{1}{2}}),\end{aligned}$$

by Lemma 21. Hence, $t_{11}c+ct_{11}=t_{11}a_{11}+a_{11}t_{11}+t_{11}a_{12}+2t_{11}a_{\frac{1}{2}\frac{1}{2}}$ which implies $t_{11}c_{11}+c_{11}t_{11}=t_{11}a_{11}+a_{11}t_{11}$, $t_{11}c_{12}=t_{11}a_{12}$, $t_{11}c_{\frac{1}{2}\frac{1}{2}}=t_{11}a_{\frac{1}{2}\frac{1}{2}}$ and $c_{21}t_{11}=0$. So $c_{11}=a_{11}$, $c_{12}=a_{12}$, $c_{\frac{1}{2}\frac{1}{2}}=a_{\frac{1}{2}\frac{1}{2}}$ and $c_{21}=0$. Now, for arbitrary $t_{22}\in\mathfrak{R}_{22}$, by Lemma 5 again, we have

$$\begin{aligned}M^{*-1}(t_{22}c+ct_{22})&=M^{*-1}(t_{22}a_{11}+a_{11}t_{22})+M^{*-1}(t_{22}a_{12}+a_{12}t_{22})\\&\quad+M^{*-1}(t_{22}b_{\frac{1}{2}\frac{1}{2}}+a_{\frac{1}{2}\frac{1}{2}}t_{22})+M^{*-1}(t_{22}a_{22}+a_{22}t_{22})\\&=M^{*-1}(a_{12}t_{22}+2t_{22}a_{\frac{1}{2}\frac{1}{2}}+t_{22}a_{22}+a_{22}t_{22}),\end{aligned}$$

by Lemma 22. Hence, $t_{22}c+ct_{22}=a_{12}t_{22}+2t_{22}a_{\frac{1}{2}\frac{1}{2}}+t_{22}a_{22}+a_{22}t_{22}$ which implies $t_{22}c_{22}+c_{22}t_{22}=t_{22}a_{22}+a_{22}t_{22}$. So $c_{22}=a_{22}$, by (v) (resp., (iv′)), of Theorem.

Similarly, we prove the case $(i=2,j=1)$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 24. *Let* $a_{ij}\in\mathfrak{R}_{ij}$ $(i,j=1,2;i\neq j)$, $t_{22}\in\mathfrak{R}_{22}$ *and* $a_{\frac{1}{2}\frac{1}{2}}\in\mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. *Then*

(i) $M(a_{12}t_{22}+2a_{\frac{1}{2}\frac{1}{2}}t_{22}+t_{22}a_{21})=M(a_{12}t_{22})+M(2a_{\frac{1}{2}\frac{1}{2}}t_{22})+M(t_{22}a_{21})$;

(ii) $M^{*-1}(a_{12}t_{22}+2a_{\frac{1}{2}\frac{1}{2}}t_{22}+t_{22}a_{21})=M^{*-1}(a_{12}t_{22})+M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{22})+M^{*-1}(t_{22}a_{21})$.

Proof. First at all, let us note that

$$(e_1+a_{12}+a_{21})(e_1+a_{\frac{1}{2}\frac{1}{2}})+(e_1+a_{\frac{1}{2}\frac{1}{2}})(e_1+a_{12}+a_{21}) = 2e_1+a_{12}+a_{\frac{1}{2}\frac{1}{2}}+a_{21}.$$

Hence,

$$\begin{aligned}
M^{*-1}&(2e_1 + a_{12} + a_{\frac{1}{2}\frac{1}{2}} + a_{21}) \\
&= M^{*-1}\big((e_1 + a_{12} + a_{21})(e_1 + a_{\frac{1}{2}\frac{1}{2}}) + (e_1 + a_{\frac{1}{2}\frac{1}{2}})(e_1 + a_{12} + a_{21})\big) \\
&= M^{*-1}\big((e_1 + a_{12} + a_{21})M^{-1}M(e_1 + a_{\frac{1}{2}\frac{1}{2}}) \\
&\quad + M^{-1}M(e_1 + a_{\frac{1}{2}\frac{1}{2}})(e_1 + a_{12} + a_{21})\big) \\
&= M^{*-1}(e_1 + a_{12} + a_{21})M(e_1 + a_{\frac{1}{2}\frac{1}{2}}) \\
&\quad + M(e_1 + a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(e_1 + a_{12} + a_{21}) \\
&= \big(M^{*-1}(e_1) + M^{*-1}(a_{12}) + M^{*-1}(a_{21})\big)\big(M(e_1) + (M(a_{\frac{1}{2}\frac{1}{2}})\big) \\
&\quad + \big(M(e_1) + M(a_{\frac{1}{2}\frac{1}{2}})\big)\big(M^{*-1}(e_1) + M^{*-1}(a_{12}) + M^{*-1}(a_{21})\big) \\
&= \big(M^{*-1}(e_1)M(e_1) + M(e_1)M^{*-1}(e_1)\big) \\
&\quad + \big(M^{*-1}(e_1)M(a_{\frac{1}{2}\frac{1}{2}}) + M(a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(e_1)\big) \\
&\quad + \big(M^{*-1}(a_{12})M(e_1) + M(e_1)M^{*-1}(a_{12})\big) \\
&\quad + \big(M^{*-1}(a_{12})M(a_{\frac{1}{2}\frac{1}{2}}) + M(a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(a_{12})\big) \\
&\quad + \big(M^{*-1}(a_{21})M(e_1) + M(e_1)M^{*-1}(a_{21})\big) \\
&\quad + \big(M^{*-1}(a_{21})M(a_{\frac{1}{2}\frac{1}{2}} + M(a_{\frac{1}{2}\frac{1}{2}})M^{*-1}(a_{21})\big) \\
&= M^{*-1}\big(e_1M^{-1}M(e_1) + M^{-1}M(e_1)e_1\big) \\
&\quad + M^{*-1}\big(e_1M^{-1}M(a_{\frac{1}{2}\frac{1}{2}}) + M^{-1}M(a_{\frac{1}{2}\frac{1}{2}})e_1\big) \\
&\quad + M^{*-1}\big(a_{12}M^{-1}M(e_1) + M^{-1}M(e_1)a_{12}\big) \\
&\quad + M^{*-1}\big(a_{12}M^{-1}M(a_{\frac{1}{2}\frac{1}{2}}) + M^{-1}M(a_{\frac{1}{2}\frac{1}{2}})a_{12}\big) \\
&\quad + M^{*-1}\big(a_{21}M^{-1}M(e_1) + M^{-1}M(e_1)a_{21}\big) \\
&\quad + M^{*-1}\big(a_{21}M^{-1}M(a_{\frac{1}{2}\frac{1}{2}}) + M^{-1}M(a_{\frac{1}{2}\frac{1}{2}})a_{21}\big) \\
&= M^{*-1}(e_1e_1 + e_1e_1) + M^{*-1}(e_1a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}e_1) \\
&\quad + M^{*-1}(a_{12}e_1 + e_1a_{12}) + M^{*-1}(a_{12}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}a_{12}) \\
&\quad + M^{*-1}(a_{21}e_1 + e_1a_{21}) + M^{*-1}(a_{21}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}a_{21}) \\
&= M^{*-1}(2e_1) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(a_{12}) + M^{*-1}(a_{21}).
\end{aligned}$$

