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Preradicals, closure operators in R-Mod and connection between them

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Abstract. For a module category R-Mod the class $\mathbb{PR}$ of preradicals and the class $\mathbb{CO}$ of closure operators are studied. The relations between these classes are realized by three mappings: $\Phi : \mathbb{CO} \to \mathbb{PR}$ and $\Psi_1, \Psi_2 : \mathbb{PR} \to \mathbb{CO}$. The impact of these mappings on the operations in $\mathbb{PR}$ and $\mathbb{CO}$ (meet, join, product, coproduct) is investigated. It is established that in most cases the considered mappings preserve the lattice operations (meet and join), while the other two operations are converted one into another (i.e. the product into the coproduct and vice versa).

0. Introduction and preliminary notions

In this work the preradicals and closure operators of a module category R-Mod are studied ([1–6]). Three known mappings $\Phi : \mathbb{CO} \to \mathbb{PR}$, $\Psi_1, \Psi_2 : \mathbb{PR} \to \mathbb{CO}$ realize the connection between the class $\mathbb{PR}$ of preradicals and the class $\mathbb{CO}$ of closure operators of R-Mod ([1]). We study the influence of these mappings on the operations in $\mathbb{PR}$ and $\mathbb{CO}$: meet, join, product and coproduct ([1, 2]).

At first we remind some necessary notions. Let R be a ring with unity and R-Mod be the category of unitary left R-modules. For every module $M \in R$-Mod we denote by $\mathbb{L}(M)$ the lattice of submodules of M.

A *preradical* r of R-Mod is a subfunctor of identity functor of R-Mod, i.e. $r(M) \subseteq M$ and $f(r(M)) \subseteq r(M')$ for every R-morphism $f : M \to M'$ ([3, 4]).

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Let $\mathbb{PR}$ be the class of all preradicals of R-Mod. In $\mathbb{PR}$ four operations are defined ([3]):

– the *meet* $\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha$ of the family $\{r_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{PR}$:

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\Big)(M) = \bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M);$$

– the *join* $\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha$ of the family $\{r_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{PR}$:

$$\Big(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\Big)(M) = \sum_{\alpha\in\mathfrak{A}} r_\alpha(M);$$

– the *product* $r \cdot s$ of preradicals $r, s \in \mathbb{PR}$:

$$(r \cdot s)(M) = r\big(s(M)\big);$$

– the *coproduct* $r : s$ of preradicals $r, s \in \mathbb{PR}$:

$$[(r : s)(M)] \,/\, s(M) = r\big(M/\, s(M)\big),$$

for every module $M \in R$-Mod.

In $\mathbb{PR}$ the order relation "$\leqslant$" is defined by the rule:

$$r \leqslant s \Leftrightarrow r(M) \subseteq s(M) \text{ for every } M \in R\text{-Mod}.$$

The class $\mathbb{PR}$ is a "big" complete lattice with respect to the operations "$\wedge$" and "$\vee$".

A *closure operator* of R-Mod is a function C which associates to every pair $N \subseteq M$, where $N \in \mathbb{L}(M)$, a submodule $C_M(N) \subseteq M$ with the conditions:

(c_1) $N \subseteq C_M(N)$;

(c_2) if $N_1 \subseteq N_2$, then $C_M(N_1) \subseteq C_M(N_2)$;

(c_3) for every R-morphism $f : M \to M'$ and $N \in \mathbb{L}(M)$ we have $f\big(C_M(N)\big) \subseteq C_{M'}\big(f(N)\big)$.

We denote by $\mathbb{CO}$ the class of all closure operators of R-Mod. In this class also four operations are defined ([1]):

– the *meet* $\bigwedge\limits_{\alpha\in\mathfrak{A}} C_\alpha$ of the family $\{C_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}$:

$$\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M(N) = \bigcap_{\alpha\in\mathfrak{A}} [(C_\alpha)_M(N)];$$

– the *join* $\bigvee\limits_{\alpha \in \mathfrak{A}} C_\alpha$ of the family $\{C_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}$:

$$\Big(\bigvee_{\alpha \in \mathfrak{A}} C_\alpha\Big)_M (N) = \sum_{\alpha \in \mathfrak{A}} [(C_\alpha)_M (N)];$$

– the *product* $C \cdot D$ of the operators $C, D \in \mathbb{CO}$:

$$(C \cdot D)_M(N) \;=\; C_M\big(D_M(N)\big);$$

– the *coproduct* $C \;\#\; D$ of the operators $C, D \in \mathbb{CO}$:

$$(C \;\#\; D)_M(N) = C_{D_M(N)}(N),$$

for every $N \subseteq M$.

