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Effective ring

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Abstract. In this paper we will investigate commutative Bezout domains whose finite homomorphic images are semipotent rings. Among such commutative Bezout rings we consider a new class of rings and call them an effective rings. Furthermore we prove that effective rings are elementary divisor rings.

Throughout this paper all rings will be commutative with nonzero identity. We say that R is a Bezout ring if every finitely generated ideal of R is principal. A nonzero element a in R is said to be *adequate* to the element $b \in R$ (${}_aA_b$ denote this fact), if we can find such two elements $r, s \in R$ that the decomposition $a = rs$ satisfies the following properties:

1) $rR + bR = R$,
2) $s'R + bR \neq R$ for any noninvertible divisor s' of element s.

If for any element $b \in R$ we have ${}_aA_b$, then we say that element a is adequate. If any nonzero element of ring R is an adequate element then R is called an adequate ring [2]. In addition we notice one simple fact: for any nonzero $a \in R$ we have ${}_aA_a$. The most obvious examples of adequate elements are units, square free elements, and factorial elements [7].

Following [4] a commutative ring R is called an exchange ring if for any element $a \in R$ one can find such idempotent $e \in R$ that $e \in aR$ and $(1 - e) \in (1 - a)R$. We say that an element $a \in R$ is clean element if $a = u + e$ for some idempotent $e = e^2$ and unit u. If all elements of a

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ring R are clean then R is called a clean ring. It is proved in [4] that for commutative rings class of exchange rings coincide with class of clean rings.

We say that commutative ring R is semipotent if for any element $a \in R$ that is not contained in Jacobson radical there is some a proper idempotent $e \in R$ that $e \in aR$ [4].

Throughout this paper $J(R)$ will denote a Jacobson radical of ring R, $mspec(a)$ - set of all maximal ideals that contain an element a and $J(aR) = \bigcap_{M \in mspec(a)} M$.

Let's start our research with few simple useful properties of adequacy.

Proposition 1. *Let R be a commutative Bezout domain, and $a \in R$ is some nonzero element. Then ${}_aA_b$ iff ${}_aA_{b+at}$ for any $t \in R$.*

Proof. Let a be adequate to element b in a Bezout ring R. Then by definition there are some elements $r, s \in R$ that $a = rs$ and $rR + bR = R$ as well as $s'R + bR \neq R$ for any element $s' \in R$ that $sR \subset s'R \neq R$. Taking arbitrary $t \in R$ we consider ideal $rR + (b + at)R$. Since R is a Bezout ring then $rR + (b + at)R = hR$ for some element $h \in R$. As $aR \subset rR \subset hR$ and $(b + at)R \subset hR$ then $bR \subset hR$, that is possible iff h is a unit because $rR + bR = R$.

Since $s'R + bR \neq R$, for any element $s' \in R$ that $sR \subset s'R \neq R$, then $s'R + (b + at)R \neq R$ for any $t \in R$.

Thus we have obtained that ${}_aA_{b+at}$ for any $t \in R$, so necessity is proved.

For the converse statement suppose that ${}_aA_{b+at}$, that means that one can find some elements $r, s \in R$ that $a = rs$, $rR + (b + at)R = R$, and $s'R + (b + at)R \neq R$ for any element $s' \in R$ that $sR \subset s'R \neq R$. If $rR + bR = hR \neq R$ for some element $h \in R$, then as $aR \subset rR \subset hR$ and $bR \subset hR$ we have $(b + at)R \subset hR$ that is impossible since $rR \subset hR$. Next, assume that there is some element $s' \in R$ that $sR \subset s'R \neq R$ and $s'R + bR = R$. Let we have $s'R + (b + at)R = hR \neq R$. Then $bR \subset hR$ and this contradicts assumption $s'R + bR = R$. Proposition is proved. □

As a consequence this result motivates us to consider cosets $\overline{b} = b + aR$ in quotient ring $\overline{R} = R/aR$. The following statement determines the correspondence between property ${}_aA_b$ of a commutative Bezout ring R and structure of homomorphic image of element b in $\overline{R} = R/aR$.

