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RESEARCH ARTICLE

On the Lie ring of derivations of a semiprime ring

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Abstract. We prove that the Lie ring of derivations of a semiprime ring is either trivial or non-nilpotent.

1. Preliminaries and introduction

Throughout the text R stands for an associative ring (possibly without identity) and n for a positive integer. By $\mathrm{Z}(R)$ we denote the center of R. The ring R is called *semiprime*, if it has no nonzero nilpotent ideals. Equivalently, $aRa = \{0\}$ with any $a \in R$ implies $a = 0$. We refer the reader to [1] for terminology, definitions and basic facts in ring theory.

A map $d : R \longrightarrow R$ is called a *derivation*, if it is additive and satisfies the Leibniz rule

$$d(xy) = d(x)y + xd(y)$$

for all $x, y \in R$. The set $\mathrm{Der}(R)$ of all derivations $d : R \longrightarrow R$ is a Lie ring under the pointwise addition and the Lie multiplication defined by

$$[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1.$$

A set $E \subseteq \mathrm{Der}(R)$ is *abelian*, if

$$[d_1, d_2] = 0$$

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for all $d_1, d_2 \in E$. For $n \geqslant 3$ and $d_1, \dots, d_n \in \mathrm{Der}(R)$ we define inductively

$$[d_1, \dots, d_n] = [[d_1, \dots, d_{n-1}], d_n].$$

A Lie ring D of derivations on R (i.e., D is a Lie subring of $\mathrm{Der}(R)$) is said to be *nilpotent*, if there exists some n such that $[d_1, \dots, d_{n+1}] = 0$ for all $d_1, \dots, d_{n+1} \in D$. We define the *nilpotency class* of D as the infimum of the set

$$\{n \in \mathbb{N} \setminus \{0\} : [d_1, \dots, d_{n+1}] = 0 \text{ for all } d_1, \dots, d_{n+1} \in D\}.$$

Notice that the Lie ring D is abelian if and only if it is nilpotent of class 1.

Let $a \in R$. It is easy to see that

$$\partial_a : R \ni x \mapsto [a, x] \in R$$

is a derivation. This derivation is referred to as the *inner derivation* generated by a. One can prove that $\mathrm{IDer}(R) = \{\partial_a : a \in R\}$ is a Lie ideal of $\mathrm{Der}(R)$.

In [2], the following theorem has been proved.

Theorem 1. *Suppose that R is semiprime. Then the Lie ring $\mathrm{IDer}(R)$ is nilpotent if and only if R is commutative.*

The purpose of the present note is to show that if R is semiprime, then either $\mathrm{Der}(R) = \{0\}$, or $\mathrm{Der}(R)$ is not nilpotent. Notice that many authors studied commuting derivations in polynomial rings (see [3–5] for references).

2. Some lemmas and useful facts

We start with a simple and well known lemma.

Lemma 1. *For any $d \in \mathrm{Der}(R)$ and any $x \in R$ we have $[d, \partial_x] = \partial_{d(x)}$.*

The above lemma implies

Proposition 1. *If d is a central element of $\mathrm{Der}(R)$ (i.e., $[d, \delta] = 0$ for every $\delta \in \mathrm{Der}(R)$), then $d(R) \subseteq \mathrm{Z}(R)$.*

Proof. For arbitrary elements $a, x \in R$ we have

$$0 = [d, \partial_x](a) = \partial_{d(x)}(a) = [d(x), a].$$ □

Notice that there exists a ring R with a derivation d such that $d(R) \subseteq \mathrm{Z}(R)$ and d is not a central element of $\mathrm{Der}(R)$.

Example 1. Let $d_1, d_2 \in \mathrm{Der}(\mathbb{R}[X])$ be defined by

$$d_1(f) = X\frac{\mathrm{d}f}{\mathrm{d}X} \quad \text{and} \quad d_2(f) = (X+1)\frac{\mathrm{d}f}{\mathrm{d}X}.$$

Then

$$d_1(d_2(X)) = d_1(X+1) = X,$$

and

$$d_2(d_1(X)) = d_2(X) = X+1.$$

Consequently, d_1 and d_2 are not central elements of $\mathrm{Der}(\mathbb{R}[X])$.

The following lemma is the key tool in the note.

Lemma 2. *Suppose that* $\mathrm{Der}(R)$ *is abelian. Let* $d, \delta \in \mathrm{Der}(R)$. *Then*

(i) $\mathrm{Ker}(d) = \{a \in R : d(a) = 0\}$ *is* δ*-stable,*

(ii) $d(R) \subseteq \mathrm{Z}(R)$,

(iii) $d(\mathrm{Z}(R))\delta(R) = \{0\}$,

(iv) $[a, y]d(x) = -[a, x]d(y)$ *for all* $a, x, y \in R$.

Proof. Pick arbitrary elements $a, b, x, y \in R$ and $c \in \mathrm{Z}(R)$. If $a \in \mathrm{Ker}(d)$, then $d(\delta(a)) = \delta(d(a)) = 0$, and hence $\delta(a) \in \mathrm{Ker}(d)$. Property (i) follows. Property (ii) is an immediate consequence of Proposition 1. Next, since $c \in \mathrm{Z}(R)$, the map

$$c\delta : R \ni r \mapsto c\delta(r) \in R$$

is a derivation. Therefore,

$$c\delta(d(a)) = d(c\delta(a)) = d(c)\delta(a) + cd(\delta(a)) = d(c)\delta(a) + c\delta(d(a)),$$

which yields $d(c)\delta(a) = 0$. Property (iii) follows. Finally, by (ii), we have $ad(xy) = d(xy)a$, and hence

$$ad(x)y + axd(y) = d(x)ya + xd(y)a.$$

Consequently,

$$\begin{aligned}[a, y]d(x) &= ayd(x) - yad(x) = ad(x)y - d(x)ya \\ &= xd(y)a - axd(y) = xad(y) - axd(y) = -[a, x]d(y). \qquad \square\end{aligned}$$

Let us proceed to some corollaries of Lemma 2. The first corollary will not be used in the sequel, but it seems to be of separate interest.