In $\mathbb{CO}$ the order relation "$\leqslant$" is defined as follows:

$$C \;\leqslant\; D \Leftrightarrow C_M(N) \;\subseteq\; D_M(N) \text{ for every } N \subseteq M.$$

The class $\mathbb{CO}$ is a "big" complete lattice with respect to the operations "$\wedge$" and "$\vee$".

The connection between the classes $\mathbb{CO}$ and $\mathbb{PR}$ for a fixed module category R-Mod is realized by the following three mappings ([1, 4]):

1) $\Phi : \mathbb{CO} \to \mathbb{PR}$, where we denote $\Phi(C) = r_C$ for every $C \in \mathbb{CO}$ and define:
$$r_C(M) = C_M(0)$$
for every $M \in R$-Mod;

2) $\Psi_1 : \mathbb{PR} \to \mathbb{CO}$, where $\Psi_1(r) = C^r$ for every $r \in \mathbb{PR}$ and
$$[(C^r)_M(N)] \,/\, N = r(M/\,N)$$
for every $N \subseteq M$;

3) $\Psi_2 : \mathbb{PR} \to \mathbb{CO}$, where $\Psi_2(r) = C_r$ for every $r \in \mathbb{PR}$ and
$$(C_r)_M(N) = N + r(M)$$
for every $N \subseteq M$.

It is well known that C^r is the *greatest* among the operators $C \in \mathbb{CO}$ with the property $\Phi(C) = r$ and, dually, C_r is the *least* among the operators $C \in \mathbb{CO}$ for which $\Phi(C) = r$. So, every preradical $r \in \mathbb{PR}$ gives the equivalence class $[C_r, C^r]$ in $\mathbb{CO}$, and every operator $C \in \mathbb{CO}$ defines the class $[C_{r_C}, C^{r_C}]$ in which it is contained. The closure operators of the form C^r are called *maximal*. An operator $C \in \mathbb{CO}$ is maximal if and only if $C_M(N)\,/\,N = C_{M/N}(\bar{0})$ for every $N \subseteq M$ $\big($or: $C_M(N)\,/\,K = C_{M/K}(N/K)$ for every $K \subseteq N \subseteq M$ [6]$\big)$.

We denote by $\mathbb{M}ax\,(\mathbb{CO})$ the class of maximal closure operators of R-Mod. The mappings Φ and Ψ_1 define a monotone bijection $\mathbb{M}ax\,(\mathbb{CO}) \cong \mathbb{PR}$. Dually, the closure operators of the form C_r are called *minimal* and the mappings Φ and Ψ_2 define a monotone bijection $\mathbb{M}in\,(\mathbb{CO}) \cong \mathbb{PR}$.

We remind also that the preradical $r \in \mathbb{PR}$ is called *cohereditary* if $r(M/N) = \big(r(M) + N\big)\,/\,N$ for every $N \subseteq M$ ([3, 4]).

1. The mapping Φ and its impact on the operations of $\mathbb{CO}$

We begin with the mapping $\Phi : \mathbb{CO} \to \mathbb{PR}$, where $\Phi(C) = r_C$ and $r_C(M) = C_M(0)$ for every $C \in \mathbb{CO}$ and $M \in R$-Mod. We will study the behaviour of this mapping relative to the operations in $\mathbb{CO}$ ([1, 2]).

Proposition 1.1. *The mapping Φ preserves the operation of* ***meet****, i.e.*

$$\Phi\,\Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big) = \bigwedge_{\alpha\in\mathfrak{A}} \big[\Phi\,(C_\alpha)\big]$$

for every family of operators $\{C_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}$.