Proposition 2. *Let R be a commutative Bezout domain and ${}_aA_b$. Then element $\overline{b} = b + aR$ is a clean element in $\overline{R} = R/aR$.*

Proof. We start with obvious equality $(-1)R + bR + aR = R$ and hence $\overline{-1}\overline{R} + \overline{b}\overline{R} = \overline{R}$. Since a is adequate to element b in a Bezout ring R then there are such elements $r, s \in R$ that $a = rs$ and $rR + bR = R$ as well as $s'R + bR \neq R$ for any element $s' \in R$ that $sR \subset s'R \neq R$. Going down to quotient ring $\overline{R}$ we have that $\overline{r}\overline{R} + \overline{b}\overline{R} = \overline{R}$ and $\overline{s'}\overline{R} + \overline{b}\overline{R} \neq \overline{R}$. Let $\overline{t}$ be some noninvertible in $\overline{R}$ divisor of element $\overline{s}$. Then there is some element $k \in R$ that $(s + ak)R \subset tR$. Let's show that $sR + tR \neq R$. Suppose the contrary, that is $sR + tR = R$. Since $(s + ak)R \subset tR$ then $s + ak = t\beta$ for some element $\beta \in R$. Equality $s + rsk = t\beta$ implies $s(1 + rk) = t\beta$. As $sR + tR = R$ then $(1 + rk)R \subset tR$, so $tR + rR = R$. Since $tR + sR = R$ and $tR + rR = R$ then $tR + rsR = R$, thus $tR + aR = R$ and then $\overline{t}\overline{R} = \overline{R}$ that contradicts assumption about noninvertibility of $\overline{t}$. Thus we have proved that $sR + tR = uR \neq R$. This also means that $\overline{u}\overline{R} + \overline{b}\overline{R} \neq \overline{R}$. Since $\overline{u}$ is a divisor of an element $\overline{t}$ then $\overline{t}\overline{R} + \overline{b}\overline{R} \neq \overline{R}$. So, we obtained that $\overline{0} = \overline{rs}$ is an adequate element to an element $\overline{b}$.

Moreover, we notice that $\overline{r}\overline{R} + \overline{s}\overline{R} = \overline{R}$. In fact, if $\overline{r}\overline{R} + \overline{s}\overline{R} = \overline{h}\overline{R} \neq \overline{R}$, then according adequacy of an element $\overline{0}$ to an element $\overline{b} \in \overline{R}$ we know that $\overline{h}\overline{R} + \overline{b}\overline{R} = \overline{R}$ (as $\overline{h}$ is a divisor of $\overline{r}$) and on the other hand $\overline{h}\overline{R} + \overline{s}\overline{R} \neq \overline{R}$ (as $\overline{h}$ is noninvertible divisor of $\overline{s}$). But this is impossible. So, $\overline{r}\overline{R} + \overline{s}\overline{R} = \overline{R}$ and there are such elements $\overline{u}, \overline{v} \in \overline{R}$ that $\overline{ru} + \overline{sv} = \overline{1}$.

Additionally, we are going to prove that elements $\overline{ru}$ and $\overline{sv}$ are idempotents in ring $\overline{R}$. This can be shown by next expression: $(\overline{ru})^2 = \overline{ru}(\overline{1} - \overline{sv}) = \overline{ru} - \overline{rusv} = \overline{ru}$. Similarly we have $(\overline{sv})^2 = \overline{sv}$.

