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On weakly semisimple derivations of the polynomial ring in two variables

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathbb{K}[x,y]$ the polynomial ring. Every element $f \in \mathbb{K}[x,y]$ determines the Jacobian derivation D_f of $\mathbb{K}[x,y]$ by the rule $D_f(h) = detJ(f,h)$, where $J(f,h)$ is the Jacobian matrix of the polynomials f and h. A polynomial f is called weakly semisimple if there exists a polynomial g such that $D_f(g) = \lambda g$ for some nonzero $\lambda \in \mathbb{K}$. Ten years ago, Y. Stein posed a problem of describing all weakly semisimple polynomials (such a description would characterize all two dimensional nonabelian subalgebras of the Lie algebra of all derivations of $\mathbb{K}[x,y]$ with zero divergence). We give such a description for polynomials f with the separated variables, i.e. which are of the form: $f(x,y) = f_1(x)f_2(y)$ for some $f_1(t), f_2(t) \in \mathbb{K}[t]$.

Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathbb{K}[x,y]$ the polynomial ring. For any polynomials $f, g \in \mathbb{K}[x,y]$ let us denote $[f,g] = detJ(f,g)$, where $detJ(f,g)$ is their Jacobian matrix. The vector $\mathbb{K}$-space $\mathbb{K}[x,y]$ with operation $(f,g) \to [f,g]$ is a Lie algebra over $\mathbb{K}$. The center of this algebra coincides with $\mathbb{K}$, the quotient algebra $\mathbb{K}[x,y]/\mathbb{K}$ is isomorphic to the Lie algebra $sa_2(\mathbb{K})$ of all derivations of $\mathbb{K}[x,y]$ with zero divergence (see, for example, [2]). For a fixed polynomial f, the linear

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operator D_f on $\mathbb{K}[x, y]$ defined by the rule $D_f(h) = [f, h]$ is a $\mathbb{K}$-derivation of the ring $\mathbb{K}[x, y]$. The derivation D_f is called the Jacobian derivation.

In [7], a polynomial $f \in \mathbb{K}[x, y]$ was called weakly semisimple if there exists a polynomial g such that $D_f(g) = \lambda g$ for a nonzero $\lambda \in \mathbb{K}$, the polynomial g here is an eigenfunction for f with respect to λ. The latter means that every weakly semisimple polynomial f induces the Jacobian derivation D_f which can be included in a two-dimensional nonabelian subalgebra $L_f =< D_f > \leftthreetimes < D_g >$ of the Lie algebra $sa_2(\mathbb{K})$. The structure of two-dimensional (abelian and nonabelian) subalgebras of the Lie algebra $sa_2(\mathbb{K})$ is very important for better understanding the structure of subalgebras of $sa_2(\mathbb{K})$, and it is closely connected with the jacobian conjecture for $n = 2$.

In [7], some properties of weakly semisimple polynomials were pointed out and a question was asked about their description. We give a description of weakly semisimple polynomials f with separated variables, i.e. of the form $f(x, y) = f_1(x)f_2(y)$.

We use standard notations. Let us remind that a polynomial $f \in \mathbb{K}[x, y]$ is called closed if there exist no polynomials $F(t) \in \mathbb{K}[t]$ and $g(x, y) \in \mathbb{K}[x, y]$ such that $degF(t) \geqslant 2$, and that $f(x, y) = F(g(x, y))$. The polynomial f has a Jacobian mate g if $[f, g] = 1$. If D is a derivation of $\mathbb{K}[x, y]$ and $D(f) = hf$ for some $h \in \mathbb{K}[x, y]$, then f will be called a Darboux polynomial for D and h its cofactor.

1. Preliminaries

We will often use the next statements which can be found in [7].

Lemma 1. *Let $f, g \in \mathbb{K}[x, y]$ be such polynomials that g is irreducible and $[f, g] = hg$ for some $h \in \mathbb{K}[x, y]$. Then there exists $c \in \mathbb{K}$ such that g divides the polynomial $f - c$.*

The first statement of the next lemma summarizes Proposition 2.1 from [7]

Lemma 2. *Let $f \in \mathbb{K}[x, y]$ be a polynomial such that there exist a polynomial $g \in \mathbb{K}[x, y], g \neq 0$ and $\lambda \in \mathbb{K}^\star$ with $D_f^n(g) = \lambda g$ for some $n \geqslant 1$. Then:*

1) f is a closed polynomial;

2) the polynomial $f - c$ is square-free for any $c \in \mathbb{K}$, i.e $f - c$ is not divisible by square of any irreducible polynomial.

