Algebra and Discrete Mathematics
Volume **18** (2014). Number 2, pp. 163–170
RESEARCH ARTICLE

On types of local deformations of quadratic forms

Vitaliy M. Bondarenko

Communicated by V. V. Kirichenko

Abstract. We consider some aspects of the theory of quadratic forms concerning their local deformations.

Throughout this paper, by a quadratic form we mean that over the field of real numbers $\mathbb{R}$

$$f(z) = f(z_1, \dots, z_n) = \sum_{i=1}^{n} f_{ii} z_i^2 + \sum_{i<j} f_{ij} z_i z_j. \tag{1}$$

The set of all such forms is denoted by $\mathcal{R}$. Because with each such quadratic form one can associate a natural graph (with the points $1, \dots, n$ and the edges (i, j), $i < j$, $f_{ij} \neq 0$), we will also use geometric terms.

We call deformation of the quadratic form $f(z)$ a family of quadratic forms parameterized by points of a manifold, one of which corresponds to $f(z)$. In this paper we consider deformations of the form

$$f^{(s)}(z, t) = t f_{ss} z_s^2 + \sum_{i \neq s} f_{ii} z_i^2 + \sum_{i<j} f_{ij} z_i z_j \ \text{ with } \ f_{ss} \neq 0 \tag{2}$$

and of the form

$$f^{(p,q)}(z, t) = \sum_{i=1}^{n} f_{ii} z_i^2 + t f_{pq} z_p z_q + \sum_{(i,j) \neq (p,q)} f_{ij} z_i z_j \ \text{ with } \ f_{pq} \neq 0 \tag{3}$$

2010 MSC: 15A.

Key words and phrases: quadratic form, matrix, graph, point-local deformation, edge-local deformation.

where the parameter t runs $\mathbb{R}$, which are called *local* (for more general definitions, see Section 4). The first and the second type of local deformations is called *point-local* and *edge-local*, respectively. The point-local deformations were introduced by the author earlier (and were called simply local) and the edge-local deformations are introduced here for the first time.

In this paper we consider some aspects of this theme.

1. P-limiting numbers

In this section we define the notion of P-limiting number of deformations, which was first introduced in [1] for point-local deformations (under the name "P-numbers").

Let $f(z,t)$ be a quadratic form of the form (1) but over the ring of polynomials $\mathbb{R}[t]$ (t is a parameter running through $\mathbb{R}$), which can be considered as a deformation of $f(z,b)$ for each $b \in \mathbb{R}$. An $a \in \mathbb{R}$ is said to be a *P-limiting number of $f(z,t)$* if the quadratic form $f(z,a)$ is not positive definite and every neighbourhood of a contains c such that $f(z,c)$ is positive definite. When $f(z,a)$ is not positive definite for any $a \in \mathbb{R}$, we will assume that ∞ or $-\infty$ or both are P-limiting numbers, depending on the specific formal arguments in each case. If $b \in \mathbb{R}$ is fixed, a P-limiting numbers of $f(z,t)$ may also be called (in concrete cases) a *P-limiting numbers of $f(z,b)$* (with indicating additional data on a form of the deformation, or without them if there is no confusion).

In the study of P-limiting numbers it is convenient to use the Sylvester's criterion of positive definiteness of quadratic forms. By this criterion the determination of the P-limiting number is reduced to analysing the solutions of the equations of the form $\Delta_i(t) = 0$ for all corner minors $\Delta_i(t)$ of the symmetric matrix of $f(z,t)$.

2. Point-local deformations

Let $f = f(z)$ be a quadratic form of the form (1) and $1 \leqslant s \leqslant n$. A point-local deformation $f^{(s)}(z,t)$ of the form (2) is also called the *local deformation of $f(z)$ with respect to z_s* or the *s-deformation of $f(z)$*.

Throughout this section we suppose that $f_{ss} > 0$. The case $f_{ss} < 0$ reduces to the case $f_{ss} > 0$ by replacing t to $-t$.