This implies that

$$M^{*-1}(2e_1 + a_{12} + a_{\frac{1}{2}\frac{1}{2}} + a_{21}) \\ = M^{*-1}(2e_1) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(a_{12}) + M^{*-1}(a_{21}).$$

Thus, for an arbitrary element $t_{22} \in \mathfrak{R}_{22}$, we have

$$M(a_{12}t_{22} + 2a_{\frac{1}{2}\frac{1}{2}}t_{22} + t_{22}a_{21}) = M(a_{12}t_{22}) + M(2a_{\frac{1}{2}\frac{1}{2}}t_{22}) + M(t_{22}a_{21}),$$

by Corollary 1.

(ii) By Corollary 2 and using the same identity in the precedent case, we prove that (ii) holds. □

Lemma 25. *Let $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$ and $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$. Then*

(i) $M(a_{ii} + a_{ij} + a_{\frac{1}{2}\frac{1}{2}} + a_{ji}) = M(a_{ii}) + M(a_{ij}) + M(a_{\frac{1}{2}\frac{1}{2}}) + M(a_{ji})$;

(ii) $M^{*-1}(a_{ii}+a_{ij}+a_{\frac{1}{2}\frac{1}{2}}+a_{ji}) = M^{*-1}(a_{ii})+M^{*-1}(a_{ij})+M^{*-1}(a_{\frac{1}{2}\frac{1}{2}})+ M^{*-1}(a_{ji})$.

Proof. (i) For the case $(i = 1, j = 2)$, suppose that $M(c) = M(a_{11}) + M(a_{12}) + M(a_{\frac{1}{2}\frac{1}{2}}) + M(a_{21})$ for some $c \in \mathfrak{R}$. For arbitrary $t_{22} \in \mathfrak{R}_{12}$, by Lemma 5, we have

$$M^{*-1}(t_{22}c + ct_{22}) = M^{*-1}(t_{22}a_{11} + a_{11}t_{22}) + M^{*-1}(t_{22}a_{12} + a_{12}t_{22}) \\ + M^{*-1}(t_{22}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{22}) + M^{*-1}(t_{22}a_{21} + a_{21}t_{22})$$

implying

$$M^{*-1}(t_{22}c + ct_{22}) = M^{*-1}(a_{12}t_{22} + 2a_{\frac{1}{2}\frac{1}{2}}t_{22} + t_{22}a_{21}),$$

by Lemma 24. Hence, $t_{22}c + ct_{22} = a_{12}t_{22} + 2a_{\frac{1}{2}\frac{1}{2}}t_{22} + t_{22}a_{21}$. It follows that $c_{12}t_{22} = a_{12}t_{22}$, $c_{\frac{1}{2}\frac{1}{2}}t_{22} = a_{\frac{1}{2}\frac{1}{2}}t_{22}$, $t_{22}c_{21} = t_{22}a_{21}$ and $t_{22}c_{22}+c_{22}t_{22} = t_{22}a_{22} + a_{22}t_{22}$. Thus, $c_{12} = a_{12}$, $c_{\frac{1}{2}\frac{1}{2}} = a_{\frac{1}{2}\frac{1}{2}}$, $c_{21} = a_{21}$ and $c_{22} = a_{22}$, by (i) (resp., (i′)), in Lemma 2, and (iv) and (v) (resp., (iii′) and (iv′)), in Theorem.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(vi), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, by Lemmas 5 and 10, we have

$$M^{*-1}(t_{12}c + ct_{12}) = M^{*-1}(a_{11}t_{12}) + M^{*-1}(t_{12}a_{21} + a_{21}t_{12}) \\ = M^{*-1}(a_{11}t_{12} + t_{12}a_{21} + a_{21}t_{12})$$

which implies $c_{11}t_{12} = a_{11}t_{12}$. Thus, $c_{11} = a_{11}$, by (i), in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in R_{\frac{1}{2}\frac{1}{2}}$, we have

$$M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})$$

which implies $M^{*-1}(2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}) = M(2a_{11}t_{\frac{1}{2}\frac{1}{2}} + 2a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})$, by Lemma 20. Hence, $c_{11}t_{\frac{1}{2}\frac{1}{2}} + c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}} = a_{11}t_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}$, which implies $c_{11}t_{\frac{1}{2}\frac{1}{2}} = a_{11}t_{\frac{1}{2}\frac{1}{2}}$. So $c_{11} = a_{11}$, by (ii′), in Theorem.

