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On a factorization of an iterated wreath product of permutation groups

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Abstract. We show that if each group of permutations (G_i, M_i), $i \in \mathbb{N}$ has a factorization then their infinite iterated wreath product $\underset{i=1}{\overset{\infty}{\wr}} G_i$ also has a factorization. We discuss some properties of this factorization and give examples.

1. Introduction

We say that a group G is **factorized** by its subgroups $G_1, ..., G_n$ if

$$G = G_1 \cdots G_n = \{g_1 g_2 \cdots g_n,\ g_i \in G_i,\ i = 1, ..., n\} \tag{1}$$

i.e. G is a product of G_i's [1]. Then equality (1) is called a **factorization** of G. We call G factorizable if there exist a natural number $n \geqslant 2$ and subgroups $G_1, ..., G_n$ of G satisfying (1).

The factorization (1) of G is called exact if each pair of G_i's intersects trivially, that is $G_i \cap G_j = \{1\}$, $i \neq j$. One can examine the nature of that factorization. Firstly, let us consider an exact factorization of a group by two subgroups, namely $G = KH$. If both K and H are normal, then G is just their direct product. If one is normal, then G is a semidirect product, and if none is normal then G is a Zappa-Szép product of subgroups K, H (see [10, 12]). It can be extended to any finite number of groups in the natural way. Note, however, that G is a Zappa-Szép product of its subgroups $G_1, ..., G_n$ if $G_i \cap \langle G_j,\ j \neq i\rangle = \{1\}$, $i = 1, .., n$, where $\langle X\rangle$

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denotes a group generated by X. So the exactness of a factorization is not sufficient for G to be a Zappa-Szép product of its subgroups (if there are more than two).

In case of infinite iterated wreath products of permutation groups, say $G = \wr_{i=1}^{\infty} G_i$ (defined in section 2.1), two kinds of factorizations are considered:

1) if we partition $\mathbb{N}$ (i.e. the set of indices) into several (m, say) subsets N_i and define corresponding subgroups of G, namely

$$G_{N_i} = \{g \in G, \quad [g]_j = 1 \text{ for every } j \in \mathbb{N} - N_i\}$$

then $G = G_{N_1} \cdots G_{N_m}$ (see e.g. [9]).

2) if each group G_i is factorized by its subgroups then (for both the finite and infinite iterated wreath products) we define the corresponding subgroups factorizing G (see e.g. [6, 8]).

Factorizations done in the first manner were used mainly to construct subgroups of $\wr_{i=1}^{\infty} \mathbb{Z}_p$ for some prime p having some extra properties and giving a negative answer to some open problems, for instance a non-locally finite π-group factorized by locally finite subgroups or a group factorized by two locally finite p-groups which contains an element of an infinite order [7–9].

Since factorizations of wreath products seem to be a nice source of examples, it may be interesting to investigate their nature in a systematic way. Our paper is meant to be one step forward.

The paper is organized as follows. In section 2 we recall a definition of an infinite iterated wreath product and two of its standard subgroups. Section 3 is devoted to investigating the factorization (of the second type) and its basic properties connected with the properties of factorization of each constituent. Also, the factorization of the two standard subgroups is considered. In section 4 we describe the factorization under the assumption that each constituent is a soluble group and give some examples.

The notions we use are standard or defined. For more details the reader is referred to the bibliography, e.g. [1, 2, 11].

2. Preliminaries

2.1. An infinite iterated wreath product

Let $(G_1, M_1),\ (G_2, M_2), \ldots$ be an infinite sequence of permutation groups and let $M = \prod_{i\in\mathbb{N}} M_i$. Then the **infinite iterated wreath product** of groups in this sequence [4], denoted by $\overset{\infty}{\underset{i=1}{\wr}} G_i$ (or $\overset{\infty}{\underset{i=1}{\wr}} (G_i, M_i)$ if the sets are important), is a permutation group on the set M consisting of all maps g satisfying the following two conditions:

1) for every $x = (x_1, x_2, \ldots) \in M$, if $y = x^g$, then i-th coordinate of y depends only on the first i coordinates of x and on g
2) for any fixed sequence $(x_1^0, \ldots, x_{i-1}^0) \in M_1 \times \cdots M_{i-1}$ the map $g_i(x_1^0, \ldots, x_{i-1}^0) : \ x_i \to y_i$ induced by g is a permutation of M_i belonging to G_i

Therefore g can be uniquely written as an infinite sequence, called an **array**, namely

$$g = [g_1, g_2(x_1), g_3(x_1, x_2), \ldots.],$$

where $g_1 \in G_1$ and for every $i \geqslant 2$ g_i is a function on $M_1 \times \cdots \times M_{i-1}$ taking values in G_i, that is $g_i \in G_i^{M_1 \times \cdots \times M_{i-1}}$.