Proof. For every $M \in R$-Mod from the definitions it follows:

$$\big(r_{\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha}\big)\,(M) = \Big(\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M(0) = \bigcap_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_M(0)\big] =$$
$$= \bigcap_{\alpha\in\mathfrak{A}} \big[r_{C_\alpha}(M)\big] = \Big(\bigwedge_{\alpha\in\mathfrak{A}} r_{C_\alpha}\Big)(M).$$

Thus $r_{\bigwedge_{\alpha\in\mathfrak{A}} C_\alpha} = \bigwedge_{\alpha\in\mathfrak{A}} r_{C_\alpha}$, so by our notations this proves the proposition. □

Proposition 1.2. *The mapping Φ preserves the operation of* ***join****, i.e.*

$$\Phi\,\Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big) = \bigvee_{\alpha\in\mathfrak{A}} \big[\Phi\,(C_\alpha)\big]$$

for every family of operators $\{C_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{CO}$.

Proof. For every $M \in R$-Mod by definitions we have:

$$\big(r_{\bigvee_{\alpha\in\mathfrak{A}} C_\alpha}\big)\,(M) = \Big(\bigvee_{\alpha\in\mathfrak{A}} C_\alpha\Big)_M(0) = \sum_{\alpha\in\mathfrak{A}} \big[(C_\alpha)_M(0)\big] =$$
$$= \sum_{\alpha\in\mathfrak{A}} \big[r_{C_\alpha}(M)\big] = \Big(\bigvee_{\alpha\in\mathfrak{A}} r_{C_\alpha}\Big)(M),$$

therefore $r_{\bigvee_{\alpha\in\mathfrak{A}} C_\alpha} = \bigvee_{\alpha\in\mathfrak{A}} r_{C_\alpha}$, proving the proposition. □

Thus the mapping Φ preserves the lattice operations "$\wedge$" and "$\vee$". Now we consider the other two operations: product and coproduct.

Proposition 1.3. *The mapping Φ converts the coproduct of $\mathbb{CO}$ into the product of $\mathbb{PR}$, i.e.*

$$\Phi\,(C \,\#\, D) = \Phi\,(C) \cdot \Phi\,(D)$$

for every operators $C, D \in \mathbb{CO}$.

Proof. For every $M \in R$-Mod we have:

$$r_{C\#D}(M) = (C \,\#\, D)_M(0) = C_{D_M(0)}(0),$$

$$(r_C \cdot r_D)\,(M) = r_C\big(r_D(M)\big) = r_C\big(D_M(0)\big) = C_{D_M(0)}(0).$$

Thus $r_{C\,\#\,D}\,(M) = (r_C \cdot r_D)\,(M)$ for every $M \in R$-Mod, i.e. $r_{C\,\#\,D} = r_C \cdot r_D$, as affirms the proposition. $\square$

Proposition 1.4. *For every operators $C, D \in \mathbb{CO}$ the relation is true:*

$$\Phi\,(C \cdot D) \leqslant \Phi\,(C) \,:\, \Phi\,(D).$$

*If C is a **maximal** closure operator, then for every operator $D \in \mathbb{CO}$ the equality holds:*

$$\Phi\,(C \cdot D) = \Phi\,(C) \,:\, \Phi\,(D).$$

Proof. By definitions we have for every $M \in R$-Mod:

$$r_{C\,\cdot\,D}\,(M) = (C \cdot D)_M(0) = C_M\big(D_M(0)\big),$$

$$[\,(r_C : r_D)\,(M)\,]\,/\,r_D(M) = r_C\big(M/\,r_D(M)\big) =$$

$$= r_C\big(M/D_M(0)\big) = C_{M/D_M(0)}(\bar{0}).$$

Therefore $[r_{C\,\cdot\,D}\,(M)\,]\,/\,r_D\,(M) = \big[C_M\big(D_M(0)\big)\big]/\,D_M(0)$. From the definition of closure operator (condition (c_3)) it follows that $\big[\,C_M\big(D_M(0)\big)\,\big]\,/\,D_M(0) \subseteq C_{M/D_M(0)}(\bar{0})$. So from the foregoing it follows that $[r_{C\,\cdot\,D}\,(M)]\,/\,r_D\,(M) \subseteq [(r_C : r_D)\,(M)]\,/\,r_D(M)$. Hence $r_{C\,\cdot\,D}\,(M) \subseteq (r_C : r_D)\,(M)$ for every $M \in R$-Mod, i.e. $r_{C\,\cdot\,D} \leqslant r_C : r_D$, which proves the first statement.