Let's denote $\overline{e} = \overline{ru}$. Now we want to prove that $\overline{b} - \overline{e}$ is a unit in $\overline{R}$. Suppose that $(\overline{b} - \overline{e})\overline{R} = \overline{h}\overline{R} \neq \overline{R}$. Consider an ideal $\overline{h}\overline{R} + \overline{r}\overline{R} = \overline{t}\overline{R}$. If $\overline{t}$ is nonunit in $\overline{R}$ then $(\overline{b} - \overline{e})\overline{R} \subset \overline{h}\overline{R} \subset \overline{t}\overline{R}$, hence $\overline{b}\overline{R} \subset \overline{t}\overline{R}$ that is impossible since $\overline{r}\overline{R} \subset \overline{t}\overline{R}$, $\overline{b}\overline{R} \subset \overline{t}\overline{R}$, and $\overline{b}\overline{R} + \overline{r}\overline{R} = \overline{R}$. So, $\overline{h}\overline{R} + \overline{r}\overline{R} = \overline{R}$. Now we'll prove that $\overline{h}\overline{R} + \overline{s}\overline{R} = \overline{R}$. Assume that $\overline{s}\overline{R} + \overline{h}\overline{R} = \overline{t}\overline{R} \neq \overline{R}$. Since $\overline{t}$ is nonunit and it divides $\overline{s}$, then by defining property of an element $\overline{s}$ we have $\overline{t}\overline{R} + \overline{b}\overline{R} = \overline{k}\overline{R} \neq \overline{R}$. On the other hand $(\overline{b} - \overline{e})\overline{R} = \overline{h}\overline{R}$ and $\overline{e}\overline{R} + \overline{s}\overline{R} = \overline{R}$, so we have that $\overline{e}\overline{R} + \overline{t}\overline{R} = \overline{R}$ implies $\overline{e}\overline{R} \subset \overline{k}\overline{R}$. Last inclusion is impossible since $\overline{b}\overline{R} \subset \overline{k}\overline{R}$ and $\overline{b}\overline{R} + (\overline{-1})\overline{R} = \overline{R}$. At last we have that $\overline{s}\overline{R} + \overline{h}\overline{R} = \overline{R}$. Since $\overline{r}\overline{R} + \overline{h}\overline{R} = R$ then $\overline{rs}\overline{R} + \overline{h}\overline{R} = \overline{R}$. As we know that $\overline{rs} = \overline{0}$ then we'll obtain that $\overline{h}$ is a unit of ring $\overline{R}$.

So, we have proved that $\overline{b} - \overline{e} = \overline{u}$ is a unit in a ring $\overline{R}$, and hence $\overline{b} = \overline{u} + \overline{e}$ is a clean element. Proposition is proved completely. □

Definition 1. Element $a \in R$ is said to be an exchange element if there exists idempotent $e \in R$ such that $e \in aR$ and $(1 - e) \in (1 - a)R$ [4]. We say that an element $a \in R$ is element of idempotent stable range 1 if for every element $b \in R$ such that $aR + bR = R$ exists idempotent $e \in R$ such that $a + be$ is invertible element of R.

By [3, 4, 5] and proposition 2 we have

Theorem 1. *Let R be a commutative Bezout domain and ${}_aA_b$. Then:*

1) $\bar{b} = b + aR$ *is a clean element of* R/aR.
2) $\bar{b} = b + aR$ *is an exchange element of* R/aR.
3) $\bar{b} = b + aR$ *is element of idempotent stable range 1 of* R/aR.

Note that entry ${}_aA_b$ implies, in particular, that $a \neq 0$

Theorem 2. *Let R be a commutative Bezout domain and let $a \in R \setminus \{0\}$. If ${}_aA_b$ and $b \notin J(aR)$ then an ideal $\overline{bR}$ contains a proper idempotent.*

Proof. Since a is adequate to b then $a = rs$, and $rR + bR = R$ as well as $s'R + bR \neq R$, for any element $s' \in R$ that $sR \subset s'R \neq R$. Since $b \notin J(aR)$ then by simple observation element r is not a unit in a ring R. Then equality $rR + bR = R$ implies that $ru + bv = 1$ for some elements $u, v \in R$. Then $\overline{rsu} + \overline{bsv} = \bar{s}$. Since $a = rs$ then $\overline{rs} = \bar{0}$ and we have $\bar{s}\overline{R} \subset \overline{bR}$ and $\bar{s}\bar{R} \neq (\bar{0})$. Moreover, since $rR + sR = R$ then $rx + sy = 1$ for some elements $x, y \in R$. As a consequence we have $\bar{s}^2\bar{v} = \bar{s}$ that means that element $\bar{s}$ is a regular element. As $\overline{R}$ is commutative then there is such idempotent $\bar{e}$ that $\bar{s}\overline{R} = \bar{e}\overline{R}$. Last fact means that we have found a proper idempotent in an ideal $\overline{bR}$ as was desired. Theorem is proved. ☐

It is natural to ask about the answer to the converse statement.