The action of $g \in \overset{\infty}{\underset{i=1}{\wr}} G_i$ on $m = (m_1, m_2, m_3, \ldots) \in M$ is defined componentwise, namely

$$m^g = \left(m_1^{g_1}, m_2^{g_2(m_1)}, m_3^{g_3(m_1, m_2)}, \ldots\right) \tag{2}$$

Thus the product of $g = [g_1, g_2(x_1), \ldots]$ and $h = [h_1, h_2(x_1), \ldots]$ is given by the rule

$$gh = \left[g_1 h_1,\ g_2(x_1) h_2\left(x_1^{g_1}\right),\ g_3(x_1, x_2) h_3\left(x_1^{g_1}, x_2^{g_2(x_1)}\right), \ldots.\right] \tag{3}$$

The identity is

$$e = [1, 1, 1, \ldots.], \tag{4}$$

where each component is a constant function taking identity of each group as the value and the inverse of g is

$$g^{-1} = \left[g_1^{-1}, g_2^{-1}\left(x_1^{g_1^{-1}}\right), g_3^{-1}\left(x_1^{g_1^{-1}}, x_2^{g_2^{-1}\left(x_1^{g_1^{-1}}\right)}\right), \ldots\right] \tag{5}$$

To make the notation shorter, in order to define multiplication in the wreath product it is enough to refer to the to the n-th component of the array instead of writing the whole one. So, let $\bar{g}_n$ denote the initial segment of g of length n and let $\bar{m}_n$ denote the initial segment of m of length n, that is

$$\bar{g}_n = [g_1, ..., g_n(x_1, ..., x_{n-1})], \qquad \bar{m}_n = (m_1, m_2, ..., m_n) \tag{6}$$

Then (2) implies the following action of $\bar{g}_n$ on $\bar{m}_n$

$$\bar{m}_n^{\bar{g}_n} = \left(m_1^{g_1}, m_2^{g_2(m_1)}, m_3^{g_3(m_1,m_2)}, ..., m_n^{g_n(m_1,...,m_{n-1})}\right) \tag{7}$$

Now, if $[g]_n$ denotes the n-th component of the array g, then, taking (7) under consideration, (3) and (5) can be written as

$$\begin{aligned} &[gh]_1 = g_1 h_1, && [gh]_n = g_n(\bar{x}_{n-1}) h_n \left(\bar{x}_{n-1}^{\bar{g}_{n-1}}\right), \ n \geqslant 2 \\ &[g^{-1}]_1 = g_1^{-1}, && [g^{-1}]_n = g_n^{-1} \left(\bar{x}_{n-1}^{\bar{g}_{n-1}}\right), \ n \geqslant 2 \end{aligned} \tag{8}$$

2.2. Standard subgroups in an iterated wreath product

1. There is another way to extend the definition of a wreath product of a finite number groups into an infinite one, $(G_i, M_i), i \in \mathbb{N}$ say. This time we take groups $G^{(n)} := \wr_{i=1}^{n} G_i$ together with the natural embeddings $G^{(n)} \to G^{(n+1)}$ defined as $[g_1, ..., g_n(\bar{x}_{n-1})] \to [g_1, ..., g_n(\bar{x}_{n-1}), 1]$. Then the direct limit of this system of groups also is an iterated wreath product. We shall denote it by $\wr_{\mathrm{f}\, i=1}^{\ n} G_i$ and call it a **finitary iterated wreath product** of groups $(G_i, M_i), i \in \mathbb{N}$ [2]. Note that it is isomorphic to the proper subgroup of $\wr_{i=1}^{\infty} G_i$, namely the one containing all elements of the form $g = [g_1, g_2(x_1), ..., g_n(x_1, ..., x_{n-1}), 1, 1, ...]$ for some natural n.

It is easy to see that $\wr_{\mathrm{f}\, i=1}^{\ n} G_i$ is locally finite if and only if every G_i is finite.