Proof. 1) Let on the contrary $f = F(h)$ for some $F(t) \in \mathbb{K}[t], degF \geqslant 2$ and $h \in \mathbb{K}[x, y]$. Then

$$D_f(g) = [F(h), g] = F'(h)[h, g] = F'(h)D_h(g).$$

Analogously we get the next relation

$$D_f^2(g) = D_f(F'(h)D_h(g)) = [F(h), F'(h)D_h(g)] =$$

$$= F'(h)[F(h), D_h(g)] = F'(h)^2 D_h^2(g).$$

Using induction on k one can easily show that $D_f^k(g) = F'(h)^k D_h^k(g)$. By the conditions of this lemma

$$D_f^n(g) = F'(h)^n D_h^n(g) = \lambda g \tag{1}$$

Write g as $g = F'(h)^m \cdot u$, where $m \geqslant n$ by (1) and the polynomial u is not divisible by any nonconstant polynomial of h. Then

$$D_h(g) = [h, F'(h)^m u] = F'(h)^m [h, u] = F'(h)^m D_h(u).$$

Using induction on k it is easily to show that $D_h^k(g) = F'(h)^m D_h^k(u)$. Inserting the last equality in (1) we obtain

$$D_f^n(g) = F'(h)^n D_h^n(g) = F'(h)^n F'(h)^m D_h^n(u) = \lambda g.$$

But $g = F'(h)^m u$ and therefore $F'(h)^{m+n} D_h(u) = \lambda F'(h)^m u$
The last equality shows that u divides on $F'(h)$ which contradicts to our choice of the polynomial u. The obtained contradiction proves that f is a closed polynomial.

2) Let on the contrary $c \in \mathbb{K}$ be such an element that we have equality $f - c = w^k f_1$, where $w, f_1 \in \mathbb{K}[x, y], k \geqslant 2$ and f_1 is not divisible by w. Write $g = w^m g_1$, where $m \geqslant 0, g_1 \in \mathbb{K}[x, y]$ and g_1 is not divisible by w. Then

$$D_f(g) = [f, g] = [f - c, g] = [w^k f_1, w^m g_1] = w^k [f_1, w^m g_1] + f_1 [w^k, w^m g_1]$$

$$= w^{k+m}[f_1, g_1] + w^k g_1 m w^{m-1}[f_1, w] + f_1 w^m k w^{k-1}[w, g_1] = w^{k+m-1} t_1$$

for some $t_1 \in \mathbb{K}[x, y]$. Thus, $D_f(g) = w^{k+m-1} t_1$.

Using induction on i it is easily to show that $D_f^i(g) = w^{m+ki-1} t_i$ for some $t_i \in \mathbb{K}[x, y]$, $i = 2, 3, \ldots$. But then for $i = n$ we obtain $D_f^n(g) = w^{m+kn-1} t_n = \lambda w^m g_1$. From here, it follows that $\lambda g_1 = w^{kn-1} t_n$ which is impossible since $k \geqslant 2$ and g_1 is not divisible by w because of choice g_1. This contradiction shows that $f - c$ is not divisible by square of any irreducible polynomial. □

Corollary 1. *If $f = f_1(x)f_2(y)$ is a weakly semisimple polynomial then the polynomials $f_1(t), f_2(t) \in \mathbb{K}[t]$ have no multiple roots.*

Proof. Since $[f, g] = g$ then putting in Lemma 2 $n = 1$ and $c = 0$ we have that $f(x, y) = f_1(x)f_2(y)$ is not divisible by square of any irreducible polynomial. The latter means that $f_1(t)$ and $f_2(t)$ do not have multiple roots. □

Lemma 3 (see [3])**.** *Let $f(x, y) = \prod_{i=1}^{n} l_i(x, y)^{r_i}$ be a product of linear polynomials $l_i(x, y), i = 1, \ldots, n$. Assume that $n \geqslant 2$ and the lines $l_1(x, y) = 0$ and $l_2(x, y) = 0$ do intersect. If $\gcd(r_1, \ldots, r_n) = 1$ then $f(x, y)$ is a closed polynomial and the polynomial $f(x, y) + c$ is irreducible for any $c \in \mathbb{K}^\star$.*

