

Connectivity and planarity of power graphs of finite cyclic, dihedral and dicyclic groups

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ABSTRACT. The power graph of a finite group is the graph whose vertices are the elements of the group and two distinct vertices are adjacent if and only if one is an integral power of the other. In this paper we discuss the planarity and vertex connectivity of the power graphs of finite cyclic, dihedral and dicyclic groups. Also we apply connectivity concept to prove that the power graphs of both dihedral and dicyclic groups are not Hamiltonian.

1. Introduction

The study of power graph associated to a semigroup or group has been done by several authors, for instance, see [1–4, 7]. Keralev and Quinn [7] defined the directed power graph $\text{Pow}(S)$ of a semigroup S as a directed graph which has all elements of S as vertices and has arcs from u to v for all $u, v \in S$ such that $u \neq v$ and $v = u^m$ for some positive integer m . Following this Chakrabarty et al. [3] defined the (undirected) power graph $\mathcal{G}(S)$ of a semigroup S as an (undirected) graph whose vertex set is S and two vertices $u, v \in S$ are adjacent if and only if $u \neq v$ and $u^m = v$ or $v^m = u$ for some positive integer m . Also in [3] it was shown that for any finite group G the power graph of a subgroup of G is an induced subgraph of $\mathcal{G}(G)$ and $\mathcal{G}(G)$ is complete if and only if G is a cyclic group

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of order 1 or p^m , for some prime p and for some $m \in \mathbb{N}$. In [1] Cameron has proved that for a finite cyclic group G of non-prime-power order n , the set of vertices T of $\mathcal{G}(G)$ which are adjacent to all other vertices of $\mathcal{G}(G)$ consists of the identity and the generators of G , so that $|T| = 1 + \phi(n)$, where $\phi(n)$ is the Euler's totient or phi function.

The vertex connectivity (or simply connectivity) of a graph G , denoted by $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is said to be k -connected if its connectivity is at least k . In [3] it was shown that $\mathcal{G}(G)$ is connected for any finite group G as the identity element of G is adjacent to all other vertices of $\mathcal{G}(G)$. This paper concerns on the problem: if we delete the identity element from $\mathcal{G}(G)$ then the resulting graph is connected or not; if connected then deletion of which vertices make the graph disconnected? In other words our main concern is the connectivity of $\mathcal{G}(G)$. In this paper we study the connectivity of the power graph of finite cyclic group and we also prove that the power graph of dicyclic group is 2-connected whereas of dihedral group is 1-connected. Moreover in [3] it was shown that for any finite cyclic group G with $|G| \geq 3$, $\mathcal{G}(G)$ is Hamiltonian. However we prove that the power graphs of both dihedral and dicyclic groups are not Hamiltonian by applying connectivity concept.

A graph G is said to be planar if it can be drawn on a plane without any crossing of its edges. Let $K_n, K_{m,n}$ denote the complete graph with n vertices and complete bipartite graph with a bipartition into two sets of m and n vertices.

Theorem 1 (Kuratowski). *A graph G is planar if and only if G does not contain K_5 or $K_{3,3}$ or any graph homeomorphic to them as a subgraph.*

Theorem 2 (Sylow's First Theorem). *Let G be a finite group of order $p^r q$, where p is a prime, r and q are positive integers and $\gcd(p, q) = 1$. Then G has a subgroup of order p^k for all k satisfying $0 \leq k \leq r$.*

In [3] Chakrabarty et al. proved that for any finite cyclic group G of order $n \geq 3$, $\mathcal{G}(G)$ is non-planar when $\phi(n) > 7$. They have also proved that, the power graph of a cyclic group of order 2^m , where $m \in \mathbb{N}$ and $m \geq 3$, is non-planar. However, using Kuratowski's Theorem and Sylow's First Theorem, we prove that a finite cyclic group of order n is non-planar if and only if $n \geq 5$. In [4] it was shown that for $n = p^m (n \geq 5)$ or $n = 2q$, where p is some prime, $m \in \mathbb{N}$ and q is some odd prime, the power graph of dihedral group D_n is non-planar. Whereas in this paper we find necessary and sufficient conditions for the planarity of the power graphs of dihedral and dicyclic groups.