Also, if each G_i is transitive on M_i, then generators of $G := \wr_{\mathrm{f}\, i=1}^{\ \infty} G_i$ can be defined as follows: for each i we define the set $\hat{G}_i$ of those arrays in G, all components of which except i-th are trivial and the i-th component of which is a function having a nontrivial value at exactly one point. Then $\wr_{\mathrm{f}\, i=1}^{\ \infty} G_i = \langle \hat{G}_i, \ i \in \mathbb{N} \rangle$.

However, $G := \overset{n}{\underset{i=1}{\wr_{\mathrm{f}}}} G_i$ never acts transitively on $M := \prod_{i=1}^{\infty} M_i$ and the orbits can be described as follows: elements $a = (a_1, a_2, ...)$ and $b = (b_1, b_2, ...)$ in M are called cofinal if there exists a natural number k such that for every $n \geqslant k$ we have $a_n = b_n$. Cofinality is an equivalence relation in M and equivalence classes are the orbits of action of G on M.

2. For any group (G, M) consider an iterated wreath product $\overset{\infty}{\underset{i=1}{\wr}} G$ of an infinite number of copies of G, which is sometimes called a wreath power of G, and take any $g \in \overset{\infty}{\underset{i=1}{\wr}} G$. Now, for an arbitrary $s \in \mathbb{N}$ and for an arbitrary $m^{(s)} := (m_1, ..., m_s) \in M^s$ define an **$m^{(s)}$-th remainder for g** to be

$$g_m^{(s)} := [g_1, g_2(m_1), ..., g_{s+1}(m^{(s)}), g_{s+2}(m^{(s)}, x_1), g_{s+3}(m^{(s)}, x_1, x_2), ...]$$

Note that for a given $m^{(s)}$ the constituent $g_{s+1}(m^{(s)})$ is an element of G and g_{s+k+1} is a function of $x_1, .., x_k$, that is $g_{s+k+1} \in G^{M^k}$. Therefore

$$\bar{g}_m^{(s)} := [g_{s+1}(m^{(s)}), g_{s+2}(m^{(s)}, x_1), g_{s+3}(m^{(s)}, x_1, x_2), ...] \in \overset{\infty}{\underset{i=1}{\wr}} G$$

and it is called the **state of g for $m^{(s)}$**. Now, let $\overset{\infty}{\underset{i=1}{\wr_{\mathrm{fs}}}} G$ denote the set of such elements $g \in \overset{\infty}{\underset{i=1}{\wr}} G$ that the set of all their states, i.e.

$$\{\bar{g}_m^{(s)}, \quad s \in \mathbb{N},\ m^{(s)} \in M^s\}$$

is finite. Now, if g and h have only a finite number of distinct states, so have gh and g^{-1}, thus $\overset{\infty}{\underset{i=1}{\wr_{\mathrm{fs}}}} G$ is a subgroup of $\overset{\infty}{\underset{i=1}{\wr}} G$, called the **finite state iterated wreath product** [2]. Moreover, each element of the form $[g_1, g_2(x_1), ..., g_n(x_1, ..., x_{n-1}), 1, 1, ...]$ has only a finite number of states whence we finally get

$$\overset{\infty}{\underset{i=1}{\wr_{\mathrm{f}}}} G < \overset{\infty}{\underset{i=1}{\wr_{\mathrm{fs}}}} G < \overset{\infty}{\underset{i=1}{\wr}} G$$

for every group (G, M).

3. The factorizations

The theorem below is a generalization of Theorem 1.1. in [8].

Theorem 1. *Let $G = \wr_{i=1}^{\infty} G_i$ be an infinite wreath product of permutation groups $(G_i, M_i),\ i \in \mathbb{N}$, each of which is factorized by at most m subgroups, that is*

$$\forall i \in \mathbb{N} \qquad G_i = G_{i_1} G_{i_2} \cdots G_{i_m} \tag{9}$$

with some G_{i_k} possibly trivial. Now, let $G^{[1]}G^{[2]}, ..., G^{[m]}$ be groups defined in the following way: $g^{(k)} \in G^{[k]}$ if and only if

$$g^{(k)} = [g_1^{(k)},\ g_2^{(k)}(x_1),\ g_3^{(k)}(x_1, x_2), ...], \quad g_i^{(k)} \in G_{i_k}^{M_1 \times \cdots \times M_{i-1}} \tag{10}$$

