Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **18** (2014). Number 1, pp. 8 – 13


# A new characterization of alternating groups

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Communicated by M. Sapir

Abstract. Let $G$ be a finite group and let $\pi_e(G)$ be the set of element orders of $G$. Let $k \in \pi_e(G)$ and let $m_k$ be the number of elements of order $k$ in $G$. Set nse($G$):=$\{m_k | k \in \pi_e(G)\}$. In this paper, we show that if $n = r$, $r+1$, $r+2$, $r+3$ $r+4$, or $r+5$ where $r \geqslant 5$ is the greatest prime not exceeding $n$, then $A_n$ characterizable by *nse* and *order*.

## 1. Introduction

If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a finite group. Denote by $\pi(G)$ the set of primes $p$ such that $G$ contains an element of order $p$. Also the set of element orders of $G$ is denoted by $\pi_e(G)$. A finite group $G$ is called a simple $K_n-$group, if $G$ is a simple group with $|\pi(G)| = n$. We denote by $\phi$ the Euler totient function. We say $p^k \parallel m$ if $p^k \mid m$ and $p^{k+1} \nmid m$. All other notations are standard and we refer to [8], for example.

Set $m_i = m_i(G)$=$|\{g \in G|$ the order of $g$ is $i\}|$. In fact $m_i$ is the number of elements of order $i$ in $G$, and nse($G$):=$\{m_i| \ i \in \pi_e(G)\}$, the set of sizes of elements with the same order.

In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows. For each finite group $G$ and each integer $d \geqslant 1$, let $G(d) = \{x \in G| \ x^d = 1\}$. Defining $G_1$ and $G_2$ is of the

**2010 MSC:** 20D06, 20D60.
**Key words and phrases:** finite group, simple group, alternating groups.

same order type if, and only if, $|G_1(d)| = |G_2(d)|$, $d = 1, 2, 3, \ldots$. Suppose $G_1$ and $G_2$ are of the same order type. If $G_1$ is solvable, is $G_2$ necessarily solvable? ( [11, Problem 12.37])

We see clearly that if groups $G_1$ and $G_2$ are of the same order type, then $|G_1| = |G_2|$ and $\mathrm{nse}(G_1) = \mathrm{nse}(G_2)$. So it is natural to investigate the Thompson's problem by $|G|$ and $\mathrm{nse}(G)$.

In [3, 4, 6], it is proved that symmetric group $S_r$, where $r$ is prime number, all sporadic simple groups and all simple $K_4-$groups are characterizable by nse and order. In [5, 10], it is proved that alternating groups $A_n$, where $5 \leqslant n \leqslant 8$ are characterizable by only nse. In [2], it is proved that $L_2(p)$, where $p$ is a prime number, is characterizable by $\mathrm{nse}(G)$ and $p \in \pi(G)$. Also in [1], it is proved that $\mathrm{PGL}_2(p)$, where $p > 3$ is a prime number, is characterizable by $\mathrm{nse}(G)$ and $p \in \pi(G)$.

Note that not all groups can be characterized by $\mathrm{nse}(G)$ and $|G|$. For instance, let $H_1 = C_4 \times C_4$ and $H_2 = C_2 \times Q_8$ where $C_2$ and $C_4$ are cyclic groups of orders 2 and 4, respectively and $Q_8$ is a quaternion group of order 8. It is easy to see that $\mathrm{nse}(H_1) = \mathrm{nse}(H_2)=\{1, 3, 12\}$ and $|H_1| = |H_2| = 16$, but $H_1$ is an abelian group and $H_2$ is a non-abelian group. Therefore $H_1 \ncong H_2$.

In this paper, it is proved that some of the alternating groups, are characterizable by nse and their order. In fact the main theorem of our paper is as follows:

**Main theorem.** Let $G$ be a group such that $|G| = |A_n|$ and let $\mathrm{nse}(G)= \mathrm{nse}(A_n)$. If $n = r$, $r + 1$, $r + 2$, $r + 3$, $r + 4$, or $r + 5$ where $r \geqslant 5$ is the greatest prime not exceeding $n$, then $G \cong A_n$.

## 2. Some lemmas

**Lemma 2.1** ([7, Lemma 2.2]). Let $G$ be a group and $P$ be a cyclic Sylow $p$-subgroup of $G$ of order $p^a$. If there is a prime $r$ such that $p^a r \in \pi_e(G)$, then $m_{p^a r} = m_r(C_G(P))m_{p^a}$. In particular, $\phi(r)m_{p^a} \mid m_{p^a r}$.

**Lemma 2.2** ([9]). Let $G$ be a finite group, $n \geqslant 4$ with $n \neq 8$, 10 and $r$ be the greatest prime not exceeding $n$. If $|G|=|A_n|$ and $|N_G(R)| = |N_{A_n}(S)|$ where $R \in \mathrm{Syl}_r(G)$ and $S \in \mathrm{Syl}_r(A_n)$, then $G \cong A_n$.