Denote by $F_+^{(s)}$ the set of all $a \in \mathbb{R}$ such that the quadratic form $f^{(s)}(z,a)$ is positive definite, and put $F_-^{(s)} = \mathbb{R} \setminus F_+^{(s)}$. Obviously, $F_-^{(s)} \neq \varnothing$

(since $f^{(s)}(z,0)$ is not positive definite), and if $F_-^{(s)} \neq \mathbb{R}$ then

$$m_f^{(s)} = \sup F_-^{(s)} \in F_-^{(s)}$$

is the only P-limiting number of $f^{(s)}(z,t)$, which we call the *P-limiting number of $f(z)$ for z_s* or the *s-th P-limiting number of $f(z)$*. In the case $F_-^{(s)} = \mathbb{R}$ we put $m_f^{(s)} = \infty$.

Theorem 1. *Let $F_-^{(s)} \neq \mathbb{R}$. Then $m_f^{(s)}$ is a rational function from the coefficients of $f(z)$ and the following hold:*

1) *$m_f^{(s)} \in [0,1)$ if $f(z)$ is positive definite;*
2) *$m_f^{(s)} = 1$ if $f(z)$ is non-negative definite, but is not positive definite;*
3) *$m_f^{(s)} \in (1,\infty)$ if $f(z)$ is not non-negative definite.*

Proof. We assume, without loss of generality, that $s = n$.

Consider the symmetric matrix of the quadratic form $f^{(n)}(z,a)$, multiplied for convenience by 2:

$$S(a) = S_f^{(n)}(a) = \begin{pmatrix} 2f_{11} & f_{12} & \dots & f_{1,n-1} & f_{1n} \\ f_{12} & 2f_{22} & \dots & f_{2,n-1} & f_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{1,n-1} & f_{2,n-1} & \dots & 2f_{n-1,n-1} & f_{n-1,n} \\ f_{1n} & f_{2n} & \dots & f_{n-1,n} & 2af_{nn} \end{pmatrix}$$

(here a belongs to $\mathbb{R}$). Denote by Δ_k $(k = 1,\dots,n-1)$ the corner $k \times k$ minor of $S(a)$ and by Δ_{in} the $(n-1)\times(n-1)$ minor of $S(a)$ which is obtained from it by deleting the i-th arrow and the n-th column. Note that all these minors do not depend on a. Expanding the determinant $\Delta(a)$ of $S(a)$ along the last column, we get

$$\Delta(a) = 2af_{nn}\Delta_{n-1} + N_n, \tag{4}$$

where

$$N_n = \Big[(-1)^{n+1} f_{1n}\Delta_{1n} + (-1)^{n+2} f_{2n}\Delta_{2n} + \cdots \\ \cdots + (-1)^{2n-1} f_{n-1,n}\Delta_{n-1,n}\Big],$$

Let $\overline{f}(z \setminus z_n)$ denotes the quadratic form in the variables z_i, where $i \in \{1,\dots,n-1\}$, determined by the formula

$$\overline{f}(z \setminus z_n) = f(z_1,\dots,z_{n-1},0).$$

Obviously, $\overline{f}(z\backslash z_n)$ is positive definite, since otherwise, for each a, $f^{(n)}(z,a)$ would not be positive definite. Hence, by the Sylvester's criterion (for positive definiteness of quadratic forms), $\Delta_1 > 0, \dots, \Delta_{n-1} > 0$. Then from this criterion it follows that $f(z,a)$ is positive definite if $\Delta(a) > 0$, and is not positive definite if $\Delta(a) \leqslant 0$. Consequently, using (4), we have

$$\begin{aligned} F_-^{(n)} &= \{a \in \mathbb{R} \,|\, \Delta(a) \leqslant 0\} = \{a \in \mathbb{R} \,|\, 2af_{nn}\Delta_{n-1} \leqslant -N_n\} \\ &= \{a \in \mathbb{R} \,|\, a \leqslant -N_n/2f_{nn}\Delta_{n-1}\}. \end{aligned}$$

So

$$m_f^{(n)} = -N_n/2f_{nn}\Delta_{n-1} \tag{5}$$

From this equality, in particular, it follows that $m_f^{(n)}$ is a rational function from the coefficients of $f(z)$.