Then

$$G = G^{[1]} G^{[2]} \cdots G^{[m]} \tag{11}$$

Proof. Observe first that $G^{[k]},\ k = 1, ..., m$ are subgroups of G, which is obvious since $G_{i_k},\ k = 1, ..., m$ are subgroups of G_i. Taking (9) under consideration for every $g \in G$ we have

$$[g]_1 = g_1^{(1)} \cdots g^{(m)}, \qquad [g]_n = g_n^{(1)}(\bar{x}_{n-1}) \cdots g_n^{(m)}(\bar{x}_{n-1}),\ n \geqslant 2 \tag{12}$$

Now, for a given $g \in G$ we define $A^{(k)} \in G^{[k]}$ for $k = 1, 2, ..., m$ in the following way: $[A^{(k)}]_1 = g_1^{(k)}$ and for every natural $n \geqslant 2$

$$[A^{(1)}]_n = g_n^{(1)}(\bar{x}_{n-1}), \qquad [A^{(k)}]_n = g_n^{(k)}\left(\bar{x}_{n-1}^{\left(\bar{g}_{n-1}^{(1)} \cdots \bar{g}_{n-1}^{(k-1)}\right)^{-1}}\right) \tag{13}$$

Note that the definition of $[A^{(k)}]_n$ is correct since $\bar{x}_{n-1}^{\bar{u}_{n-1}} \in M_1 \times \cdots \times M_{n-1}$ for every $u \in G$. Thus by the rule (3) of multiplication in the infinite iterated wreath product we have $[A^{(1)} \cdots A^{(m)}]_1 = g_1^{(1)} \cdots g^{(m)} = [g]_1$ and for every natural $n \geqslant 2$

$$\begin{aligned} &[A^{(1)} \cdots A^{(m)}]_n = \\ &g_n^{(1)}(\bar{x}_{n-1}) g_n^{(2)}\left(\left(\bar{x}_{n-1}^{(g_1^{(1)})^{-1}}\right)^{g_1^{(1)}}\right) \cdots g_n^{(m)}\left(\left(\bar{x}_{n-1}^{\left(\bar{g}_{m-1}^{(1)} \cdots \bar{g}_{m-1}^{(m-1)}\right)^{-1}}\right)^{\bar{g}_m^{(1)} \cdots \bar{g}_{m-1}^{(m-1)}}\right) \\ &= g_n^{(1)}(\bar{x}_{n-1}) g_n^{(2)}(\bar{x}_{n-1}) \cdots g_n^{(m)}(\bar{x}_{n-1}) = [g]_n \end{aligned} \tag{14}$$

whence $g = A^{(1)} A^{(2)} \cdots A^{(m)}$ which proves (11) as required. $\square$

A little bit more can be said about the nature of factorization (11) if the nature of a factorization of each constituent is known. Namely,

Corollary 1. *Let* $G = \overset{\infty}{\underset{i=1}{\wr}} G_i$ *be an infinite iterated wreath product of permutation groups* (G_i, M_i), $i \in \mathbb{N}$, *each of which is factorized by at most* m *permutation groups, that is*

$$\forall i \in \mathbb{N} \qquad G_i = G_{i_1} G_{i_2} \cdots G_{i_m}$$

1) *If the factorization of each* G_i *is exact, then the factorization of* G *is exact.*
2) *If each* G_i *is a Zappa-Szép product of its subgroups, then* G *is a Zappa-Szép product of its subgroups.*

Proof. By Theorem 1, $G = G^{[1]} G^{[2]} \cdots G^{[m]}$.

1) Take $g = [g_1, g_2(x_1), ...] \in G^{[j]} \cap G^{[l]}$, $j \neq l$. Then g_1 belongs both to G_{1_j} and G_{1_l} and functions $g_i(\bar{x}_{i-1})$ take values both in G_{i_j} and G_{i_l} for every $\bar{x}_{i-1} \in M_1 \times \cdots \times M_{i-1}$ and every natural $i \geqslant 2$. That means that the only possible value of each function g_i is 1, so by (4) $g = e$, which means that the factorization is exact.