Corollary 1. *If* $\mathrm{Der}(R)$ *is abelian, then*

(i) $[R,R] \subseteq \mathrm{Z}(R)$,

(ii) $[R,R] \subseteq \bigcap_{d\in \mathrm{Der}(R)} \mathrm{Ker}(d)$.

Proof. Property (i) follows from the definition of inner derivation and Lemma 2 (ii). Now, for any $d \in \mathrm{Der}(R)$ and any $a, x \in R$ we have $d(x) \in \mathrm{Z}(R)$, and hence

$$d([a,x]) = d(\partial_a(x)) = \partial_a(d(x)) = 0.$$

This proves property (ii). □

Corollary 2. *Let* R *be a commutative ring. Suppose that* $\mathrm{Der}(R)$ *is abelian. Then*

(i) $d(R)\delta(R) = \{0\}$ *for all* $d, \delta \in \mathrm{Der}(R)$,

(ii) $\mathrm{Der}(R) = \{0\}$ *whenever* R *is reduced.*

Proof. Property (i) is an obvious consequence of Lemma 2 (iii). If R is reduced, $d \in \mathrm{Der}(R)$ and $x \in R$, then by (i) we have $d(x)^2 = 0$, and hence $d(x) = 0$. Property (ii) follows. □

3. Main results

Theorem 2. *Let* R *be a semiprime ring. Then* $\mathrm{Der}(R)$ *is abelian if and only if* $\mathrm{Der}(R) = \{0\}$.

Proof. ($\Leftarrow$) is obvious.

($\Rightarrow$) Assume that $\mathrm{Der}(R)$ is abelian. Then by Theorem 1, the ring R is commutative. The commutativity and semiprimeness yield that R is reduced. By applying Corollary 2 (ii), we get therefore $\mathrm{Der}(R) = \{0\}$. □

We will need one more lemma.

Lemma 3. *Let* R *be a commutative ring and* $n \geqslant 2$. *Suppose that*

$$\forall\, d_1, \dots, d_{n+1} \in \mathrm{Der}(R) : \; [d_1, \dots, d_{n+1}] = 0.$$

Then $[d_1, \dots, d_n](R)\delta(R) = \{0\}$ *for any* $d_1, \dots, d_n, \delta \in \mathrm{Der}(R)$.

Proof. Pick arbitrary $d_1, \ldots, d_n, \delta \in \mathrm{Der}(R)$ and arbitrary $a, c \in R$. Recall that

$$c\delta : R \ni x \mapsto c\delta(x) \in R$$

is a derivation of R. Consequently,

$$\begin{aligned} 0 &= [[d_1, \ldots, d_n], c\delta](a) = [d_1, \ldots, d_n](c\delta(a)) - c\delta([d_1, \ldots, d_n](a)) \\ &= [d_1, \ldots, d_n](c)\delta(a) + c[d_1, \ldots, d_n](\delta(a)) - c\delta([d_1, \ldots, d_n](a)) \\ &= [d_1, \ldots, d_n](c)\delta(a) + c[[d_1, \ldots, d_n], \delta](a) = [d_1, \ldots, d_n](c)\delta(a). \end{aligned}$$

The assertion follows. □

By making use of the above lemma with $\delta = [d_1, \ldots, d_n]$, we obtain

Corollary 3. *Let $n \geqslant 2$. Suppose that R is commutative and reduced. If the Lie ring* $\mathrm{Der}(R)$ *is nilpotent of class at most n, then it is nilpotent of class at most $n - 1$.*

We are ready to prove our most important theorem.

Theorem 3. *Suppose that R is semiprime and the Lie ring* $\mathrm{Der}(R)$ *is nilpotent. Then* $\mathrm{Der}(R) = \{0\}$.

Proof. Since $\mathrm{Der}(R)$ is nilpotent, so is $\mathrm{IDer}(R)$. Therefore, by Theorem 1, the ring R is commutative, and hence reduced. It follows now from Corollary 3 that $\mathrm{Der}(R)$ is abelian. Consequently, Theorem 2 yields $\mathrm{Der}(R) = \{0\}$. □

Let us conclude the note by a natural example of a ring whose Lie ring of derivations is nontrivial and abelian.

Example 2. Consider the ring $\mathbb{Q}[a] = \{s + ta : s, t \in \mathbb{Q}\}$, where a is a nonzero element such that $a^2 = 0$. (This ring is isomorphic to the quotient $\mathbb{Q}[X]/\langle X^2 \rangle$). For an arbitrary $\lambda \in \mathbb{Q}$, we define $d_\lambda : \mathbb{Q}[a] \longrightarrow \mathbb{Q}[a]$ by the rule

$$d_\lambda(s + ta) = \lambda ta.$$

It is easy to see that $d_\lambda \in \mathrm{Der}(\mathbb{Q}[a])$. Moreover,

$$\mathrm{Der}(\mathbb{Q}[a]) = \{d_\lambda : \lambda \in \mathbb{Q}\}.$$

Since $[d_\lambda, d_\mu] = 0$ for all $\lambda, \mu \in \mathbb{Q}$, the Lie ring $\mathrm{Der}(\mathbb{Q}[a])$ is abelian.

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