Corollary 2. *Let $p(x) \in \mathbb{K}[x]$ and $q(y) \in \mathbb{K}[y]$ be nonconstant polynomials such that $p(x)$ has no multiple roots. Then the polynomial $p(x)q(y) + c$ is irreducible for any $c \in \mathbb{K}^\star$.*

Lemma 4. *(see, for instance [1]). Let D be a derivation of the polynomial ring $\mathbb{K}[x, y]$ and g a Darboux polynomial for D. Then any divisor of g is a Darboux polynomial for D. If f_1, f_2 are Darboux polynomials for D with cofactors h_1 and h_2 respectively, then $f_1 f_2$ is a Darboux polynomial for D with the cofactor $h_1 + h_2$.*

2. Weakly semisimple polynomials of the form $f_1(x)f_2(y)$

In this section we give a description of all weakly semisimple polynomials $f \in \mathbb{K}[x, y]$ of the form $f(x, y) = f_1(x)f_2(y)$ (that is with separated variables) and their eigenfunctions $g(x, y) \in \mathbb{K}[x, y]$ such that $[f, g] = \lambda g, \lambda \in \mathbb{K}^\star$. In fact, we can assume that $\lambda = 1$ because in other case we can consider $\lambda^{-1} f$ instead of f. So, we will consider only the case $[f, g] = g$.

Lemma 5. *Let a polynomial $g = g(x, y)$ satisfies the relation $[f, g] = g$, where $f(x, y) = f_1(x)f_2(y) \in \mathbb{K}[x, y]$. Then the only irreducible factors of the polynomial $g(x, y)$ are the polynomials of the form $\delta(f(x, y) + c)$ with $\delta, c \in \mathbb{K}^\star$ or $\beta(x - c_1), \gamma(y - c_2)$ with $\beta, \gamma \in \mathbb{K}^\star$ and c_1, c_2 satisfying $f_1(c_1) = 0, f_2(c_2) = 0$.*

Proof. Let h be an irreducible factor of g. By Lemma 4 h is a Darboux polynomial for D_f, i.e. $[f, h] = hu, u \in \mathbb{K}[x, y]$, so by Lemma 1 there exists $c \in \mathbb{K}$ such that $f - c$ is divisible by h. If $c \neq 0$, then the polynomial

$f - c$ is irreducible by Lemma 3 and therefore $h = (f + c)\delta$ for some $\delta \in \mathbb{K}^\star$. Let $c = 0$. As h is irreducible and divides $f(x, y) = f_1(x)f_2(y)$, then h is linear of the form $\beta(x - c_1)$ or $\gamma(y - c_2)$. □

Lemma 6. *Let $f(x, y) = f_1(x)f_2(y)$ and $g = g(x, y)$ be polynomials from $\mathbb{K}[x, y]$ satisfying the relation $[f, g] = g$. Then $g = F(f)g_1g_2$, where $F(t) \in \mathbb{K}[t]$ and $g_1 = g_1(x), g_2 = g_2(y)$ satisfying the equality $[f, g_1g_2] = g_1g_2$.*

Proof. Write the polynomial g as a product $g = h_1^{k_1} \cdots h_s^{k_s}$ of powers of irreducible polynomials $h_1, \dots, h_s$. By Lemma 5 the polynomial h_i is either of the form $h_i = \delta_i(f + c_i)$ for $\delta_i \in \mathbb{K}^*, c_i \in \mathbb{K}$ or of the form $h_i = \alpha_i(x - d_i)$, or $h_i = \beta_i(y - r_i)$ for $\alpha_i, \beta_i \in \mathbb{K}^*$. The elements $d_i, r_i \in \mathbb{K}$ are such that $f_1(d_i) = 0, f_2(r_i) = 0$. Denote by $F(f)$ the product of all irreducible divisors of the polynomial g which are of the form $\delta_i(f + c_i)$ (this is a polynomial of f). Group other irreducible divisors of g and write their product in the form $g_1(x)g_2(y)$. Then

$$[f, F(f)g_1g_2] = F(f)[f, g_1g_2] = F(f)g_1g_2.$$

From this it follows the equality $[f, g_1g_2] = g_1g_2$. □

Thus the problem of finding polynomials f and g satisfying the relation $[f, g] = g$, is reduced to searching polynomials $f_1(x)f_2(y), g_1(x)g_2(y)$ with separated variables such that

$$[f_1(x)f_2(y), g_1(x)g_2(y)] = g_1(x)g_2(y).$$

We have from the last relation that

$$[f_1f_2, g_1g_2] = f_1[f_2, g_1g_2] + f_2[f_1, g_1g_2] = f_1g_2[f_2, g_1] + f_2g_1[f_1, g_2]. \quad (2)$$