In both cases we conclude that $c_{11} = a_{11}$. Similarly, we prove the case $(i = 2, j = 1)$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 26. *Let $a_{ij} \in \mathfrak{R}_{ij}$ $(i, j = 1, 2; i \neq j)$ and $a_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$ then*

(i) $M(a_{11} + a_{12} + a_{\frac{1}{2}\frac{1}{2}} + a_{21} + a_{22}) = M(a_{11}) + M(a_{12}) + M(a_{\frac{1}{2}\frac{1}{2}}) + M(a_{21}) + M(a_{22})$;

(ii) $M^{*-1}(a_{11} + a_{12} + a_{\frac{1}{2}\frac{1}{2}} + a_{21} + a_{22}) = M^{*-1}(a_{11}) + M^{*-1}(a_{12}) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(a_{21}) + M^{*-1}(a_{22})$.

Proof. (i) Suppose that $M(c) = M(a_{11}) + M(a_{12}) + M(a_{\frac{1}{2}\frac{1}{2}}) + M(a_{21}) + M(a_{22})$. Then $M^{*-1}(e_1c+ce_1) = M^{*-1}(2a_{11})+M^{*-1}(a_{12})+M^{*-1}(a_{\frac{1}{2}\frac{1}{2}})+M^{*-1}(a_{21}) = M^{*-1}(2a_{11} + a_{12} + a_{\frac{1}{2}\frac{1}{2}} + a_{21})$, by Lemmas 5 and 25. Hence, $2c_{11} + c_{12} + c_{\frac{1}{2}\frac{1}{2}} + c_{21} = 2a_{11} + a_{12} + a_{\frac{1}{2}\frac{1}{2}} + a_{21}$ which implies $c_{11} = a_{11}$, $c_{12} = a_{12}$, $c_{\frac{1}{2}\frac{1}{2}} = a_{\frac{1}{2}\frac{1}{2}}$ and $c_{21} = a_{21}$. Now, for arbitrary $t_{22} \in \mathfrak{R}_{22}$, we have

$$\begin{aligned} M^{*-1}(t_{22}c + ct_{22}) &= M^{*-1}(t_{22}a_{11} + a_{11}t_{22}) + M^{*-1}(t_{22}a_{12} + a_{12}t_{22}) \\ &\quad + M^{*-1}(t_{22}a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}t_{22}) + M^{*-1}(t_{22}a_{21} + a_{21}t_{22}) \\ &\quad + M^{*-1}(t_{22}a_{22} + a_{22}t_{22}) \\ &= M^{*-1}(a_{12}t_{22}) + M^{*-1}(2t_{22}a_{\frac{1}{2}\frac{1}{2}}) \\ &\quad + M^{*-1}(t_{22}a_{21}) + M^{*-1}(t_{22}a_{22} + a_{22}t_{22}) \\ &= M^{*-1}(a_{12}t_{22} + 2t_{22}a_{\frac{1}{2}\frac{1}{2}} + t_{22}a_{21} + t_{22}a_{22} + a_{22}t_{22}), \end{aligned}$$

again by Lemmas 5 and 25 again, which implies $t_{22}c_{22} + c_{22}t_{22} = t_{22}a_{22} + a_{22}t_{22}$. By (v) (resp., (iv′)), in the Theorem, we obtain $c_{22} = a_{22}$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 27. (i) $M(a_{12}+b_{12}c_{22})=M(a_{12})+M(b_{12}c_{22})$;

(ii) $M^{*-1}(a_{12}+b_{12}c_{22})=M^{*-1}(a_{12})+M^{*-1}(b_{12}c_{22})$;

(iii) $M(a_{21}+b_{22}c_{21})=M(a_{21})+M(b_{22}c_{21})$;

(iv) $M^{*-1}(a_{21}+b_{22}c_{21})=M^{*-1}(a_{21})+M^{*-1}(b_{22}c_{21})$.

Proof. First of all, let us note that

$$a_{12}+b_{12}c_{22}=(e_1+b_{12})(a_{12}+c_{22})+(a_{12}+c_{22})(e_1+b_{12}).$$

Hence, $M(a_{12}+b_{12}c_{22})=M((e_1+b_{12})(a_{12}+c_{22})+(a_{12}+c_{22})(e_1+b_{12}))=M((e_1+b_{12})M^*M^{*-1}(a_{12}+c_{22})+M^*M^{*-1}(a_{12}+c_{22})(e_1+b_{12}))=M(e_1+b_{12})M^{*-1}(a_{12}+c_{22})+M^{*-1}(a_{12}+c_{22})M(e_1+b_{12})=M(e_1+b_{12})M^{*-1}(a_{12})+M(e_1+b_{12})M^{*-1}(c_{22})+M^{*-1}(a_{12})M(e_1+b_{12})+M^{*-1}(c_{22})M(e_1+b_{12})=M((e_1+b_{12})a_{12}+a_{12}(e_1+b_{12}))+M((e_1+b_{12})c_{22}+c_{22}(e_1+b_{12}))=M(a_{12})+M(b_{12}c_{22})$. Similarly, we prove $M(a_{21}+b_{22}c_{21})=M(a_{21})+M(b_{22}c_{21})$, from the identity

$$a_{21}+b_{22}c_{21}=(e_1+c_{21})(a_{21}+b_{22})+(a_{21}+b_{22})(e_1+c_{21}).$$

The identities *(ii)* and *(iv)* follow from *(i)* and *(iii)*, respectively, by Lemma 4. □