Now we suppose that the operator C is *maximal*, i.e. $\big[C_M(N)\big]/\,N = C_{M/N}(\bar{0})$ for every $N \subseteq M$. For $N = D_M(0)$ we obtain $\big[C_M\big(D_M(0)\big)\big]/\,D_M(0) = C_{M/D_M(0)}(\bar{0})$, which means that $[\,r_{C\,\cdot\,D}\,(M)\,]/\,r_D\,(M) = [\,(r_C : r_D)\,(M)\,]\,/\,r_D\,(M)$. Therefore $r_{C\,\cdot\,D}\,(M) = (r_C : r_D)\,(M)$ for every $M \in R$-Mod, i.e. $r_{C\,\cdot\,D} = r_C : r_D$, proving the second statement. $\square$

2. The mapping Ψ_1 and its impact on the operations of $\mathbb{PR}$

In continuation we consider the mapping $\Psi_1 : \mathbb{PR} \to \mathbb{CO}$, which is defined by the rule $\Psi_1(r) = C^{\,r}$ and $[\,(C^{\,r})_M(N)\,]\,/\,N = r(M/\,N)$ for every $r \in \mathbb{PR}$ and $N \subseteq M$. We verify the influence of this mapping on the operations of $\mathbb{PR}$.

Proposition 2.1. *The mapping Ψ_1 preserves the operation of **meet**, i.e.*

$$\Psi_1\,\big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big) = \bigwedge_{\alpha\in\mathfrak{A}}\,[\Psi_1\,(r_\alpha)]$$

for every family of preradicals $\{r_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{PR}$.

Proof. By definitions for every $N \subseteq M$ we have:

$$[(C^{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha})_M(N)\,]\,/\,N = (\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha)(M/\,N) = \bigcap_{\alpha\in\mathfrak{A}}\,[r_\alpha(M/\,N)] =$$

$$= \bigcap_{\alpha\in\mathfrak{A}}[((C^{\,r_\alpha})_M(N))/\,N] = [\bigcap_{\alpha\in\mathfrak{A}}((C^{\,r_\alpha})_M(N))]/\,N = [(\bigwedge_{\alpha\in\mathfrak{A}} C^{\,r_\alpha})_M(N)]/\,N.$$

Hence $(C^{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha})_M(N) = (\bigwedge_{\alpha\in\mathfrak{A}} C^{\,r_\alpha})_M(N)$ for every $N \subseteq M$, i.e. $C^{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha} = \bigwedge_{\alpha\in\mathfrak{A}} C^{\,r_\alpha}$, which proves the proposition. □

Proposition 2.2. *The mapping Ψ_1 preserves the operation of **join**, i.e.*

$$\Psi_1\,\big(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\big) = \bigvee_{\alpha\in\mathfrak{A}}\,[\Psi_1\,(r_\alpha)]$$

for every family of preradicals $\{r_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{PR}$.

Proof. For every $N \subseteq M$ by definitions we have:

$$[(C^{\bigvee_{\alpha\in\mathfrak{A}} r_\alpha})_M(N)]/\,N = (\bigvee_{\alpha\in\mathfrak{A}} r_\alpha)(M/\,N) = \sum_{\alpha\in\mathfrak{A}}\,[r_\alpha(M/\,N)] =$$

$$= \sum_{\alpha\in\mathfrak{A}}[((C^{r_\alpha})_M(N))/\,N] = [\sum_{\alpha\in\mathfrak{A}}((C^{\,r_\alpha})_M(N))]/\,N = [(\bigvee_{\alpha\in\mathfrak{A}} C^{\,r_\alpha})_M(N)]/\,N.$$

Hence $(C^{\bigvee_{\alpha\in\mathfrak{A}} r_\alpha})_M(N) = (\bigvee_{\alpha\in\mathfrak{A}} C^{\,r_\alpha})_M(N)$ for every $N \subseteq M$, which means that $C^{\bigvee_{\alpha\in\mathfrak{A}} r_\alpha} = \bigvee_{\alpha\in\mathfrak{A}} C^{\,r_\alpha}$, proving the proposition. □

Remark. Since the mappings Φ and Ψ_1 preserve the lattice operations "$\wedge$" and "$\vee$", the bijection $\mathbb{M}ax\,(\mathbb{CO}) \cong \mathbb{PR}$, which is defined by these mappings, is an *isomorphism* of complete "big" lattices.