We now proceed directly to the proofs of statements 1) –3) (assuming as before that $s = n$).

1) The belonging of $m_f^{(n)}$ to $[0,1)$ follows from the facts that $f^{(n)}(z,1)$ is positive definite and $f^{(n)}(z,0)$ is not.

2) Obviously, $1 \in F_-^{(n)}$. On the other hand, each $a > 1$ does not belong to $F_-^{(n)}$, otherwise, there exists a vector $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ with $r_n \neq 0$ such that $f^{(n)}(r,a) \leqslant 0$, whence $f(r) \leqslant -(a-1)f_{nn}r_n^2 < 0$ (since $f^{(n)}(r,a) = f(r) + (a-1)f_{nn}r_n^2$), a contradiction.

3) Fix a non-zero vector $r = (r_1, \dots, r_n) \in \mathbb{R}^n$ such that $f(r) < 0$. Then $r_n \neq 0$ (by the positive definiteness of $\overline{f}(z \setminus z_n)$) and $f^{(n)}(r,a) = 0$ for $a = 1 - \frac{f(r)}{f_{nn}r_n^2} > 1$. Consequently, $m_f^{(n)} > 1$. Inequality $m_f^{(n)} < \infty$ follows from (5). □

From the above proof it follows the following simple but practically useful corollary.

Corollary 1. $F_-^{(s)} \neq \mathbb{R}$ *if and only if the matrix, obtained from the symmetric matrix of* $f(z)$ *by deleting the* s*-th arrow and the* s*-th column, has all positive corner minors. In this case* $m_f^{(s)}$ *is determined by* (5) *with replacement* n *on* s*.*

We have also the following theorem.

Theorem 2. *If* $m_f^{(s)} \neq \infty$*, then the quadratic form* $f^{(s)}(z, m_f^{(s)})$ *is non-negative definite.*

Proof. Supposing again $s = n$, we have from the proof of Theorem 1 that $\Delta(m_f^{(n)}) = 0$ (see (4) and (5)) and that $\Delta_1 > 0, \dots, \Delta_{n-1} > 0$. It only remains to use (for $r = n - 1$) the following statement, which is a generalization of Sylvester's criterion to the case of non-negative definite quadratic forms: if the matrix of a quadratic form has rank r and the first r corner minors are positive definite, then the quadratic form is non-negative definite. □

Clearly the most interesting case is the case of positive definite quadratic forms among which we highlight integral ones, often occurring in the modern theory of representations. We state one of the results of [2] for such quadratic forms that are unit (i.e. $f_{11} = \cdots = f_{nn} = 1$).

Theorem 3. *Let $\mathcal{L}^+ \subset \mathbb{R}$ be the set of P-limiting numbers for the variables of all unit integral positive definite quadratic forms. Then*

$$\mathcal{L}^+ = \left\{ 1 - \frac{1}{2a} - \frac{1}{2b} \;\middle|\; a, b \in \mathbb{N} \right\} \bigcup \left\{ 1 - \frac{2}{c+1} \;\middle|\; c \in \mathbb{N} \right\}.$$

3. Edge-local deformations

Let $f = f(z)$ be a quadratic form of the form (1) and $f_{pq} \neq 0$, where $1 \leqslant p < q \leqslant n$. An edge-local deformation $f^{(p,q)}(z,t)$ of the form (2) is also said to be the *local deformation of $f(z)$ with respect to $z_p z_q$* or the *(p,q)-deformation of $f(z)$*.