Lemma 28. *The following are true.*

(i) $M(a_{12}+b_{12})=M(a_{12})+M(b_{12})$;

(ii) $M^{*-1}(a_{12}+b_{12})=M^{*-1}(a_{12})+M^{*-1}(b_{12})$.

Proof. (i) Let us suppose that $c\in\mathfrak{R}$ satisfies $M(c)=M(a_{12})+M(b_{12})$. For any $t_{22}\in\mathfrak{R}_{22}$, we have $M^{*-1}(t_{22}c+ct_{22})=M^{*-1}(t_{22}a_{12}+a_{12}t_{22})+M^{*-1}(t_{22}b_{12}+b_{12}t_{22})=M^{*-1}(a_{12}t_{22})+M^{*-1}(b_{12}t_{22})=M^{*-1}(a_{12}t_{22}+b_{12}t_{22})$, by (ii) in Lemma 27. Therefore we have $t_{22}c+ct_{22}=a_{12}t_{22}+b_{12}t_{22}\in\mathfrak{R}_{12}$. It follows that $c_{\frac{1}{2}\frac{1}{2}}=0$, $c_{21}=0$ and $c_{22}=0$, by (iv) (resp., (iii′)), in Lemma 2. It follows yet, from identity above, that $c_{12}t_{22}=a_{12}t_{22}+b_{12}t_{22}$ which implies $c_{12}=a_{12}+b_{12}$, by (i) (resp., (i′)) in Lemma 2.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then for arbitrary $t_{12}\in\mathfrak{R}_{12}$, we have $M^{*-1}(t_{12}c+ct_{12})=M^{*-1}(t_{12}a_{12}+a_{12}t_{12})+M^{*-1}(t_{12}b_{12}+b_{12}t_{12})=0$, by Lemma 5. Hence, $t_{12}c+ct_{12}=0$ implying $c_{11}t_{12}=0$. So $c_{11}=0$, by *(i)* in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then for arbitrary $t_{\frac{1}{2}\frac{1}{2}}\in\mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$, we have $M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}c+ct_{\frac{1}{2}\frac{1}{2}})=0$. Hence, $t_{\frac{1}{2}\frac{1}{2}}c+ct_{\frac{1}{2}\frac{1}{2}})=0$ which implies $c_{11}t_{\frac{1}{2}\frac{1}{2}}=0$. So $c_{11}=0$, by (ii′), of the Theorem.

In both cases we conclude that $c_{11} = 0$, and therefore $c = a_{12} + b_{12}$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Similarly, we can get the following result.

Lemma 29. *The following hold:*

(i) $M(a_{21} + b_{21}) = M(a_{21}) + M(b_{21})$;

(ii) $M^{*-1}(a_{21} + b_{21}) = M^{*-1}(a_{21}) + M^{*-1}(b_{21})$.

Lemma 30. *For arbitrary* $a_{\frac{1}{2}\frac{1}{2}}, b_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$ *and* $t_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, *the following hold:*

(i) $M(2a_{\frac{1}{2}\frac{1}{2}}t_{jj} + 2^2(t_{ii}b_{\frac{1}{2}\frac{1}{2}})t_{jj}) = M(2a_{\frac{1}{2}\frac{1}{2}}t_{jj}) + M(2^2(t_{ii}b_{\frac{1}{2}\frac{1}{2}})t_{jj})$;

(ii) $M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{jj} + 2^2(t_{ii}b_{\frac{1}{2}\frac{1}{2}})t_{jj}) = M^{*-1}(2a_{\frac{1}{2}\frac{1}{2}}t_{jj}) + M^{*-1}(2^2(t_{ii}b_{\frac{1}{2}\frac{1}{2}})t_{jj})$.

Proof. (i) For the case $(i = 1)$. First of all, note that $a_{\frac{1}{2}\frac{1}{2}} + 2t_{11} + 2t_{11}b_{\frac{1}{2}\frac{1}{2}} + 2a_{\frac{1}{2}\frac{1}{2}}b_{\frac{1}{2}\frac{1}{2}} = (e_1 + b_{\frac{1}{2}\frac{1}{2}})(a_{\frac{1}{2}\frac{1}{2}} + t_{11}) + (a_{\frac{1}{2}\frac{1}{2}} + t_{11})(e_1 + b_{\frac{1}{2}\frac{1}{2}})$. Hence,