Further, notice that $[f_2, g_1] = \begin{vmatrix} 0 & \frac{\partial f_2}{\partial y} \\ \frac{\partial g_1}{\partial x} & 0 \end{vmatrix} = -f_2'g_1'$ and analogously $[f_1, g_2] = f_1'g_2'$ (we omit here signs of variables while differentiating). Therefore we obtain from (2) that

$$[f_1f_2, g_1g_2] = -f_1f_2'g_2g_1' + f_2f_1'g_1g_2' = g_1g_2.$$

Dividing both parts of this equality by g_1g_2 we get

$$-f_1f_2'\frac{g_1'}{g_1} + f_2f_1'\frac{g_2'}{g_2} = 1. \quad (3)$$

Note that every linear factor of g_1 is a divisor of f_1 by Lemma 5, so f_1g_1' is divisible by g_1 and analogously f_2g_2' is divisible by g_2. Therefore (3) is in fact a relation for polynomials with separated variables.

Lemma 7. *Let $a(x), c(x) \in \mathbb{K}[x]$ and $b(y), d(y) \in \mathbb{K}[y]$ be polynomials such that*

$$a(x)b(y) + c(x)d(y) = 1. \tag{4}$$

Then either $a, c \in \mathbb{K}$ or $b, d \in \mathbb{K}$.

Proof. Let us differentiate the equality (4) on the variable x. Then we have $a'(x)b(y) + c'(x)d(y) = 0$. If $a'(x) = 0$ and $c'(x) = 0$ then $a, c \in \mathbb{K}$ and all is done. If $c'(x) \neq 0$, then $\frac{a'(x)}{c'(x)} = -\frac{d(y)}{b(y)}$ and therefore there exists $\lambda \in \mathbb{K}$ such that $d = \lambda b$. Substituting this equality into (4) we obtain $b(y)(a(x) + \lambda c(x)) = 1$. The latter equality implies $b, d \in \mathbb{K}$. $\square$

Lemma 8. *Let $f_1(x), g_1(x) \in \mathbb{K}[x]$ and $f_2(y), g_2(y) \in \mathbb{K}[y]$ be polynomials satisfying the equality (3). Then either $f_1(x)$ is linear and then $g_1(x) = c_1 f_1^l(x)$, or $f_2(y)$ is linear and then $g_2(y) = c_2 f_2^k(y)$ for some $c_1, c_2 \in \mathbb{K}^*, k, l, \in \mathbb{N}$.*

Proof. By Lemma 7 we obtain from (3) that either $f_1', \frac{f_1 g_1'}{g_1} \in \mathbb{K}$ or $f_2', \frac{f_2 g_2'}{g_2} \in \mathbb{K}$. For example, let the second case hold. Then $f_2 = \alpha y + \beta$ and $\frac{f_2 g_2'}{g_2} = \gamma \in \mathbb{K}$, for some $\alpha, \gamma \in \mathbb{K}^\star, \beta \in \mathbb{K}$. From the relation $f_2 g_2' = \lambda g_2$ we have that g_2 is divisible by g_2'. But the latter is possible only in the case $g_2 = \lambda(y + \delta)^k$ for some $\lambda \in \mathbb{K}^\star, \delta \in \mathbb{K}, k \in \mathbb{N}$.