Further we consider the effect of the mapping Ψ_1 on the rest of operations: product and coproduct.

Proposition 2.3. *The mapping Ψ_1 converts the product of $\mathbb{PR}$ into the coproduct of $\mathbb{CO}$, i.e.*

$$\Psi_1\,(r \cdot s) = \Psi_1\,(r) \,\#\, \Psi_1\,(s)$$

for every preradicals $r, s \in \mathbb{PR}$.

Proof. Let $r, s \in \mathbb{PR}$ and $N \subseteq M$. Then:

$$\big[(C^{\,r\cdot s})_{M}(N)\big]/\,N = (r \cdot s)\,(M/\,N) = r\big(s(M/\,N)\big).$$

From the other hand, by the definition of coproduct in $\mathbb{CO}$ we obtain:

$$\big[(C^{\,r} \# \, C^{\,s})_M(N)\big]/\,N = \big[\,(C^{\,r})_{(C^{\,s})_M(N)}(N)\,\big]/\,N =$$
$$= r\big[((C^{\,s})_M(N))/\,N\big] = r\big(s(M/\,N)\big).$$

Therefore $\big[(C^{\,r\cdot s})_{M}(N)\big]/\,N = \big[(C^{\,r} \# \, C^{\,s})_M(N)\big]/\,N$, and so $(C^{\,r\cdot s})_M(N) = (C^{\,r} \# \, C^{\,s})_M(N)$ for every $N \subseteq M$. This means that $C^{\,r\cdot s} = C^{\,r} \# \, C^{\,s}$, proving the proposition. $\square$

Proposition 2.4. *The mapping Ψ_1 converts the coproduct of $\mathbb{PR}$ into the product of $\mathbb{CO}$, i.e.*

$$\Psi_1\,(r : s) = \Psi_1\,(r) \cdot \Psi_1\,(s)$$

for every preradicals $r, s \in \mathbb{PR}$.

Proof. By definition $[\,(C^{\,r:s})_M(N)\,]\,/\,N = (r : s)\,(M/\,N)$ and

$$\big[(r : s)\,(M/\,N)\big]/\,s(M/\,N) = r\big[(M/\,N)\,/\,s(M/\,N)\big] =$$
$$= \big[(C^{\,r})_{M/N}\big(s(M/\,N)\big)\big]/\,s(M/N) \ \text{ for every } N \subseteq M.$$

Therefore $(r : s)\,(M/\,N) = (C^{\,r})_{M/N}\big(s(M/\,N)\big)$, i.e. $\big[(C^{\,r:s})_{M}(N)\big]/\,N = (C^{\,r})_{M/N}\big(s(M/\,N)\big)$.

Now we will use the fact that $C^{\,r}$ is a *maximal* closure operator (see Section 1), which in the situation $N \subseteq (C^{\,s})_M(N) \subseteq M$ implies the relations:

$$\big[(C^{\,r})_M\big((C^{\,s})_M(N)\big)\big]/\,N = (C^{\,r})_{M/N}\big[((C^{\,s})_M(N))/\,N\big] =$$
$$= (C^{\,r})_{M/N}\big(s(M/\,N)\big).$$

Comparing with the relation obtained above, now we have:

$$\left[(C^{r:s})_M(N)\right]/\,N = \left[(C^{r})_M\big((C^{s})_M(N)\big)\right]/\,N,$$

thus $(C^{r:s})_M(N) = (C^{r}\cdot C^{s})_M(N)$ for every $N \subseteq M$. This means that $C^{r:s} = C^{r}\cdot C^{s}$, proving the proposition. $\square$

3. The mapping Ψ_2 : impact on the operations of $\mathbb{PR}$

Finally, we study the mapping $\Psi_2 : \mathbb{PR} \to \mathbb{CO}$, where $\Psi_2(r) = C_r$ and $(C_r)_M(N) = N + r(M)$ for every $r \in \mathbb{PR}$ and $N \subseteq M$. We show the influence of Ψ_2 to the operations of $\mathbb{PR}$.