$$
\begin{aligned}
&M(a_{\frac{1}{2}\frac{1}{2}} + 2t_{11}b_{\frac{1}{2}\frac{1}{2}} + 2t_{11} + 2a_{\frac{1}{2}\frac{1}{2}}b_{\frac{1}{2}\frac{1}{2}})\\
&\quad = M\big((e_1 + b_{\frac{1}{2}\frac{1}{2}})(a_{\frac{1}{2}\frac{1}{2}} + t_{11}) + (a_{\frac{1}{2}\frac{1}{2}} + t_{11})(e_1 + b_{\frac{1}{2}\frac{1}{2}})\big)\\
&\quad = M\big((e_1 + b_{\frac{1}{2}\frac{1}{2}})M^*M^{*-1}(a_{\frac{1}{2}\frac{1}{2}} + t_{11})\\
&\qquad + M^*M^{*-1}(a_{\frac{1}{2}\frac{1}{2}} + t_{11})(e_1 + b_{\frac{1}{2}\frac{1}{2}})\big)\\
&\quad = M(e_1 + b_{\frac{1}{2}\frac{1}{2}})M^{*-1}(a_{\frac{1}{2}\frac{1}{2}} + t_{11}) + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}} + t_{11})M(e_1 + b_{\frac{1}{2}\frac{1}{2}})\\
&\quad = M(e_1 + b_{\frac{1}{2}\frac{1}{2}})M^{*-1}(a_{\frac{1}{2}\frac{1}{2}}) + M(e_1 + b_{\frac{1}{2}\frac{1}{2}})M^{*-1}(t_{11})\\
&\qquad + M^{*-1}(a_{\frac{1}{2}\frac{1}{2}})M(e_1 + b_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(t_{11})M(e_1 + b_{\frac{1}{2}\frac{1}{2}})\\
&\quad = M\big((e_1 + b_{\frac{1}{2}\frac{1}{2}})a_{\frac{1}{2}\frac{1}{2}} + a_{\frac{1}{2}\frac{1}{2}}(e_1 + b_{\frac{1}{2}\frac{1}{2}})\big)\\
&\qquad + M\big((e_1 + b_{\frac{1}{2}\frac{1}{2}})t_{11} + t_{11}(e_1 + b_{\frac{1}{2}\frac{1}{2}})\big)\\
&\quad = M(a_{\frac{1}{2}\frac{1}{2}} + 2a_{\frac{1}{2}\frac{1}{2}}b_{\frac{1}{2}\frac{1}{2}}) + M(2t_{11} + 2t_{11}b_{\frac{1}{2}\frac{1}{2}})\\
&\quad = M(a_{\frac{1}{2}\frac{1}{2}}) + M(2a_{\frac{1}{2}\frac{1}{2}}b_{\frac{1}{2}\frac{1}{2}}) + M(2t_{11}) + M(2t_{11}b_{\frac{1}{2}\frac{1}{2}}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&M^{*-1}(a_{\frac{1}{2}\frac{1}{2}} + 2t_{11}b_{\frac{1}{2}\frac{1}{2}} + 2^2t_{11} + 2^2(a_{\frac{1}{2}\frac{1}{2}}b_{\frac{1}{2}\frac{1}{2}})_{11})\\
&\quad = M^{*-1}(a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2t_{11}b_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2^2t_{11})\\
&\qquad + M^{*-1}(2^2(a_{\frac{1}{2}\frac{1}{2}}b_{\frac{1}{2}\frac{1}{2}})_{11}).
\end{aligned}
$$

So

$$M(2a_{\frac{1}{2}\frac{1}{2}}t_{22} + 2^2(t_{11}b_{\frac{1}{2}\frac{1}{2}})t_{22}) = M(2a_{\frac{1}{2}\frac{1}{2}}t_{22}) + M(2^2(t_{11}b_{\frac{1}{2}\frac{1}{2}})t_{22}).$$

Similarly, we prove the case $(i = 2)$, from the identity

$$a_{\frac{1}{2}\frac{1}{2}} + 2t_{22}b_{\frac{1}{2}\frac{1}{2}} + 2a_{\frac{1}{2}\frac{1}{2}}b_{\frac{1}{2}\frac{1}{2}} = (e_1 + b_{\frac{1}{2}\frac{1}{2}})(a_{\frac{1}{2}\frac{1}{2}} + t_{22}) + (a_{\frac{1}{2}\frac{1}{2}} + t_{22})(e_1 + b_{\frac{1}{2}\frac{1}{2}}).$$

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 31. *For arbitrary* $a_{\frac{1}{2}\frac{1}{2}}, b_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$ *and* $t_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, *the following hold:*

(i) $M(2t_{ii}a_{\frac{1}{2}\frac{1}{2}} + 2t_{ii}b_{\frac{1}{2}\frac{1}{2}}) = M(2t_{ii}a_{\frac{1}{2}\frac{1}{2}}) + M(2t_{ii}b_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(2t_{ii}a_{\frac{1}{2}\frac{1}{2}} + 2t_{ii}b_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2t_{ii}a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2t_{ii}b_{\frac{1}{2}\frac{1}{2}})$.

Proof. (i) For the case $i = 1$. Let c be an element of $\mathfrak{R}$, satisfying $M(c) = M(2t_{11}a_{\frac{1}{2}\frac{1}{2}}) + M(2t_{11}b_{\frac{1}{2}\frac{1}{2}})$. For any $t_{22} \in \mathfrak{R}_{22}$, we have

$$M^{*-1}(t_{22}c + ct_{22}) = M^{*-1}(2^2(t_{11}a_{\frac{1}{2}\frac{1}{2}})t_{22} + 2^2(t_{11}b_{\frac{1}{2}\frac{1}{2}})t_{22}),$$