2) To prove that it is enough to show that for each i, k the group G_{i_k} intersects trivially with the group generated by $G_{i_j}, j \neq k$. The proof is analogous to that of 1). $\square$

Corollary 2. *Let* $G = \overset{\infty}{\underset{i=1}{\wr_f}} G_i$ *be a finitary iterated wreath product of permutation groups* $(G_i, M_i), i \in \mathbb{N}$ *each of which is factorized by at most* m *permutation subgroups. Then* $\overset{\infty}{\underset{i=1}{\wr_f}} G_i$ *can be factorized by* m *subgroups. Moreover, if each* G_i *is a Zappa-Szép product of its subgroups, so is* $\overset{\infty}{\underset{i=1}{\wr_f}} G_i$.

Proof. By Theorem 1, $G = G^{[1]} G^{[2]} \cdots G^{[m]}$, where each $G^{[k]}$ consists of

$$g^{(k)} = [g_1^{(k)},\ g_2^{(k)}(x_1),\ g_3^{(k)}(x_1, x_2), ...], \quad g_i^{(k)} \in G_{i_k}^{M_1 \times \cdots \times M_{i-1}}$$

Now consider any element of $\overset{\infty}{\underset{i=1}{\wr_f}} G_i$. Then it can be written as a product of those elements from each $G^{[k]}$ which have only a finite number of nontrivial components. Thus if $G_f^{[k]}$ is a subset of $G^{[k]}$ consisting of all elements with only a finite number of nontrivial components, then it is actually a subgroup of $G^{[k]}$, which gives the required factorization. The last part follows from Corollary 1, part 2). $\square$

Corollary 3. *Let* $\mathop{\wr_{fs}}\limits_{i=1}^{\infty} G$ *be a finite state iterated wreath power of a permutation group* (G, M)*, which is factorized by* m *permutation subgroups. Then* $\mathop{\wr_{fs}}\limits_{i=1}^{\infty} G$ *can be factorized by* m *subgroups. Moreover, if* G *is a Zappa-Szép product of its subgroups, so is* $\mathop{\wr_{fs}}\limits_{i=1}^{\infty} G$*.*

Proof. If $G = G_1 \cdots G_m$, then by Theorem 1, $\mathop{\wr}\limits_{i=1}^{\infty} G = G^{[1]}G^{[2]} \cdots G^{[m]}$, where each $G^{[k]}$ consists of elements of the form

$$g^{(k)} = [g_1^{(k)},\ g_2^{(k)}(x_1),\ g_3^{(k)}(x_1, x_2), \ldots.], \quad g_i^{(k)} \in G_k^{M^{i-1}}$$

Now consider any element of $\mathop{\wr_{\mathrm{fs}}}\limits_{i=1}^{\infty} G_i$. Then it can be written as a product of those elements from each $G^{[k]}$ which have only a finite number of states. Thus if $G_{fs}^{[k]}$ is a subset of $G^{[k]}$ consisting of all elements with only a finite number of states, then it is actually a subgroup of $G^{[k]}$, which gives the required factorization. The last part follows from Corollary 1, part 2). $\square$

Example 1. Since every finite soluble group has a finite normal series with cyclic factors, by theorem of Krasner and Kaloujnine [5] it can be embedded into a finite iterated wreath product of cyclic groups. Thus every group having an infinite normal series with cyclic factors, all terms of which intersect trivially (e.g. a residually finite or a residually soluble group) can be embedded into an infinite iterated wreath product $\mathop{\wr}\limits_{i=1}^{\infty} C_{n_i}$ of cyclic groups. Now, each C_{n_i} is either a cyclic group of prime power order or is isomorphic to a direct product of cyclic subgroups of prime power orders. More exactly, if $n_i = q_1^{\alpha_1} \cdots q_s^{\alpha_s}$, where q_i are primes and $s \in \mathbb{N}$ then $C_{n_i} \cong C_{q_1^{\alpha_1}} \times \cdots \times C_{q_s^{\alpha_s}}$. Therefore if there exists a natural number n such that $n_i \leqslant n$ for every $i \in \mathbb{N}$, then each C_{n_i} can be viewed as a direct product with the same finite number of factors, each of which is either a cyclic group of prime power order less than n or is the trivial group. Hence by Theorem 1 G can be written as a Zappa-Szép product of a finite number of infinite iterated wreath products of cyclic groups of some prime power orders less than or equal to n.

4. Iterated wreath products of soluble groups

4.1. Sylow subgroups in an iterated wreath product

Note that an infinite wreath product $G := \wr_{i=1}^{\infty} G_i$ can be seen as the inverse limit of groups $G^{(n)} := \wr_{i=1}^{n} G_i$ together with the natural projections $G^{(n+1)} \to G^{(n)}$ defined as $[g_1, ..., g_{n+1}(\bar{x}_n)] \to [g_1, ..., g_n(\bar{x}_{n-1})]$. Thus if every G_i is finite, G is profinite. If moreover each G_i is a p-group, so is each $G^{(n)}$, whence G is a pro-p-group.