From the equality

$$\frac{f_2 g_2'}{g_2} = \frac{(\alpha y + \beta) k \lambda (y + \delta)^{k-1}}{\lambda (y + \delta)^k} = \gamma \in \mathbb{K}$$

it follows that $(\alpha y + \beta)k = \gamma(y + \delta)$. But then $\gamma = k\alpha$ and $\gamma\delta = k\beta$ which gives us $\delta = \frac{\beta}{\alpha}$. The latter equality means that $g_2(y) = c_2 f_2^k(y)$ for some $c_2 \in \mathbb{K}^\star$. The case $f_1', \frac{f_1 g_1'}{g_1} \in \mathbb{K}$ can be analogously considered. $\square$

Theorem 1. *A polynomial $f(x, y) = f_1(x) f_2(y) \in \mathbb{K}[x, y]$ is weakly semisimple if and only if it has no multiple roots and at least one of the polynomials $f_1(x), f_2(y)$ is linear, and if for example $f_2(y) = ay + b, a, b \in \mathbb{K}$ and $\alpha_1, \cdots, \alpha_n$ are the roots of $f_1(x)$, then $l_i = \frac{1}{a f_1'(\alpha_i)} \in \mathbb{Z}, i = 1, \ldots, n$. Besides, if $g = g(x, y)$ is an eigenfunction for D_f with eigenvalue 1, then $g = F(f) f_2(y)^k \prod_{i=1}^n (x - \alpha_i)^{k - l_i}$, where $F(t) \in \mathbb{K}[t], k \in \mathbb{N}$ such that $k \geqslant l_i, i = 1, \ldots, n$.*

Proof. $\Rightarrow$ Let $f = f(x, y)$ be a weakly semisimple polynomial of the form $f(x, y) = f_1(x) f_2(y)$, i.e. such that for some $g \in \mathbb{K}[x, y]$ it holds

$[f,g] = detJ(f,g) = g$. By Corollary 1 f has no multiple roots and taking into account Lemma 8 we see that at least one of the polynomials $f_1(x), f_2(y)$ is linear. Let for instance $f_2(y) = ay+b$ for some $a, b \in \mathbb{K}$. By Lemma 6 one can assume $g = g_1(x)g_2(y)$ and any nonconstant polynomial of f does not divide g. Using Lemma 8 we can assume that $g_2(y) = df_2(y)^k$ for some $d \in \mathbb{K}^\star$ and $k \in \mathbb{N}$. Then the equality (3) can be written in the form

$$-f_1 a\frac{g_1'}{g_1} + \frac{f_1' f_2 dk f_2^{k-1} a}{df_2^k} = 1$$

Rewriting the latter relation we obtain

$$\frac{-af_1g_1'}{g_1} + akf_1' = 1,$$

and as consequence

$$\frac{akf_1' - 1}{f_1} = \frac{ag_1'}{g_1}. \tag{5}$$

The polynomial $f_1(x)$ has no multiple roots (by Lemma 6), let $\alpha_1, \ldots, \alpha_n$ be all the roots of $f_1(x)$. Then $f_1(x) = c_1(x-\alpha_1)\cdots(x-\alpha_n)$ for some element $c_1 \in \mathbb{K}^\star$, and taking into account the relation (5) we have $g_1(x) = d_1(x-\alpha_1)^{m_1}\cdots(x-\alpha_n)^{m_n}$ for some $d_1 \in \mathbb{K}, m_i \in \mathbb{N}\cup\{0\}$. The relation (5) can be rewritten in the form $\frac{akf_1'}{f_1} - \frac{1}{f_1} = \frac{ag_1'}{g_1}$. Substituting f_1 and g_1 from the last expressions, we have

$$\sum_{i=1}^{n} \frac{ak}{(x-\alpha_i)} - \sum_{i=1}^{n} \frac{1}{f_1'(\alpha_i)(x-\alpha_i)} = \sum_{i=1}^{n} \frac{am_i}{(x-\alpha_i)} \tag{6}$$

(We used the decomposition of the rational function $\frac{1}{f_1} = \frac{1}{c_1(x-\alpha_1)\cdots(x-\alpha_n)}$ into the sum of elementary fraction of the form $\frac{A_i}{x-\alpha_i}, i = 1, \ldots, n$).

The relation (6) implies $m_i = k - \frac{1}{af_1'(\alpha_i)}$ and since $m_i \geqslant 0, k \geqslant 1$ we have $l_i = \frac{1}{af_1'(\alpha_i)} \in \mathbb{Z}, i = 1, \ldots, n$. Since $m_i \geqslant 0$ we obtain $k \geqslant l_i$, $i = 1, \ldots, n$. But then $g_1(x) = d_1 \prod_{i=1}^{n} (x-\alpha_i)^{k-l_i}, g_2(y) = (ay+b)^k, k \geqslant l_i$ and therefore by Lemma 6 $g(x,y) = F(f)(ay+b)^k \prod_{i=1}^{n} (x-\alpha_i)^{k-l_i}$.