Proposition 3.1. *For every family of preradicals $\{r_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{PR}$ the relation is true:*

$$\Psi_2\,\big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big) \leqslant \bigwedge_{\alpha\in\mathfrak{A}} \left[\Psi_2\,(r_\alpha)\right].$$

If the lattice $\mathbb{L}(M)$ is infinite distributive (relative to meets) for every $M \in R$-Mod, then the equality holds:

$$\Psi_2\,\big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big) = \bigwedge_{\alpha\in\mathfrak{A}} \left[\Psi_2\,(r_\alpha)\right].$$

Proof. By definitions $\Psi_2\,\big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big) = C_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}$, where $\big(C_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}\big)_M(N) = N + \left[\big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\big)(M)\right] = N + \left[\bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M)\right]$ for every $N \subseteq M$.

From the other hand, $\bigwedge_{\alpha\in\mathfrak{A}} \left[\Psi_2\,(r_\alpha)\right] = \bigwedge_{\alpha\in\mathfrak{A}} C_{r_\alpha}$, where

$$\big(\bigwedge_{\alpha\in\mathfrak{A}} C_{r_\alpha}\big)_M(N) = \bigcap_{\alpha\in\mathfrak{A}} \left[(C_{r_\alpha})_M(N)\right] = \bigcap_{\alpha\in\mathfrak{A}} \left[N + r_\alpha(M)\right]$$

for every $N \subseteq M$. Since $N + \left[\bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M)\right] \subseteq \bigcap_{\alpha\in\mathfrak{A}} \left[N + r_\alpha(M)\right]$, we have $\big(C_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}\big)_M(N) \subseteq \big(\bigwedge_{\alpha\in\mathfrak{A}} C_{r_\alpha}\big)_M(N)$, i.e. $C_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha} \leqslant \bigwedge_{\alpha\in\mathfrak{A}} C_{r_\alpha}$, proving the first statement.

If $\mathbb{L}(M)$ is infinite distributive (relative to meets) for every $M \in R$-Mod, then $N + \left[\bigcap_{\alpha\in\mathfrak{A}} r_\alpha(M)\right] = \bigcap_{\alpha\in\mathfrak{A}} \left[N + r_\alpha(M)\right]$ for every $N \subseteq M$, which implies the equality $C_{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha} = \bigwedge_{\alpha\in\mathfrak{A}} C_{r_\alpha}$, proving the second statement. $\square$

Proposition 3.2. *The mapping* Ψ_2 *preserves the operation of* ***join****, i.e.*

$$\Psi_2\,\big(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\big) = \bigvee_{\alpha\in\mathfrak{A}} \big[\Psi_2\,(r_\alpha)\big]$$

for every family of preradicals $\{r_\alpha | \alpha \in \mathfrak{A}\} \subseteq \mathbb{PR}$*.*

Proof. For every $N \subseteq M$ we have:

$$\big(C_{\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha}\big)_M(N) = N + \big[\big(\bigvee_{\alpha\in\mathfrak{A}} r_\alpha\big)(M)\big] = N + \big[\sum_{\alpha\in\mathfrak{A}} r_\alpha(M)\big].$$

From the other hand:

$$\big(\bigvee_{\alpha\in\mathfrak{A}} C_{r_\alpha}\big)_M(N) = \sum_{\alpha\in\mathfrak{A}} \big[(C_{r_\alpha})_M(N)\big] = \sum_{\alpha\in\mathfrak{A}} \big[N + r_\alpha(M)\big] =$$

$$= N + \big[\sum_{\alpha\in\mathfrak{A}} r_\alpha(M)\big].$$

Therefore $\big(C_{\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha}\big)_M(N) = \big(\bigvee\limits_{\alpha\in\mathfrak{A}} C_{r_\alpha}\big)_M(N)$ for every $N \subseteq M$, i.e. $C_{\bigvee\limits_{\alpha\in\mathfrak{A}} r_\alpha} = \bigvee\limits_{\alpha\in\mathfrak{A}} C_{r_\alpha}$, which proves the proposition. $\square$

It remains to verify the impact of the mapping Ψ_2 on the operations of product and coproduct of $\mathbb{PR}$.