Now, following [11], the index $|G : H|$ of a (closed) subgroup H of a profinite group G is defined as the least common multiple of the indices of the open subgroups of G containing H and the order $|G|$ of G is defined as $|G : 1|$. Therefore for every prime p a **p-Sylow subgroup**[1] of G is a subgroup P such that $|P|$ is a (possibly infinite) power of p and $|G : P|$ is a supernatural number coprime to $|P|$. From that it follows that (similarly as in the finite case) p-Sylow subgroups of a profinite group G are maximal pro-p-subgroups of G.

Moreover, p-Sylow subgroups of an infinite wreath product $\wr_{i=1}^{\infty} G_i$ can be characterized by Sylow p-subgroups of each G_i, if they are finite. Indeed, if (G_1, M_1), (G_2, M_2) are finite groups of permutations and (P_1, M_1), (P_2, M_2) are their Sylow p-subgroups for some prime p then $P_1 \wr P_2$ is the Sylow p-subgroup of $G_1 \wr G_2$ (which follows from the comparison of orders). And conversely - each Sylow p-subgroup of a wreath product of two finite groups is a wreath product of Sylow p-subgroups of constituents. This property can be extended to a finite iterated wreath product of finite permutation groups $\wr_{i=1}^{n}(G_i, M_i)$ for every natural n by induction whence we finally get

Proposition 1. *Let $G = \wr_{i=1}^{\infty} G_i$ be an infinite iterated wreath product of finite permutation groups (G_i, M_i), $i \in \mathbb{N}$. Then a p-Sylow subgroup of G is an infinite iterated wreath product of Sylow p-subgroups of G_i's.* $\square$

[1] In case of profinite groups one rather say "p-Sylow subgroup" than "Sylow p-subgroup" since they are usually not p-subgroups in a standard sense

4.2. The factorization in wreath products of soluble groups

Recall that if π is any set of primes, then a (pro)finite group G is called (pro-)π-group if every prime divisor of $|G|$ belongs to π.

Theorem 2. *Let π be any finite set of primes and let $G = \wr_{i=1}^{\infty} G_i$ be an infinite iterated wreath product of finite soluble permutation π-groups (G_i, M_i), $i \in \mathbb{N}$. Then a pro-π group G is a Zappa-Szép product of its Sylow subgroups.*

Proof. By theorem of P. Hall [3], every finite soluble group has a Sylow basis, which is a set of pairwise permutable Sylow subgroups, one for each prime p (note that a Sylow basis contains a trivial subgroup). Since π is finite, say $\pi = \{p_1, ..., p_m\}$, then a Sylow basis for each constituent contains at most $m + 1$ elements, say $\{1, P_1, ..., P_m\}$ with P_k being a Sylow p_k-subgroup. Since the subgroups in a Sylow basis are pairwise permutable, a group generated by all P_i's except j-th is a product of all P_i's except j-th, that is

$$\langle P_i, i \neq j\rangle = \prod_{i \neq j} P_i$$

Moreover, P_i's are Sylow subgroups of G, thus $|G| = |P_1 \cdots P_m| = |P_1| \cdots |P_m|$. From that and from the equality $|HK| = |H||K|/|H \cap K|$ which is true for every pair of subgroups of a given group we finally get that

$$\forall j = 1, ..., m \qquad P_j \cap \langle P_i, i \neq j\rangle = \{1\} \tag{15}$$

Now, let $G = \wr_{i=1}^{\infty} G_i$ where each G_i is a soluble π-group. Then G is pro-π and (15) means that each G_i is a Zappa-Szép product of at most m subgroups, G_{i_k} say, each of which is a Sylow p_k-subgroup of G_i. Therefore by Corollary 1 the product $G = G^{[1]} \cdots G^{[m]}$ is a Zappa-Szép product. Next, by Proposition 1 each group $G^{[k]}$ is a p_k-Sylow subgroup of G, which finishes the proof. □

We give some examples of iterated wreath products of soluble groups.