Theorem 3. *Let R be a commutative Bezout domain and let $a \in R \setminus \{0\}$. If quotient ring $\overline{R} = R/aR$ is semipotent then for any element $b \in R \setminus J(aR)$ there is some element $u \in R$ that ${}_aA_{bu}$.*

Proof. Let $\overline{R} = R/aR$ be a semipotent ring and $b \in R \setminus J(aR)$. By semipotency of $\overline{R}$ for a coset $\bar{b} = b + aR$ there is some nonzero idempotent $\bar{e} \in \overline{bR}$. Hence exist some elements $u, t \in R$ that $e - bu = at$. Moreover, since $\bar{e} = \bar{e}^2$ then $e(1 - e) = as$, for some element $s \in R$. Let's show that ${}_aA_e$. In fact, as R is a commutative Bezout ring then $eR + aR = dR$, for some element $d \in R$. By [6] we have $e = e_0 d$,$a = a_0 d$ and $e_0 p + a_0 q = 1$, for some elements $e_0, a_0, p, q \in R$. Furthermore, equality $e(1 - e) = as$ implies $e_0(1 - e) = a_0 s$. Since $e_0 p + a_0 q = 1$ and $e_0(1 - e) = a_0 s$ then one can deduce that $a_0 k = 1 - e$ for some element $k \in R$. Hence $e + a_0 k = 1$.

Taking $r = a_0$ and $s = d$ we obtain a decomposition $a = rs$, where $rR + eR = R$ as well as $s'R + eR \neq R$ for any element $s' \in R$ that $sR \subset s'R \neq R$. Thus ${}_aA_e$ and since $bu = e + at$ we conclude ${}_aA_{bu}$ by Proposition 1. Theorem is proved. □

As a corollary of two previous theorems we have next result.

Theorem 4. *Let R be a commutative Bezout domain and let $a \in R \setminus \{0\}$. Then quotient ring $\overline{R} = R/aR$ is semipotent iff for any element $b \in R \setminus J(aR)$ there is some element $u \in R$ that ${}_aA_{bu}$, $bu \notin aR$.*

All mentioned results allows us to define a new subclass of commutative Bezout ring that are so called effective ring, that are also elementary divisor rings.

Recall that a commutative ring R is said to be an elementary divisor ring if for every matrix A over R one can find such invertible matrices P and Q of appropriate sizes that $PAQ = diag(\varepsilon_1, ..., \varepsilon_r, 0, ..., 0)$, where $\varepsilon_{i+1}R \subset \varepsilon_i R$, for any $i \in \overline{1, r-1}$ [2].

Definition 2. A commutative Bezout domain R is said to be effective if for any elements $a, b, c \in R$ that $aR + bR + cR = R$ and $aR + bR \neq R$ there exists such element $p \in R$ that ${}_cA_{pa}$ and $pR + bR + cR = R$.

An obvious example of effective ring is any adequate ring. Henriksen's example [1], that is commutative Bezout domain $R = \{z_0 + a_1x + a_2x^2 + ...|z_0 \in \mathbb{Z}, a_i \in \mathbb{Q}\}$, is also an effective ring that is not adequate. Notice that in this ring $aR + bR + cR = R$ implies that at least one of these elements is an adequate element of R.