2. Power graph of finite cyclic group

Theorem 3. *For any finite cyclic group G of order n , the vertex connectivity $\kappa(\mathcal{G}(G))$ of the power graph of G satisfies the following:*

- (i) $\kappa(\mathcal{G}(G)) = n - 1$ when $n = 1$ or p^m for some prime p and for some positive integer m .
- (ii) $\kappa(\mathcal{G}(G)) \geq \phi(n) + 1$ when n is not a prime-power. The equality holds for $n = pq$ where p, q are distinct primes.

Proof. (i) Since G is a cyclic group of order 1 or p^m for some prime p and for some $m \in \mathbb{N}$ so $\mathcal{G}(G)$ is a complete graph on n vertices [3]. Therefore $\kappa(\mathcal{G}(G)) = n - 1$.

(ii) Number of generators of G is $\phi(n)$. So when n is not a prime-power, $\phi(n) + 1$ vertices of $\mathcal{G}(G)$ are adjacent to all the other vertices of $\mathcal{G}(G)$ [1]. Hence $\kappa(\mathcal{G}(G)) \geq \phi(n) + 1$.

Let $n = pq$ where p, q are distinct primes. It is known [6] that every finite cyclic group G of order n is isomorphic to the additive group \mathbb{Z}_n . So we prove the equality only for the additive group $\mathbb{Z}_n = \mathbb{Z}_{pq}$.

There are exactly $pq - \phi(pq) - 1 = (q - 1) + (p - 1)$ elements of $\mathbb{Z}_n = \mathbb{Z}_{pq}$, namely, $\bar{p}, 2\bar{p}, \dots, (q - 1)\bar{p}, \bar{q}, 2\bar{q}, \dots, (p - 1)\bar{q}$ which are neither identity nor generators. If possible suppose that $r\bar{p}$ is adjacent to $s\bar{q}$ for some r, s satisfying $1 \leq r \leq q - 1, 1 \leq s \leq p - 1$. Then for some positive integers m_1 and m'_1 ,

$$r\bar{p} = m_1 s\bar{q} \quad \text{or} \quad s\bar{q} = m'_1 r\bar{p},$$

We first consider $r\bar{p} = m_1 s\bar{q}$. This implies $rp = m_1 sq + m_2 pq$ for some $m_2 \in \mathbb{Z}$. Then $q \mid rp$. Since q is a prime and $q \mid rp$ then either $q \mid r$ or $q \mid p$. But as $1 \leq r < q$ so $q \nmid r$. Also by our assumption $q \nmid p$. A contradiction arises and so $r\bar{p} \neq m_1 s\bar{q}$. Similarly it can be proved that $s\bar{q} \neq m'_1 r\bar{p}$.

Thus none of the vertices in $\{\bar{p}, 2\bar{p}, \dots, (q - 1)\bar{p}\}$ is adjacent to any vertex in $\{\bar{q}, 2\bar{q}, \dots, (p - 1)\bar{q}\}$.

So after deleting identity and all the generators of $\mathbb{Z}_n = \mathbb{Z}_{pq}$ we get a vertex deleted subgraph of $\mathcal{G}(\mathbb{Z}_{pq})$ which is a disconnected graph with two components, namely,

- (a) the graph induced by $\{\bar{p}, 2\bar{p}, \dots, (q - 1)\bar{p}\}$, which is the complete graph on $q - 1$ vertices, and
- (b) the graph induced by $\{\bar{q}, 2\bar{q}, \dots, (p - 1)\bar{q}\}$, which is the complete graph on $p - 1$ vertices.