by Lemmas 5 and 30. Therefore, $t_{22}c + ct_{22} = 2^2(t_{11}a_{\frac{1}{2}\frac{1}{2}})t_{22} + 2^2(t_{11}b_{\frac{1}{2}\frac{1}{2}})t_{22}$. It follows that $c_{12}t_{22} = 0$, $c_{\frac{1}{2}\frac{1}{2}}t_{22} = (2t_{11}a_{\frac{1}{2}\frac{1}{2}} + 2t_{11}b_{\frac{1}{2}\frac{1}{2}})t_{22}$, $t_{22}c_{21} = 0$ and $t_{22}c_{22} + c_{22}t_{22} = 0$, which implies $c_{12} = 0$, $c_{\frac{1}{2}\frac{1}{2}} = 2t_{11}a_{\frac{1}{2}\frac{1}{2}} + 2t_{11}b_{\frac{1}{2}\frac{1}{2}}$, $c_{21} = 0$ and $c_{22} = 0$, by (i) and (iv) (resp., (i′) and (iii′)), in Lemma 2, and (v) (resp., (iv′)) in Theorem. This implies that $M(c_{11} + c_{\frac{1}{2}\frac{1}{2}}) = M(2t_{11}a_{\frac{1}{2}\frac{1}{2}}) + M(2t_{11}b_{\frac{1}{2}\frac{1}{2}})$.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, we have $c_{11}t_{12} = 0$, Lemma 5. So $c_{11} = 0$, by (i) in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$,

$$\begin{aligned} &M^{*-1}(2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}) \\ &\qquad = M^{*-1}(2^2(t_{11}a_{\frac{1}{2}\frac{1}{2}})t_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2^2(t_{11}b_{\frac{1}{2}\frac{1}{2}})t_{\frac{1}{2}\frac{1}{2}}), \end{aligned}$$

by Lemma 5, which implies

$$\begin{aligned} &M(2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2^2(c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}) \\ &\qquad = M(2^3((t_{11}a_{\frac{1}{2}\frac{1}{2}})t_{\frac{1}{2}\frac{1}{2}})_{11}) + M(2^3((t_{11}b_{\frac{1}{2}\frac{1}{2}})t_{\frac{1}{2}\frac{1}{2}})_{11}), \end{aligned}$$

by Corollary 1. Hence, for arbitrary $u_{22} \in R_{22}$, we obtain

$$M^{*-1}(2^2(c_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}) = 0$$

and so $(c_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22} = 0$. From the hypothesis (ii′) and (iii′), of Theorem, we conclude that $c_{11} = 0$.

In both cases we conclude that $c_{11} = 0$. So $c = 2t_{11}a_{\frac{1}{2}\frac{1}{2}} + 2t_{11}b_{\frac{1}{2}\frac{1}{2}}$.

Similarly, we prove the case $i = 2$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 32. *For arbitrary $a_{\frac{1}{2}\frac{1}{2}}$ and $b_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$, the following hold:*

(i) $M(a_{\frac{1}{2}\frac{1}{2}} + b_{\frac{1}{2}\frac{1}{2}}) = M(a_{\frac{1}{2}\frac{1}{2}}) + M(b_{\frac{1}{2}\frac{1}{2}})$;

(ii) $M^{*-1}(a_{\frac{1}{2}\frac{1}{2}} + b_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(b_{\frac{1}{2}\frac{1}{2}})$.

Proof. (i) Let c be an element of $\mathfrak{R}$, satisfying $M(c) = M(a_{\frac{1}{2}\frac{1}{2}}) + M(b_{\frac{1}{2}\frac{1}{2}})$. For any $t_{22} \in \mathfrak{R}_{22}$, we have

$$M^{*-1}(t_{22}c + ct_{22}) = M^{*-1}(2t_{22}a_{\frac{1}{2}\frac{1}{2}} + 2t_{22}b_{\frac{1}{2}\frac{1}{2}}),$$

by Lemmas 5 and 31. Therefore, we have $t_{22}c + ct_{22} = 2t_{22}a_{\frac{1}{2}\frac{1}{2}} + 2t_{22}b_{\frac{1}{2}\frac{1}{2}}$. It follows that $c_{12}t_{22} = 0$, $t_{22}c_{\frac{1}{2}\frac{1}{2}} = t_{22}(a_{\frac{1}{2}\frac{1}{2}} + b_{\frac{1}{2}\frac{1}{2}})$, $t_{22}c_{21} = 0$ and $t_{22}c_{22} + c_{22}t_{22} = 0$, which implies $c_{12} = 0$, $c_{\frac{1}{2}\frac{1}{2}} = a_{\frac{1}{2}\frac{1}{2}} + b_{\frac{1}{2}\frac{1}{2}}$, $c_{21} = 0$ and $c_{22} = 0$, by (i) and (iv) (resp., (i′) and (iii′)), in Lemma 2, and (v) (resp., (iv′)) in Theorem.

Next, by Lemma 5 again, we have

$$M^{*-1}(2c_{11} + c_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(a_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(b_{\frac{1}{2}\frac{1}{2}}). \tag{6}$$

If $\mathfrak{R}$ satisfies the conditions (i)–(v), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, we have $M(2c_{11}t_{12}) = 0$, by (6) and Corollary 1, which implies $c_{11}t_{12} = 0$. So $c_{11} = 0$, by (i) in Lemma 2.

If $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$, we have

$$M(2^2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}) = M(2a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}) + M(2b_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}}),$$

by (6) and Corollary 1, which implies that

$$\begin{aligned} M^{*-1}&(2^2c_{11}t_{\frac{1}{2}\frac{1}{2}} + 2^2(c_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}) \\ &= M^{*-1}(2^2(a_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}) + M^{*-1}(2^2(b_{\frac{1}{2}\frac{1}{2}}t_{\frac{1}{2}\frac{1}{2}})_{11}), \end{aligned}$$

Lemma 5. Hence, for arbitrary $u_{22} \in R_{22}$, by Corollary 1, we obtain

$$M(2^3(c_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22}) = 0$$

and so $(c_{11}t_{\frac{1}{2}\frac{1}{2}})u_{22} = 0$. From the hypothesis (ii′) and (iii′), of Theorem, we conclude that $c_{11} = 0$.