Example 2. Consider a wreath power of C_p^n, that is $G = \wr_{i=1}^{\infty} C_p^n$, where C_p^n acts on $M^n = \{1, ..., p\}^n$. Since C_p^n is a direct product of cyclic groups, then G is a Zappa-Szép product of groups $G^{[k]}$, each of which consists of elements of the form

$$g^{(k)} = [g_1^{(k)},\ g_2^{(k)}(x_1),\ g_3^{(k)}(x_1, x_2), ...],\quad g_i^{(k)} \in \left(\bar{C}_p^{(k)}\right)^{M^{i-1}}$$

where $\bar{C}_p^{(k)}$ is a product of $n-1$ trivial groups and the group C_p, which stands as the k-th constituent in this product. So $G^{[k]} \cong \underset{i=1}{\overset{\infty}{\wr}}(C_p, M^n)$ for every k and hence by Theorem 2 G is isomorphic to the Zappa-Szép product of n copies of $\underset{i=1}{\overset{\infty}{\wr}}(C_p, M^n)$, that is

$$\underset{i=1}{\overset{\infty}{\wr}}(C_p^n, M^n) \cong \left(\underset{i=1}{\overset{\infty}{\wr}}(C_p, M^n)\right) \bowtie \cdots \bowtie \left(\underset{i=1}{\overset{\infty}{\wr}}(C_p, M^n)\right).$$

Example 3. Take $\underset{i=1}{\overset{\infty}{\wr}}(S_3, X)$, where $X = \{1, 2, 3\}$. Since S_3 is soluble of order 6, it can be factorized by its Sylow 2- and 3-subgroups, which are both cyclic. Although S_3 is a semidirect product of $(\langle(1, 2, 3)\rangle, X)$, a cyclic group of order 3 (which is normal), by $(\langle(1, 2)\rangle, X)$, a cyclic group of order 2, the group $\underset{i=1}{\overset{\infty}{\wr}}(S_3, X)$ is not a semidirect product of $\underset{i=1}{\overset{\infty}{\wr}}(\langle(1, 2, 3)\rangle, X)$ and $\underset{i=1}{\overset{\infty}{\wr}}(\langle(1, 2)\rangle, X)$. But it is a Zappa-Szép product, which follows from 2). in Corollary 1, that means

$$\underset{i=1}{\overset{\infty}{\wr}}(S_3, X) \cong \left(\underset{i=1}{\overset{\infty}{\wr}}(C_3, X)\right) \bowtie \left(\underset{i=1}{\overset{\infty}{\wr}}(C_2, X)\right).$$

Example 4. Take under consideration $\underset{i=1}{\overset{\infty}{\wr}}(A_4, X),\ X = \{1, 2, 3, 4\}$. Each of Sylow 3-subgroups of A_4 has order 3 (there are 4 of them), which makes it cyclic. Choose any and denote it by C_3. Sylow 2-subgroups has order 4 and there is only such subgroup (whence it is normal), namely $\{(1), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}$, which is the Klein 4-group. Thus $A_4 \cong C_3 \ltimes V_4$ since $|C_3||V_4| = |A_4|$ and A_4 is generated by these two subgroups. Therefore

$$\underset{i=1}{\overset{\infty}{\wr}}(A_4, X) \cong \left(\underset{i=1}{\overset{\infty}{\wr}}(V_4, X)\right) \bowtie \left(\underset{i=1}{\overset{\infty}{\wr}}(C_3, X)\right).$$

Example 5. Finally, let $G = \underset{i=1}{\overset{\infty}{\wr}}(S_4, X)$, where $X = \{1, 2, 3, 4\}$. Each of Sylow 3-subgroups of S_4 has order 3 (there are 4 of them), which makes it cyclic, and each of Sylow 2-subgroups has order 8 (there are 3 of them). Now, if we take $\{(1),(1,2,3),(1,3,2)\} \cong C_3$ and $\{(1),(1,2,3,4),$ $(1,3)(2,4),(1,4,3,2),(1,3),(2,4),(1,2)(3,4),\ (1,4)(2,3)\} \cong D_4$, then we get $S_4 \cong C_3 \bowtie D_4$ since these two subgroups generate S_4 and $|C_3||D_4| = |S_4|$. That gives us

$$\mathop{\wr}_{i=1}^{\infty}(S_4,X) \cong \left(\mathop{\wr}_{i=1}^{\infty}(D_4,X)\right) \bowtie \left(\mathop{\wr}_{i=1}^{\infty}(C_3,X)\right).$$

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