## 3. Proof of the main theorem

In [10], it is proved that the alternating groups $A_5$ and $A_6$ are characterizable only by $\mathrm{nse}(G)$. Note that if $n = 7$, 8, 9, or 10, then $A_n$ is a simple $K_4-$group. In [6], it is proved that all simple $K_4-$groups are characterizable by nse($G$) and $|G|$. Therefore if $n < 11$, then $G \cong A_n$.

Suppose that $n \geqslant 11$ and $r \geqslant 5$ is the greatest prime not exceeding $n$. Since $r \parallel |G|$ and the Sylow $r$-subgroups of $G$ have trivial pairwise intersections, it follows that $m_r = t(r-1)$ where $t$ is the number of Sylow $p$-subgroups of $G$. Futhermore since $t \equiv 1 \pmod r$ by Sylow's theorem, it follows that $m_r(G) \equiv -1 \pmod r$.

**Claim 3.1**. $m_r(G) = m_r(A_n)$.

Let $m_r(G) \neq m_r(A_n)$. Since nse($G$)=nse($A_n$), $m_r(G) \in$ nse($A_n$). Suppose that there exists $r \neq k \in \pi_e(G)$ such that $m_r(G) = m_k(A_n)$. Thus $m_k(A_n) \equiv -1 \pmod r$. We know that $m_k(A_n) = \sum |cl_{A_n}(x_i)|$ such that $|x_i| = k$ for every $i$. Since $m_k(A_n) \equiv -1 \pmod r$, $(r, m_k(A_n)) = 1$.

If for every $i$, the cyclic structure of $x_i$ is $1^{r_1}2^{r_2}\ldots l^{r_l}$ such that $r \nmid 1^{r_1}2^{r_2}\ldots l^{r_l} r_1! r_2! \ldots r_l!$, then

$$|cl_{A_n}(x_i)| = \frac{n!}{1^{r_1}2^{r_2}\ldots l^{r_l} r_1! r_2! \ldots r_l!} \equiv 0 \pmod r.$$

Hence $(r, m_k(A_n)) \neq 1$, which is a contradiction. Hence

$$r \mid 1^{r_1}2^{r_2}\ldots l^{r_l} r_1! r_2! \ldots r_l!.$$

Therefore, there exists at least a $x_i$ with cyclic structure $1^{r_1}2^{r_2}\ldots l^{r_l}$such that $r_j = r + t$ for some $1 \leqslant j \leqslant l$, where $t$ is a non-negative integer, or one of the numbers 1, 2, ..., $l$ is equal to $r$.

Now for the completion of our claim, we consider the following cases:

**Case a.** Let $n = r$. By the above discussion in this case there exists $x_i$ such that the cyclic structure of $x_i$ is $1^r$ or $r^1$.

If the cyclic structure of $x_i$ is $1^r$, then $|x_i| = 1$, which is a contradiction.

If the cyclic structure of $x_i$ is $r^1$, then $|x_i| = k = r$. Therefore $m_r(G) = m_r(A_n)$.

**Case b.** Let $n = r + 1$. In this case, there exists $x_i$ such that the cyclic structure of $x_i$ is $1^{r+1}$ or $1^1 r^1$.

If the cyclic structure of $x_i$ is $1^{r+1}$, then $|x_i| = 1$, which is a contradiction.

If the cyclic structure of $x_i$ is $1^1 r^1$. Then $|x_i| = k = r$ and $m_r(G) = m_r(A_n)$.

**Case c.** Let $n = r+2$. In this case, there exists $x_i$ such that the cyclic structure of $x_i$ is $1^2 r^1$. Thus $|x_i| = k = r$ and $m_r(G) = m_r(A_n)$.

**Case d.** Let $n = r + 3$. In this case, there exists $x_i$ such that the cyclic structure of $x_i$ is $1^r3^1$, $3^1r^1$ or $1^3r^1$. Since $m_k(A_n) \equiv -1 \pmod r$ and $m_k(A_n) = \sum |cl_{A_n}(x_i)|$ such that $|x_i| = k$ and $|cl_{A_n}(x_j)| \equiv 0 \pmod r$ for $j \neq i$, $|cl_{A_n}(x_i)| \equiv -1 \pmod r$.

If the cyclic structure of $x_i$ is $1^r3^1$, then $\frac{(r+3)!}{r!3} \equiv -1 \pmod r$. Thus $\frac{(r+3)(r+2)(r+1)}{3} \equiv -1 \pmod r$. It follows that $2 \equiv -1 \pmod r$. Then $r = 3$ and $n = 6$, a contradiction.