In both cases we conclude that $c_{11} = 0$. So $c = a_{\frac{1}{2}\frac{1}{2}} + b_{\frac{1}{2}\frac{1}{2}}$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 33. *For arbitrary* $a_{11}, b_{11} \in \mathfrak{R}_{11}$*, we have:*

(i) $M(a_{11} + b_{11}) = M(a_{11}) + M(b_{11})$;

(ii) $M^{*-1}(a_{11} + b_{11}) = M^{*-1}(a_{11}) + M^{*-1}(b_{11})$.

Proof. (i) Let $c \in \mathfrak{R}$ be such that $M(c) = M(a_{11}) + M(b_{11})$. For any $t_{22} \in \mathfrak{R}_{22}$, we have $M^{*-1}(t_{22}c + ct_{22}) = 0$, by Lemma 5. This implies that $t_{22}c + ct_{22} = 0$. Thus, $c_{12} = c_{\frac{1}{2}\frac{1}{2}} = c_{21} = c_{22} = 0$, by (iv) (resp., (iii′)), in Lemma 2.

Now, if $\mathfrak{R}$ satisfies the conditions (i)–(vi), of Theorem, then for arbitrary $t_{12} \in \mathfrak{R}_{12}$, we compute

$$\begin{aligned} M^{*-1}(t_{12}c + ct_{12}) &= M^{*-1}(t_{12}a_{11} + a_{11}t_{12}) + M^{*-1}(t_{12}b_{11} + b_{11}t_{12}) \\ &= M^{*-1}(a_{11}t_{12}) + M^{*-1}(b_{11}t_{12}) \\ &= M^{*-1}(a_{11}t_{12} + b_{11}t_{12}), \end{aligned}$$

by Lemma 28. It follows that $c_{11}t_{12} = (a_{11} + b_{11})t_{12}$. Thus, $c_{11} = a_{11} + b_{11}$, by (i), in Lemma 2.

if $\mathfrak{R}$ satisfies the conditions (i′)–(iv′), of Theorem, then for arbitrary $t_{\frac{1}{2}\frac{1}{2}} \in \mathfrak{R}_{\frac{1}{2}\frac{1}{2}}$, we have

$$M^{*-1}(t_{\frac{1}{2}\frac{1}{2}}c + ct_{\frac{1}{2}\frac{1}{2}}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}}) + M^{*-1}(2b_{11}t_{\frac{1}{2}\frac{1}{2}}),$$

which implies

$$M^{*-1}(2t_{\frac{1}{2}\frac{1}{2}}c_{11}) = M^{*-1}(2a_{11}t_{\frac{1}{2}\frac{1}{2}} + 2b_{11}t_{\frac{1}{2}\frac{1}{2}}),$$

by Lemma 32. It follows that $\big(c_{11} - (a_{11} + b_{11})\big)t_{\frac{1}{2}\frac{1}{2}} = 0$, which implies $c_{11} = a_{11} + b_{11}$, by (ii′), in Theorem.

In both cases we conclude that $c_{11} = a_{11} + b_{11}$.

(ii) By Lemma 4, we can infer that (ii) holds. □

Lemma 34. *For arbitrary $a_{22}, b_{22} \in \mathfrak{R}_{22}$, we have:*

(i) $M(a_{22} + b_{22}) = M(a_{22}) + M(b_{22})$;

(ii) $M^{*-1}(a_{22} + b_{22}) = M^{*-1}(a_{22}) + M^{*-1}(b_{22})$.

Proof. The proof is similar to Lemma 33. □

Proof of main Theorem

Let a and b be two arbitrary elements of $\mathfrak{R}$. Then

$$\begin{aligned} M(a+b) &= M((a_{11}+b_{11})+(a_{12}+b_{12})+(a_{\frac{1}{2}\frac{1}{2}}+b_{\frac{1}{2}\frac{1}{2}})+(a_{21}+b_{21})+(a_{22}+b_{22})) \\ &= M(a_{11}+b_{11}) + M(a_{12}+b_{12}) + M(a_{\frac{1}{2}\frac{1}{2}} + b_{\frac{1}{2}\frac{1}{2}}) \\ &\quad + M(a_{21}+b_{21}) + M(a_{22}+b_{22}) \\ &= M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) + M(a_{\frac{1}{2}\frac{1}{2}}) \\ &\quad + M(b_{\frac{1}{2}\frac{1}{2}}) + M(a_{21}) + M(b_{21}) + M(a_{22}) + M(b_{22}) \\ &= M(a_{11}+a_{12}+a_{\frac{1}{2}\frac{1}{2}}+a_{21}+a_{22}) + M(b_{11}+b_{12}+b_{\frac{1}{2}\frac{1}{2}}+b_{21}+b_{22}) \\ &= M(a) + M(b). \end{aligned}$$

