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RESEARCH ARTICLE

Minimal non-PC-groups

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Abstract. The purpose of this paper is to prove that a non-perfect group G is a minimal non-PC-group if and only if it is a minimal non-FC-group. It is shown that a perfect locally graded minimal non-PC-group is an indecomposable countable locally finite p-group.

1. Introduction

A group G is called a *PC-group* if the quotient group $G/C_G(x^G)$ is polycyclic-by-finite for all $x \in G$ [1]. The class of PC-groups is closed with respect to subgroups, quotients and direct products of its members and contains FC-groups (that is groups with finite conjugacy classes). Recall that a group G is called *non-perfect* if the derived subgroup G' is proper in G, and is called *perfect* otherwise. Moreover, a group is *locally graded* if its every finitely generated subgroup contains a proper subgroup of finite index [2]. Recall also that a group G is called *indecomposable* if any two proper its subgroups generate a proper subgroup in G, and is called *decomposable* otherwise.

If $\mathfrak{X}$ is a class of groups, then a group G is called *a minimal non-$\mathfrak{X}$-group* if it is not a $\mathfrak{X}$-group, while every proper subgroup of G is a $\mathfrak{X}$-group. Every minimal non-FC-group is a minimal non-PC-group and every torsion minimal non-PC-group is a minimal non-FC-group. It is known that finitely generated torsion-free minimal non-PC-groups there

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exist (see e.g. Theorem 28.3 from [3]). V. V. Belyaev (see [4], [5] and [6]) have proved that every minimal non-FC-group with a non-trivial finite or abelian homomorphic image is a finite extension of a divisible Černikov p-group. F. Russo and N. Trabelsi [8] have shown that a minimal non-PC-group with a non-trivial finite homomorphic image is an extension of a divisible abelian group of finite rank by a cyclic group. By Corollary 2.3 of [8], a locally graded minimal non-PC-group is not finitely generated. In this way we study the problem:"Are there non-torsion locally graded minimal non-PC-groups?" and give the answer.

Theorem 1. *Let G be a non-perfect group. Then G is a minimal non-PC-group if and only if it is a minimal non-FC-group.*

From this, in particular, it holds that every non-perfect minimal non-PC-group has a non-trivial finite homomorphic image, and so it is torsion.

The question about the structure of perfect locally graded minimal non-FC-groups discussed by V. V. Belyaev (see [5], [6] and [7]), M. Kuzucuoğlu and R. E. Phillips [10], F. Leinen [11]. It is proved (see [6] and [10]) that every perfect locally graded minimal non-FC-group must be a p-group. In this way we prove the following

Theorem 2. *A perfect locally graded minimal non-PC-group is an indecomposable countable locally finite p-group.*

Throughout this paper, p will always denote a prime, $\mathbb{C}_{p^\infty}$ the quasicyclic p-group, $\mathbb{Z}$ the integer numbers ring. For a group G, G' will indicate the derived subgroup, $Z(G)$ the center, $C_G(H)$ the centralizer of H in G, $\langle x\rangle^G$ the normal closure of a cyclic subgroup $\langle x\rangle$ in G and G^n the subgroup generated by the n-th powers of all elements in G.

Any unexplained terminology is standard as in [12] and [13].

2. Non-perfect minimal non-PC-groups

Lemma 1. *Let G be a minimal non-PC-group. If H is a normal subgroup of finite index in G, then G/H is a cyclic p-group for some prime p.*

Proof. See [8, Lemma 3.3]. □

In the next we need the fact (which contains in Theorem 1.2 of [8]) that a minimal non-PC-group with a non-trivial finite homomorphic image contains a proper divisible abelian subgroup. But the proof of Theorem 1.2 from [8] (see its part (i)) depends on the fact that any residually finite group, whose finite quotients are cyclic of prime-power orders, must be finite (that is false). Therefore preliminary we prove the following

Lemma 2. *Let G be a minimal non-FC-group. If G is non-perfect, then its derived subgroup G' is divisible abelian and G/G' is a cyclic p-group.*

Proof. Assume, by contrary, that $G'\langle x\rangle$ is proper in G for any its element x.

a) Suppose that, for every $x \in G$, $G'\langle x\rangle$ is contained in some maximal subgroup M of G. Then $\langle x\rangle^G \leqslant M$ and there exists an element $b \in M$ such that

$$\langle x\rangle^G = x^M \cdot \langle b\rangle = \langle x\rangle^M \cdot \langle b\rangle^M.$$