The P-limiting numbers of $f^{(p,q)}(z,t)$ are also called the *P-limiting numbers of $f(z)$ for $z_p z_q$* or the (p,q)-th P-limiting numbers of $f(z)$.

Denote by $F_+^{(p,q)}$ the set of all $a \in \mathbb{R}$ such that $f^{(p,q)}(z,a)$ is positive definite, and put $F_-^{(p,q)} = \mathbb{R} \setminus F_+^{(p,q)}$. Obviously, since the quadratic forms $f_{pp}z_p^2 + t z_p z_q + f_{qq} z_q^2$ is not positive definite, for instance, for $t = -f_{pp} - f_{qq}$, the set $F_-^{(p,q)}$ is not empty.

Theorem 4. *Let $F_-^{(p,q)} \neq \mathbb{R}$. Then there are exactly two P-limiting numbers of $f^{(p,q)}(z,t)$.*

Proof. We assume, without loss of generality, that $p = n-1, q = n$.

Let $a \in \mathbb{R}$ and let

$$S(a) = S_f^{(n-1,n)}(a) = \begin{pmatrix} 2f_{11} & f_{12} & \dots & f_{1,n-1} & f_{1n} \\ f_{12} & 2f_{22} & \dots & f_{2,n-1} & f_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{1,n-1} & f_{2,n-1} & \dots & 2f_{n-1,n-1} & af_{n-1,n} \\ f_{1n} & f_{2n} & \dots & af_{n-1,n} & 2f_{nn} \end{pmatrix}$$

be the symmetric matrix of the quadratic form $f^{(n-1,n)}(z,a)$, multiplied for convenience by 2. It is easy to see that the determinant $\Delta(a)$ of this matrix has the form

$$\Delta(a) = -a^2 f_{n-1,n}^2 \Delta_{n-2} + aM + N, \tag{6}$$

where Δ_{n-2} is the corner $(n-2)\times(n-2)$ minor of $S(a)$ (for $n=2$, $\Delta_{n-2}=1$) and M, N are some (depending, of course, on n) polynomials from the coefficients of $f(z)$. Note that Δ_{n-2} also does not depend on a.

Since $\overline{f}(z \setminus z_n)$ (introduced in Section 2) is positive definite (otherwise, for each a, $f^{(n-1,n)}(z,a)$ would not be positive definite), it follows from the Sylvester's criterion (of positive definiteness of quadratic forms) that the first $n-1$ corner minors of $S(a)$ are positive; in particular, $\Delta_{n-2}>0$. Then from this criterion it follows that $f(z,a)$ is not positive definite if and only if $\Delta(a)\leqslant 0$. Consequently, using (6), we have that

$$F_-^{(n-1,n)} = \{a \in \mathbb{R} \mid -a^2 f_{n-1,n-1}^2 \Delta_{n-2} + aM + N \leqslant 0\},$$

and since $f_{n-1,n-1}^2\Delta_{n-2}>0$ and $F_-^{(n-1,n)} \neq \infty$, from this it follows that the equation (of a) $-a^2 f_{n-1,n-1}^2 \Delta_{n-2} + aM + N = 0$ has two different roots. To complete the proof it only remains to use the simple fact that all P-limiting numbers are exhausted by these roots. $\square$

We denote two P-limiting numbers, which have been discussed in Theorem 4, by $\overleftarrow{m}_f^{(p,q)}$ and $\overrightarrow{m}_f^{(p,q)}$ (the first is less than the second) and call them, respectively, the *left* and *right*. It is obvious that

$$F_-^{(p,q)} = \left(-\infty,\ \overleftarrow{m}_f^{(p,q)}\right] \bigcup \left[\overrightarrow{m}_f^{(p,q)},\ \infty\right), \quad F_+^{(p,q)} = \left(\overleftarrow{m}_f^{(p,q)},\ \overrightarrow{m}_f^{(p,q)}\right).$$

In the case $F_-^{(p,q)}=\mathbb{R}$, we assume that $\overleftarrow{m}_f^{(p,q)} = -\infty$, $\overrightarrow{m}_f^{(p,q)} = \infty$.