$\Leftarrow$ Let $f = f_1(x)f_2(y)$ with $f_1(x) = c_1(x-\alpha_1)\cdots(x-\alpha_n)$ and $f_2 = (ay+b)$, and let $g(x,y) = F(f)g_1g_2$ where $g_1 = \prod_{i=1}^{n}(x-\alpha_i)^{k-l_i}$ and $g_2 = (ay+b)^k$, where $l_i = k - \frac{1}{af_1'(\alpha_i)}$ are integers. We will show that

$$[f, (ay+b)^k \prod_{i=1}^{n} (x-\alpha_i)^{k-l_i}] = (ay+b)^k \prod_{i=1}^{n} (x-\alpha_i)^{k-l_i}$$

Using the equality $f(x,y) = f_1(x)(ay+b)$ we have

$$\begin{aligned}
&[f,(ay+b)^k\prod_{i=1}^{n}(x-\alpha_i)^{k-l_i}]\\
&\qquad\qquad = [f_1(x)(ay+b),(ay+b)^k\prod_{i=1}^{n}(x-\alpha_i)^{k-l_i}]\\
&\qquad\qquad = f_1'(ay+b)ak(ay+b)^{k-1}g_1 - af_1g_1'(ay+b)^k\\
&\qquad\qquad = (ay+b)^k a(f_1'kg_1 - f_1g_1').
\end{aligned}$$

But the relation (5) yields $af_1'kg_1 - af_1g_1' = g_1$, so we get $[f, g_1g_2] = g_1g_2$ and the polynomial f is weakly semisimple. $\square$

The next statement allows us to produce infinitely many weakly semisimple polynomials from a given one.

Corollary 3. *Let $f(x,y)$ be a weakly semisimple polynomial and $g(x,y)$ be such that $[f,g] = g$. If polynomials p,q satisfy the condition $[p,q] = 1$, then $f(p,q)$ is weakly semisimple and $[f(p,q), g(p,q)] = g(p,q)$.*

Example 1. Let $f(x,y) = x(x-1)y$, $g(x,y) = x^{k+1}(x-1)^{k-1}y^k$, $k \in \mathbb{N}$. Then

$$\begin{aligned}
[f,g] &= [x(x-1)y, x^{k+1}(x-1)^{k-1}y^k]\\
&= [x(x-1)y, y^k]x^{k+1}(x-1)^{k-1} + [x(x-1)y, x^{k+1}(x-1)^{k-1}]y^k\\
&= y^k(k[x(x-1),y]x^{k+1}(x-1)^{k-1} + x(x-1)[y, x^{k+1}(x-1)^{k-1}])\\
&= y^k(x^{k+2}(x-1)^{k-1} - x^{k+1}(x-1)^k) = x^{k+1}(x-1)^{k-1}y^k.
\end{aligned}$$

So, the polynomial $f(x,y) = x(x-1)y$ is weakly semisimple and $g(x,y)$ is its eigenfunction with eigenvalue $\lambda = 1$.

Example 2. The polynomial $f(x,y) = y(x-1)(x-\frac{1}{2})\dots(x-\frac{1}{n})$, $n \in \mathbb{N}$, $n \geqslant 2$ is weakly semisimple. Really, putting $f_1(x) = (x-1)(x-\frac{1}{2})\dots(x-\frac{1}{n})$ $1 \leqslant i \leqslant n$ we obtain:

$$\begin{aligned}
f_1'\left(\frac{1}{i}\right) &= \left(\frac{1}{i}\right)^{n-2}\frac{1}{n!}(1-i)(2-i)\dots(i-1-i)(i+1-i)\dots(n-i)\\
&= (-1)^{i-1}\left(\frac{1}{i}\right)^{n-1}\frac{1}{n!}i!(n-i)! = (-1)^{i-1}\left(i^{n-1}\binom{n}{i}\right)^{-1} \in \mathbb{Z}.
\end{aligned}$$

Therefore the polynomial $f(x,y)$ satisfies the conditions of Theorem 1 and is weakly semisimple.

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