Thus $\kappa(\mathcal{G}(\mathbb{Z}_{pq})) = \phi(pq) + 1$ and hence the proof follows. \square

Theorem 4. *Let $G = \langle a \rangle$ be a finite cyclic group of order n . Then $\mathcal{G}(G)$ is non-planar if and only if $n \geq 5$.*

Proof. Let $G = \langle a \rangle$ be a finite cyclic group of order $n \geq 5$. We can write any positive integer $n \geq 5$ (except 6 and 12) as $p^r q$ with $p^r \geq 5$, where p is a prime, r and q are positive integers and $\gcd(p, q) = 1$.

Case I: Let $n \neq 6, 12$. Then $n = p^r q$ with $p^r \geq 5$, where p, q, r satisfy the above conditions. By Sylow's First Theorem G has a subgroup C of order p^r . Since every subgroup of a cyclic group is cyclic, so C is a cyclic subgroup of G of order $p^r \geq 5$. Thus $\mathcal{G}(C) = K_{p^r}$ is a supergraph of K_5 . Hence by Kuratowski's theorem $\mathcal{G}(C)$ and so $\mathcal{G}(G)$ is non-planar.

Case II: Let $n = 6$ or 12 and the identity element of G be denoted by e .

For $n = 6$, $G = \langle a \rangle$ has exactly two generators, say, a, a^5 and exactly three elements, say, a^2, a^3, a^4 which are neither identity nor generators. The vertices e, a and a^5 are adjacent to all other vertices of $\mathcal{G}(G)$ [1], in particular to all the vertices in $\{a^2, a^3, a^4\}$. Thus the power graph induced by $\{e, a, a^5\} \cup \{a^2, a^3, a^4\}$ that is $\mathcal{G}(G)$ is a supergraph of $K_{3,3}$. Hence by Kuratowski's theorem $\mathcal{G}(G)$ is non-planar.

For $n = 12$, $\langle a^2 \rangle$ is a cyclic subgroup of G of order 6 and hence from above we get that the power graph of $\langle a^2 \rangle$ is non-planar. Therefore $\mathcal{G}(G)$, being a supergraph of the power graph of $\langle a^2 \rangle$, is non-planar.

For the converse part it is sufficient to prove that $\mathcal{G}(G)$ is planar for $n < 5$. However for $n < 5$, $\mathcal{G}(G)$ is nothing but the complete graph K_n , which is planar if and only if $n < 5$. \square

3. Power graph of dihedral group

For each positive integer $n \geq 3$, the dihedral group [6] $D_n = \langle a, b \rangle$ is a non-commutative group of order $2n$ whose generators a and b satisfy:

- (i) $o(a) = n, o(b) = 2$;
- (ii) $ba = a^{-1}b = a^{n-1}b$.

3.1. Structure of the power graph of D_n

Since $o(a) = n$, $H = \langle a \rangle$ is a cyclic subgroup of D_n with $|H| = n$. So $\mathcal{G}(H)$ is a connected subgraph of $\mathcal{G}(D_n)$. Since $a^n = e = b^2$ so $(ab)^2 = a(ba)b = a(a^{n-1}b)b = e$ and $(a^{n-1}b)^2 = a^{n-1}(ba)a^{n-2}b = a^{n-1}(a^{n-1}b)a^{n-2}b = (a^{n-2}b)^2$. Continuing in this way we get $(a^{n-1}b)^2 =$

$(a^{n-2}b)^2 = \dots = (a^2b)^2 = (ab)^2 = e$. So for $1 \leq k \leq n-1$ power of any of $\{b, a^k b\}$ is either itself or e . Again for $1 \leq k \leq n-1$ power of any a^k is some power of a , which is none of $\{b, a^k b\}$. Thus b and $a^k b$ are adjacent to only e , for all k satisfying $1 \leq k \leq n-1$. Therefore the graph $\mathcal{G}(D_n)$ is of the form given in Figure 1.