We have the following theorem.

Theorem 5. *If $F_-^{(p,q)} \neq \mathbb{R}$, then the quadratic forms $f^{(p,q)}(z, \overleftarrow{m}_f^{(p,q)})$ and $f^{(p,q)}(z, \overrightarrow{m}_f^{(p,q)})$ are non-negative definite.*

The proof is analogous to that of Theorem 2.

From the above it follows that in the case $F_-^{(p,q)} \neq \mathbb{R}$ the (p,q)-limiting numbers of $f(z)$ are determined by the roots of the polynomial (6), which (after dividing by $-f_{n-1,n}^2\Delta_{n-2}$, the corresponding renumbering of the indices and replacement a on t) have the form

$$t^2 - t\frac{M}{f_{pq}^2\Delta_{(p,q)}} - \frac{N}{f_{pq}^2\Delta_{(p,q)}},$$

where $\Delta_{(p,q)}$ is the determinant of the matrix obtained from the symmetric matrix of $f(z)$ (multiplied by 2) by deleting the p-th and the q-th arrows and columns. We denote this polynomial by $\Delta_f^{(p,q)}(t)$ and call one of the following:

a) the *defining polynomial for P-limiting numbers of $f^{(p,q)}(z,t)$*, or *of $f(z)$ for z_pz_q*;

b) the *P-defining polynomial of $f^{(p,q)}(z,t)$*, or *of $f(z)$ for z_pz_q*;

c) the *P-defining (p,q)-polynomial* of $f(z)$.

We have from the above that all coefficients of this polynomial are rational functions from the coefficients of $f(z)$. It is clear that under the study of different classes of quadratic forms it may be necessary to change $\Delta_f^{(p,q)}(t)$ by multiplying it by some number; for example, for the integral quadratic forms it is natural to introduce the *integral defining polynomial.*

There may be, of course, other modifications of our notions, some of which we consider in the next section.

4. Some generalizations

As can be seen from the above, the study of P-limiting numbers (in our both cases) makes sense only if the quadratic form $f(z)$ is a one-variable extension of a positive definite quadratic form (in particular, positive definite one); otherwise each P-limiting number is equal to ∞ or $-\infty$. In this situation point-local and edge-local deformations can be considered in more general forms as follows:

$$f^{(s)}(z,t') = t'z_s^2 + \sum_{i\neq s} f_{ii}z_i^2 + \sum_{i<j} f_{ij}z_iz_j, \tag{7}$$

$$f^{(p,q)}(z,t') = \sum_{i=1}^{n} f_iz_i^2 + t'z_pz_q + \sum_{(i,j)\neq(p,q)} f_{ij}z_iz_j. \tag{8}$$

By replacing the parameter, (2) can be obtained from (7) if $f_{ss}\neq 0$, and (3) can be obtained from (8) if $f_{pq}\neq 0$. Otherwise, we have in some sense degenerate cases and it is natural to call such local deformations *singular*.

Singular deformations arise under a more deep study of quadratic forms.

References

[1] V. M. Bondarenko, Yu. M. Pereguda, *On P-numbers of quadratic forms*, Geometry, Topology, and their Applications, Proc. of the Institute of Mathematics of NAN of Ukraine **6** (2009), no. 2, 474-477.

[2] V. M. Bondarenko, V. V. Bondarenko, Yu. M. Pereguda, *Local deformations of positive-definite quadratic forms*, Ukrainian Math. J. **64** (2012), no. 7, 1019-1035 (in Russian).

Contact information

V. M. Bondarenko Institute of Mathematics, NAS, Kyiv, Ukraine
E-Mail(s): vitalij.bond@gmail.com

Received by the editors: 20.08.2014
and in final form 25.11.2014.