That is, M is additive. Now, for any $x, y \in \mathfrak{R}'$, there are elements c, d in $\mathfrak{R}$ such that $c = M^*(x) + M^*(y)$ and $d = M^*(x + y)$. For arbitrary $t_{jj} \in \mathfrak{R}_{jj}$ $(j = 1, 2)$, using the additivity of M, we compute

$$\begin{aligned} M(t_{jj}c + ct_{jj}) &= M(t_{jj}(M^*(x) + M^*(y)) + (M^*(x) + M^*(y))t_{jj}) \\ &= M(t_{jj}M^*(x)) + M(t_{jj}M^*(y)) \\ &\quad + M(M^*(x)t_{jj}) + M(M^*(y)t_{jj}) \\ &= M(t_{jj}M^*(x) + M^*(x)t_{jj}) + M(t_{jj}M^*(y) + M^*(y)t_{jj}) \\ &= M(t_{jj})x + xM(t_{jj}) + M(t_{jj})y + yM(t_{jj}) \\ &= M(t_{jj})(x + y) + (x + y)M(t_{jj}) \\ &= M(t_{jj}M^*(x + y) + M^*(x + y)t_{jj}) \\ &= M(t_{jj}d + dt_{jj}). \end{aligned}$$

Therefore, $t_{jj}c + ct_{jj} = t_{jj}d + dt_{jj}$ $(i, j = 1, 2)$. So $c = d$, by (iv) (resp., (iii′)), in Lemma 2. Consequently, $M^*(x + y) = M^*(x) + M^*(y)$, which completes the proof. □

The following two examples show that the conditions of the Theorem 1 are not artificial.

Example 1. Let $\mathfrak{F}$ be a field of characteristic different from 2, $\mathfrak{J}$ a four dimensional algebra over $\mathfrak{F}$ and a basis $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ with the multiplication table given by: $e_{ij}e_{kl} = \delta_{jk}e_{il}$ $(i, j, k, l = 1, 2)$, where δ_{jk} is the Kronecker delta. We can verify that $\mathfrak{J}$ is a standard algebra. In fact, $\mathfrak{J}$ is an associative algebra where e_{11} and e_{22} are orthogonal idempotents such that $e = e_{11} + e_{22}$ is the unity element of $\mathfrak{J}$. Moreover, if $\mathfrak{J} = \mathfrak{J}_{11} \oplus \mathfrak{J}_{12} \oplus \mathfrak{J}_{21} \oplus \mathfrak{J}_{22}$ is the Peirce decomposition of $\mathfrak{J}$, relative to e_{11}, then we have $\mathfrak{J}_{ij} = \mathfrak{F}e_{ij}$ $(i, j = 1, 2)$. From a direct calculation, we can verify that $\mathfrak{J}$ satisfies the conditions (i)–(v) of the Theorem 1.

Example 2. Let $\mathfrak{K}$ be the algebra obtained from the associative algebra $\mathfrak{J}$, in Example 1, on replacing the product xy by $x \cdot y = \frac{1}{2}(xy + yx)$. We can verify that $\mathfrak{K}$ is a standard algebra. In fact, $\mathfrak{K}$ is a Jordan algebra where e_{11} and e_{22} are orthogonal idempotents such that $e = e_{11} + e_{22}$ is the unity element of $\mathfrak{K}$. Moreover, if $\mathfrak{K} = \mathfrak{K}_1 \oplus \mathfrak{K}_{\frac{1}{2}} \oplus \mathfrak{K}_2$ is the Peirce decomposition of $\mathfrak{K}$, relative to e_{11}, then we have $\mathfrak{K}_{ii} = \mathfrak{F}e_{ii}$ $(i = 1, 2)$ and $\mathfrak{K}_{\frac{1}{2}} = \mathfrak{F}e_{12} + \mathfrak{F}e_{21}$. From a direct calculation, we can verify that the algebra $\mathfrak{K}$ satisfies the conditions (i′)–(iv′) of the Theorem 1.

We conclude with the following result.

Theorem 2. *Let $\mathfrak{J}$ and $\mathfrak{J}^{'}$ be two standard algebras. If $\mathfrak{J}$ is a unital non-degenerate prime standard algebra over a field of characteristic different from* 2 *and* 3 *containing a non-trivial idempotent* e_1, *then every surjective Jordan elementary map* (M, M^*) *of* $\mathfrak{J} \times \mathfrak{J}^{'}$ *is additive.*

Proof. Let $\mathfrak{J} = \mathfrak{J}_1 \oplus \mathfrak{J}_{\frac{1}{2}} \oplus \mathfrak{J}_2$ be the Peirce decomposition of $\mathfrak{J}$, relative to e_1. If $\mathfrak{J}$ is a prime standard algebra, then either $\mathfrak{J}$ is an associative algebra or a Jordan algebra, by [6, Theorem 1] and [7]. If $\mathfrak{J}$ is associative, then it is easy to verify that the conditions (i)–(v) of Theorem 1 hold in $\mathfrak{J}$. If $\mathfrak{J}$ is a Jordan algebra, then it is also easy to verify that the conditions *(i′), (iii′) and (iv′)* of Theorem 1 hold in $\mathfrak{J}$. In addition, for an arbitrary element $a_1 \in \mathfrak{J}_1$ such that $a_1 t_{\frac{1}{2}} = 0$, for every $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$, and $e_2 = 1 - e_1$, we have the Jordan triple product $\{a_1 \mathfrak{J} e_2\} = 0$ which results in $a_1 = 0$, by [1, Theorem 2]. Similarly, we prove that for an arbitrary element $a_2 \in \mathfrak{J}_2$, if $a_2 t_{\frac{1}{2}} = 0$ for every $t_{\frac{1}{2}} \in \mathfrak{J}_{\frac{1}{2}}$, then $a_2 = 0$. Thus, the condition (ii′) also holds in $\mathfrak{J}$. Consequently, we can conclude that every surjective Jordan elementary map (M, M^*) of $\mathfrak{J} \times \mathfrak{J}^{'}$ is additive. □

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