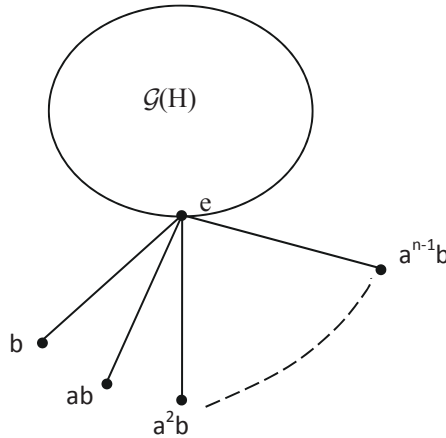


FIGURE 1. Power Graph of D_n for $n \geq 3$

Recall that a cut-vertex of a graph is a vertex whose deletion increases the number of components.

Theorem 5. *The identity element e of D_n is a cut vertex of $\mathcal{G}(D_n)$ and so $\kappa(\mathcal{G}(D_n)) = 1$. Also $\mathcal{G}(D_n)$ is not Hamiltonian.*

Proof. It is clear from the structure of $\mathcal{G}(D_n)$ (Figure 1) that deletion of the identity e increases the number of components of $\mathcal{G}(D_n)$ that makes the graph $\mathcal{G}(D_n)$ disconnected. Hence e is a cut vertex of $\mathcal{G}(D_n)$ and so $\kappa(\mathcal{G}(D_n)) = 1$.

Since every Hamiltonian graph is 2-connected [8, pp 287], $\mathcal{G}(D_n)$ is not Hamiltonian. \square

Theorem 6. *$\mathcal{G}(D_n)$ is non-planar if and only if $n \geq 5$.*

Proof. It is clear from the structure of $\mathcal{G}(D_n)$ (Figure 1) that $\mathcal{G}(D_n)$ is non-planar if and only if $\mathcal{G}(H)$ is non-planar. However $H = \langle a \rangle$ is a cyclic group with $|H| = n$ and so by Theorem 4, $\mathcal{G}(H)$ is non-planar if and only if $n \geq 5$. \square

4. Power graph of dicyclic group

For any integer $n \geq 2$, the dicyclic group [5] $Q_n = \langle a, b \rangle$ is a non-commutative group of order $4n$ whose generators a and b satisfy:

- (i) $a^{2n} = e, a^n = b^2,$
- (ii) $ab = ba^{-1} = ba^{2n-1}.$

Each element outside the big cyclic subgroup $A = \langle a \rangle$ of Q_n has order 4, and there is a unique element a^n of order 2.

4.1. Structure of the power graph of Q_n

For $1 \leq k \leq 2n - 1,$

$$(a^k b)^2 = a^{k-1}(ab)a^k b = a^{k-1}(ba^{2n-1})a^k b = (a^{k-1}b)^2.$$

Thus

$$\begin{aligned} (a^{2n-1}b)^2 &= (a^{2n-2}b)^2 = \dots = (a^n b)^2 = \dots \\ &= (a^2 b)^2 = (ab)^2 = b^2 = a^n. \end{aligned} \quad (1)$$

Using (1) and $a^{2n} = e$ we get for $1 \leq k \leq n - 1,$

$$(a^k b)^3 = (a^k b)^2(a^k b) = a^{n+k} b \quad (2)$$

$$(a^{n+k} b)^3 = (a^{n+k} b)^2(a^{n+k} b) = a^k b. \quad (3)$$

So

$$\begin{aligned} H_1 &= \{e, b, a^n, a^n b\}, H_2 = \{e, ab, a^n, a^{n+1}b\}, \dots, \\ H_n &= \{e, a^{n-1}b, a^n, a^{2n-1}b\} \end{aligned}$$

and $A = \langle a \rangle$ are the only cyclic subgroups of Q_n with $|H_i| = 4, |A| = 2n,$ for all $1 \leq i \leq n.$ Thus for $1 \leq i \leq n$ each $\mathcal{G}(H_i)$ (which is the complete graph K_4) and $\mathcal{G}(A)$ are the induced subgraphs of $\mathcal{G}(Q_n)$ on 4 and $2n$ vertices respectively. Since for $1 \leq k \leq 2n - 1$ power of any a^k is some power of $a,$ which is none of $\{b, a^k b\},$ so it implies from (1),(2),(3) that no other pair of vertices of $\mathcal{G}(Q_n)$ are adjacent. Hence $\mathcal{G}(Q_n)$ is of the form given in Figure 2.