Proposition 3.3. *For every preradicals* $r, s \in \mathbb{PR}$ *the relation is true:*

$$\Psi_2\,(r \cdot s) \;\leqslant\; \Psi_2\,(r) \,\#\, \Psi_2\,(s).$$

Proof. Let $r, s \in \mathbb{PR}$ and $N \subseteq M$. Then:

$$\big(C_{r\,\cdot\,s}\big)_M(N) = N + \big[\,(r \,\cdot\, s)\,(M)\,\big] = N + r\big(s(M)\big),$$

$$(C_r \,\#\, C_s)_M(N) = (C_r)_{(C_s)_M(N)}(N) = (C_r)_{N+s(M)}(N) =$$

$$= N + r\big(N + s(M)\big).$$

Since $N + r\big(s(M)\big) \subseteq N + r\big(N + s(M)\big)$, we have $(C_{r\,\cdot\,s})_M(N) \subseteq (C_r \# C_s)_M\,(N)$ for every $N \subseteq M$, i.e. $C_{r\,\cdot\,s} \leqslant C_r \# C_s$, proving the proposition. $\square$

Proposition 3.4. *For every preradicals* $r, s \in \mathbb{PR}$ *the relation is true:*

$$\Psi_2\,(r : s) \geqslant \Psi_2\,(r) \cdot \Psi_2\,(s).$$

If the preradical $r \in \mathbb{PR}$ *is* ***cohereditary****, then for every* $s \in \mathbb{PR}$ *the equality holds:*

$$\Psi_2\,(r : s) = \Psi_2\,(r) \cdot \Psi_2\,(s).$$

Proof. If $r, s \in \mathbb{PR}$ and $N \subseteq M$, then $(C_{r:s})_M(N) = N + [(r:s)(M)]$ and $[(r:s)(M)] / s(M) = r(M / s(M))$. Therefore:

$$[\,(C_{r:s})_M(N)\,] / s(M) = [\,N + (r:s)(M)\,] / s(M) =$$
$$= \big[(N + s(M)) / s(M)\,\big] + \big[((r:s)(M)) / s(M)\,\big] =$$
$$= \big[(N + s(M)) / s(M)\,\big] + \big[\,r(M / s(M))\big].$$

On the other hand, for the product of respective operators we have:

$$(C_r \cdot C_s)_M(N) = (C_r)_M\,[\,(C_s)_M(N)\,] =$$
$$= (C_r)_M[N + s(M)\,] = N + s(M) + r(M).$$

Hence:

$$[\,(C_r \cdot C_s)_M(N)\,] / s(M) = [N + s(M) + r(M)] / s(M) =$$
$$= \big[(N + s(M)) / s(M)\big] + \big[(r(M) + s(M)) / s(M)\big].$$

From the definition of preradical we have the inclusion:

$$[r(M) + s(M)] / s(M) \;\subseteq\; r\big(M / s(M)\big),$$

and comparing with the previous relations we obtain:

$$[\,(C_{r:s})_M(N)\,] / s(M) \;\supseteq\; [\,(C_r \cdot C_s)_M(N)\,] / s(M).$$

Therefore $(C_{r:s})_M(N) \supseteq (C_r \cdot C_s)_M(N)$ for every $N \subseteq M$, which means that $C_{r:s} \geqslant C_r \cdot C_s$, proving the first statement.

If we suppose that the preradical $r \in \mathbb{PR}$ is *cohereditary*, then for every $s \in \mathbb{PR}$ by previous calculation we obtain the equality $[(C_{r:s})_M(N)\,] / s(M) = [\,(C_r \cdot C_s)_M(N)\,] / s(M)$, which implies the equality $C_{r:s} = C_r \cdot C_s$. □

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