Theorem 5. *Effective ring R is an elementary divisor ring.*

Proof. Due to [7] it is sufficient to prove that for any coprime triple of elements $aR + bR + cR = R$ there are some elements $p, q \in R$ that $(pa + qb)R + qcR = R$. If $aR + bR = R$ then obviously, exist element $p, g \in R$ such that $(pa + gb)R + qcR = R$ [1,2,7]. By definition of an effective ring there is some element $p \in R$ that ${}_cA_{ap}$ and $pR+bR+cR = R$. As $aR + bR + cR = R$ and $pR + bR + cR = R$ then $apR + bR + cR = R$. Since ${}_cA_{ap}$ then $c = qs$, where $qR + apR = R$ and $s'R + apR \neq R$, for any $s' \in R$ that $sR \subset s'R \neq R$. Let's show that $(ap + bq)R + cqR = R$. Suppose the contrary, that is $(ap+bq)R+cqR = hR \neq R$. Since $cq = sq^2$, then let $qR + hR = dR \neq R$. As $qR \subset dR$ and $hR \subset dR$ then $apR \subset dR$ that is impossible since $qR + apR = R$. Thus $sR \subset hR$. By definition of element s we have that $hR + apR = kR \neq R$. Let $kR + aR = xR \neq R$, then $bqR \subset xR$. Since $cR \subset xR$, $aR \subset xR$ and $aR + bR + cR = R$ then $xR + bR = R$. So $qR \subset xR$, that is impossible as $qR + sR = R$. As a result we have $pR \subset kR$. Then $bqR \subset kR$. If $bR + kR = \alpha R \neq R$ then $cR \subset \alpha R$, $bR \subset \alpha R$ and $pR \subset \alpha R$. Using fact that $pR + bR + cR = R$ we obtain a contradiction, so α must be a unit. So $qR \subset kR$, but this is impossible as $qR + sR = R$ and $sR \subset kR$. So, we have proved that $(ap + bq)R + cqR = R$ that was desired. Theorem is proved. □

Neat rings are studied in [3] as rings whose homomorphic images are exchange rings. We'll show that in case of commutative Bezout ring neat rings are effective rings. To complete this purpose we need the following.

Proposition 3. *A commutative ring R is an exchange ring iff for any pair of elements $a, b \in R$ that $aR + bR = R$ there is an idempotent $e \in R$ that $e \in aR$ and $(1 - e) \in bR$.*

Proof. By [4, Prop.1.1.1] in every exchange ring equality $aR + bR = R$ implies that there are orthogonal idempotents e and $1 - e$ that $e \in aR$ and $(1 - e) \in bR$, so necessity is proved. If $aR + bR = R$ implies that there is an idempotent $e \in aR$ that $e \in aR$ and $(1 - e) \in bR$ then considering equality $a + (1 - a) = 1$ we obtain that $e \in R$ and $(1 - e) \in (1 - a)R$, so R is an exchange ring by definition in [4]. Proposition is proved. □

Theorem 6. *Let R be a commutative Bezout domain whose finite homomorphic image R/cR is an exchange ring for any $c \in R \setminus \{0\}$. Then R is an effective ring.*

Proof. Let $\overline{R} = R/cR$ be an exchange ring for any $c \in R \setminus \{0\}$. By [3] R is a neat ring. Then by Proposition 3 an equality $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$ implies that we can find an idempotent $\overline{e} \in \overline{R}$, that $\overline{e} \in \overline{a}\overline{R}$ and $\overline{1} - \overline{e} \in \overline{b}\overline{R}$. Let's notice that condition $\overline{a}\overline{R} + \overline{b}\overline{R} = \overline{R}$ implies $aR + bR + cR = R$. Since $\overline{e} \in \overline{a}\overline{R}$ then there is some element $p \in R$ that $e - ap = cs$ for some element $s \in R$. Similarly $1 - e - b\alpha = c\beta$ for some elements $\alpha, \beta \in R$. By substitution $e = cs + ap$ in $1 - e - b\alpha = c\beta$ we'll get $ap + cw + b\alpha = 1$, that means $pR + cR + bR = R$. Let's prove that ${}_cA_{ap}$. Since $\overline{e} = \overline{e}^2$ then $e(1 - e) = ct$ for some element $t \in R$. We consider an ideal $eR + cR = dR$. By [6], we have $e = de_0$ and $c = dc_0$ for some elements $e_0, c_0 \in R$ that $e_0R + c_0R = R$, then $e_0(1 - e) = c_0t$, and then $e + c_0\gamma = 1$ for some element $\gamma \in R$. Taking $r = c_0$, $s = d$ we obtain a decomposition $c = rs$, where $rR + eR = R$ and $sR \subset eR$. So, we have ${}_cA_e$. Since $e = ap + cs$

then by Proposition 1 and we conclude ${}_cA_{ap}$, as was desired. Theorem is proved. □