Theorem 7. $\kappa(\mathcal{G}(Q_n)) = 2$ for all $n \geq 2.$

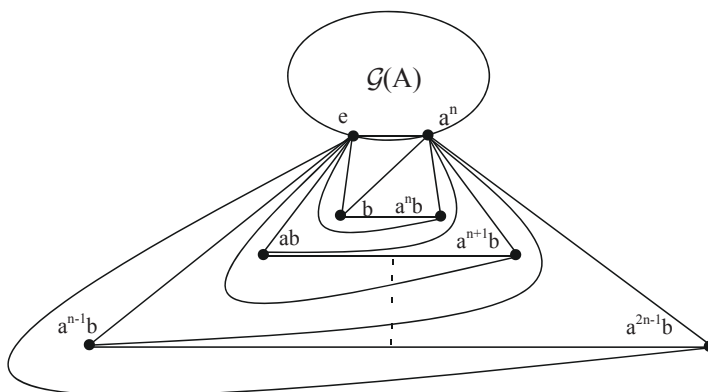


FIGURE 2. Power Graph of Q_n for $n \geq 2$

Proof. The identity element e of Q_n is adjacent to every other vertices of $\mathcal{G}(Q_n)$. If we delete e then also the vertex deleted subgraph $\mathcal{G}(Q_n) - \{e\}$ of $\mathcal{G}(Q_n)$ remains connected for the following reasons:

(i) $\mathcal{G}(A) - \{e\}$ is connected since the vertex a is adjacent to any other vertices of $\mathcal{G}(A) - \{e\}$. In particular a is adjacent to a^n and so any vertex a^r ($2 \leq r \leq n - 1, n + 1 \leq r \leq 2n - 1$) of $\mathcal{G}(A) - \{e\}$ is connected to a^n by a path $P_r : a^r a a^n$;

(ii) for $1 \leq k \leq 2n - 1$ any vertex of $\{b, a^k b\}$ is adjacent to a^n . However by deleting both the vertices e and a^n from $\mathcal{G}(Q_n)$ the resulting graph will be of the form given in Figure 3. Clearly the vertex deleted subgraph

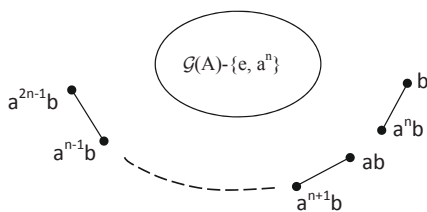


FIGURE 3. Power Graph of $Q_n - \{e, a^n\}$ for $n \geq 2$

$\mathcal{G}(Q_n) - \{e, a^n\}$ of $\mathcal{G}(Q_n)$ is disconnected with $n + 1$ components. Hence $\kappa(\mathcal{G}(Q_n)) = 2$. □

Theorem 8. $\mathcal{G}(Q_n)$ is not Hamiltonian for all $n \geq 2$.

Proof. It is known that [8, pp 287] if G is a Hamiltonian graph then for each non-empty set $S \subseteq V(G)$ the graph $G-S$ has at most $|S|$ components. Here taking $S = \{e, a^n\}$ we see that the number of components in $\mathcal{G}(Q_n) - S$ is $n + 1$ which is greater than $|S| = 2$ for all $n \geq 2$. So $\mathcal{G}(Q_n)$ is not Hamiltonian for all $n \geq 2$. \square

Theorem 9. $\mathcal{G}(Q_n)$ is non-planar if and only if $n > 2$.

Proof. It is clear from the structure of $\mathcal{G}(Q_n)$ (Figure 2) that $\mathcal{G}(Q_n)$ is non-planar if and only if $\mathcal{G}(A)$ is non-planar. However $A = \langle a \rangle$ is a cyclic group of order $2n$ and so by Theorem 4, $\mathcal{G}(A)$ is non-planar if and only if $2n > 4$ that is if and only if $n > 2$. \square

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