If the cyclic structure of $x_i$ is $3^1r^1$, then $\frac{(r+3)!}{3r} \equiv -1 \pmod r$. Thus $\frac{(r+3)(r+2)(r+1)(r-1)!}{3} \equiv -1 \pmod r$. Then $2(r-1)! \equiv -1 \pmod r$. By Wilson's theorem, $(r-1)! \equiv -1 \pmod r$. So $2 \equiv 1 \pmod r$, a contradiction. Therefore the cyclic structure of $x_i$ is $1^3r^1$. Then $|x_i| = k = r$ and $m_r(G) = m_r(A_n)$.

**Case e.** Let $n = r+4$. In this case, there exists $x_i$ such that the cyclic structure of $x_i$ is $1^{r+1}3^1$, $1^r2^2$, $2^2r^1$, $1^13^1r^1$ or $1^4r^1$. By $|cl_{A_n}(x_i)| \equiv -1 \pmod r$, we can see easily the cyclic structures $x_i$ is not equal to $1^{r+1}3^1$ or $1^r2^2$.

Let the cyclic structure of $x_i$ be $2^2r^1$ or $1^13^1r^1$. Then $\frac{(r+4)!}{8r} \equiv -1 \pmod r$ or $\frac{(r+4)!}{3r} \equiv -1 \pmod r$, respectively. Hence $3(r-1)! \equiv -1 \pmod r$ or $8(r-1)! \equiv -1 \pmod r$, respectively.

If $3(r-1)! \equiv -1 \pmod r$, then by Wilson's theorem, $3 \equiv 1 \pmod r$, a contradiction.

If $8(r-1)! \equiv -1 \pmod r$, then $8 \equiv 1 \pmod r$. Thus $r = 7$ and $n = 11$. Since $r$ is the greatest prime not exceeding $n$, we get a contradiction.

Therefore the cyclic structure of $x_i$ is $1^4r^1$. Then $|x_i| = k = r$ and $m_r(G) = m_r(A_n)$.

**Case f.** Let $n = r+5$. In this case, there exists $x_i$ such that the cyclic structure of $x_i$ is $1^{r+2}3^1$, $1^{r+1}2^2$, $1^r5^1$, $5^1r^1$, $1^12^2r^1$, $1^23^1r^1$, $1^23^1r^1$ or $1^5r^1$. By $|cl_{A_n}(x_i)| \equiv -1 \pmod r$, we can see easily the cyclic structures $x_i$ is not equal to $1^{r+2}3^1$, $1^{r+1}2^2$, $1^r5^1$, $1^12^2r^1$ or $1^23^1r^1$.

Let cyclic structure of $x_i$ be $r^15^1$. Then $\frac{(r+5)!}{5r} \equiv -1 \pmod r$. Hence $r = 23$, $n = 28$ and $|x_i| = 23 \times 5 = 115$. Therefore $m_{23}(G) = m_{115}(A_{28}) = \frac{28!}{115}$.

If $115 \notin \pi_e(G)$, then the group $P_5$ acts fixed point freely on the set of elements of order 23. Thus $|P_5| \mid m_{23}(G) = \frac{28!}{115}$. Since $|G| = |A_n|$, $|P_5| = 5^6$, we get a contradiction.

Let $115 \in \pi_e(G)$. By Lemma 2.1, $\phi(5)m_{23} \mid m_{115}$. Then $4 \times \frac{28!}{115} \mid m_{115}$. On the other hand, there are $\frac{28!}{27}$ elements of order 27 in $A_{28}$. This is the greatest member in nse($A_{28}$)=nse($G$). But $m_{115} > \frac{28!}{27}$, a contradiction.

Therefore the cyclic structure of $x_i$ is $1^5r^1$. Then $|x_i| = k = r$ and $m_r(G) = m_r(A_n)$. This proves the claim.

Hence the number of Sylow $r-$subgroups of $G$ is equal to the number of Sylow $r-$subgroups of $A_n$. Since $|G| = |A_n|$, $|N_G(R)| = |N_{A_n}(S)|$ where $R \in$Syl$_r(G)$ and $S \in$Syl$_r(A_n)$. By Lemma 2.2, $G \cong A_n$. □

For an arbitrary $n$ a question arises naturally as follows.

**Problem.** Let $G$ be a group such that $|G| = |A_n|$ and nse($G$)= nse($A_n$). Is $G$ isomorphic to $A_n$?

If $r = 83$ and $n = 83 + 6 = 89$, then $n = 89$ is the smallest $n$ for which the answer is not known.

## 4. Acknowledgment

The authors are thankful to the referee for carefully reading the paper and for his suggestions and remarks. Partial support by the Center of Excellence of Algebraic Hyper structures and its Applications of Tarbiat Modares University (CEAHA) is gratefully acknowledge by the third author (AI).

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Received by the editors: 15.01.2014
and in final form 14.02.2014.