Proposition 4. *Let R be a commutative Bezout domain, which for any elements $a, b, c \in R$ that $aR + bR + cR = R$ there exists such element $p \in R$ that ${}_cA_{pa}$. Let $c = rs$ where $rR + apR = R$ and $s'R + apR \neq R$ for any noninvertible divisor s' of element s and then $pR + bR + cR = R$ iff $sR + bR = R$.*

Proof. Let $c = rs$ where $rR + apR = R$ and $s'R + apR \neq R$ for any noninvertible divisor s' of elements s. If $aR + bR + cR = R$ and $pR + bR + cR = R$ then $apR + bR + cR = R$. If $sR + bR = \delta R \neq R$, we have $\delta R + apR = hR \neq R$. It is impossible, since $apR + bR + cR = R$.

Let $sR+bR = R$. We prove that $apR+bR+cR = R$. If $pR+bR+cR = hR \neq R$, then $pR \subset hR$, $cR \subset hR$ and $bR \subset hR$. Since $rR + apR = R$, then h is noninvertible divisor of s. By $bR \subset hR$ and $sR \subset hR$, we have $sR + bR \subset hR \neq R$. It is impossible, since $sR + bR = R$. Proposition is proved. □

Theorem 7. *Let R be a commutative Bezout domain in which for any elements $a, b, c \in R$ that $aR + bR + cR = R$ there exists such element $p \in R$ that ${}_cA_{ap}$ and $pR + bR + cR = R$. Then R/cR is an exchange ring for every $c \in R \setminus \{0\}$.*

Proof. Let $\bar{R} = R/cR$ and $\bar{a}\bar{R} + \bar{b}\bar{R} = \bar{R}$, where $\bar{a} = a + cR$, $\bar{b} = b + cR$. Then $aR+bR+cR = R$ and exists such element $p \in R$ that $c = rs$, where $rR + apR = R$ and $s'R + apR \neq R$ for each $s'R$ such that $sR \subset s'R \neq R$ and $pR+bR+cR = R$. Obviously, we have $ru+sv = 1$. Since $\bar{r}\bar{u}+\bar{s}\bar{v} = 1$ and $\bar{r}\bar{s} = \bar{0}$, we have $\bar{r}^2\bar{u} = \bar{r}$ and $\bar{s}^2\bar{v} = \bar{s}$.

Notice $\bar{e} = \bar{s}\bar{v}$. Obviously $\bar{1} - \bar{e} = \bar{r}\bar{u}$ and $\bar{s}\bar{R} = \bar{e}\bar{R}$, $(\bar{1} - \bar{e})\bar{R} = \bar{r}\bar{R}$.

Since, by proposition 4, we have $sR+bR = R$ then $s\alpha+bp = 1, \alpha, \beta \in R$. Since $rs\alpha + rbp = r$, we have $\bar{r}\bar{b}\bar{p} = \bar{r}$ and $\bar{r}\bar{R} \subset \bar{b}\bar{R}$. Since $rR + apR = R$, $t, k \in R$. Since $rsp+apsk = s$ we have $\bar{s}\bar{R} \subset \bar{a}\bar{R}$. We proved if $\bar{a}\bar{R}+\bar{b}\bar{R} = \bar{R}$ then exists idempotents $\bar{e} \in \bar{a}\bar{R}$ and $\bar{1} - \bar{e} \in \bar{b}\bar{R}$. By proposition 3, R/cR is an exchange ring. Theorem is proved completely. □

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