Since quotient groups $M/C_M(\langle x\rangle^M)$ and

$$C_M(\langle x\rangle^M)/(C_M(\langle x\rangle^M)\bigcap C_M(\langle b\rangle^M))$$

are polycyclic-by-finite, $M/(C_M(\langle x\rangle^M)\bigcap C_M(\langle b\rangle^M))$ is also polycyclic-by-finite. Then, in view of

$$C_M(\langle x\rangle^M)\bigcap C_M(\langle b\rangle^M) \leqslant C_M(\langle x\rangle^G),$$

we obtain that $M/C_M(\langle x\rangle^G)$ (and so $G/C_G(\langle x\rangle^G)$) is a polycyclic-by-finite group, a contradiction.

b) Now assume that there is an element $x \in G$ such that $G'\langle x\rangle$ is a proper subgroup of G that is not contained in a maximal subgroup of G. This means that $G/G'\langle x\rangle$ is a divisible abelian group. If it is decomposable, then $G = G_1G_2$ is a product of two proper normal PC-subgroups G_1 and G_2, each of which contains the derived subgroup G'. If the quotient group $G/G'\langle x\rangle$ is indecomposable, then it is quasicyclic.

In the first case suppose that $i, j \in \{1, 2\}$, $i \neq j$ and a_j is a non-trivial element of G_j. Then the quotient group

$$G/C_{G_j}(\langle a_j\rangle^{G_j})G_i$$

is polycyclic-by-finite and it not contains proper subgroups of finite index. Hence

$$G = C_{G_j}(\langle a_j\rangle^{G_j})G_i.$$

Since $G_i\langle a_j\rangle$ is a proper PC-subgroup in G, $G/G_G(\langle a_j\rangle^G)$ is polycyclic-by-finite. But it is divisible, and therefore $a_j \in Z(G)$. This means that G is abelian, a contradiction.

In the second case $G/G'\langle x\rangle$ is a quasicyclic p-group. If the derived subgroup G' not contains proper subgroups of finite index, then we obtain that a PC-subgroup $G'\langle x\rangle$ is abelian for any $x \in G$, which leads to a contradiction. Thus $(G')^n$ is a proper subgroup in G' for some positive integer n, and so

$$A = G/(G')^n$$

is a torsion (and consequently locally finite) group every proper subgroup of which is a FC-group. By results from [4], A is a FC-group, a contradiction.

Thus the quotient group G/G' is cyclic. By Lemma 1, G/G' is a p-group for some prime p and the derived subgroup G' not contains proper subgroups of finite index. From this it follows that G' is a divisible abelian group. □

Lemma 3. *Let G be a group with a non-trivial finite homomorphic image. If G is a minimal non-PC-group, then it is torsion.*

Proof. By Lemmas 1 and 2,

$$G = G'\langle a\rangle,$$

where G' is a divisible abelian group, $a^{p^k} \in G'$ with some $a \in G$, a prime p and a positive integer k.

The torsion part of G' is normal in G. Therefore, without loss of generality, we can assume that the derived subgroup G' is torsion-free. Let us $t \in G_{G'}(a)$ and n is a positive integer. Then there exists an element $x \in G'$ such that

$$t = x^n$$

and

$$[x, a]^n = [x^n, a] = [t, a] = 1.$$

Hence $[x, a] = 1$ and $x \in G_{G'}(a)$. This gives that $G_{G'}(a)$ is a divisible subgroup. Since any divisible PC-group is abelian, the centralizer of $aG_{G'}(a)$ in the quotient group $G/G_{G'}(a)$ is trivial. Therefore, without loss of generality, we can assume that $G_{G'}(a) = \langle 1\rangle$ is trivial.

Let r and s be different primes. Since G' is a $\mathbb{Z}[G/G']$-module, by Lemma 2.3 of [9] it contains a submodule N such that G'/N is a torsion group, which has some elements of orders r and s. Hence $G'/N = A_1 \times A_2$ is a group direct product of a non-trivial r-subgroup A_1 and a non-trivial

r'-subgroup A_2. Let B be an inverse image of A_1 in G. Then $H = B\langle a\rangle$ is a proper normal subgroup of G and the intersection

$$C_H(a) \bigcap B = \langle 1\rangle$$

is trivial. Inasmuch as $B/N = A_1$ is a non-trivial divisible group and

$$C_H(\langle a\rangle^H) \leqslant C_H(a),$$

the quotient group $H/C_H(\langle a\rangle^H)$ is not polycyclic-by-finite, a contradiction. Hence G' is a torsion subgroup. □

Proof of Theorem 1. The assertion follows from Lemmas 2 and 3. □

3. Perfect minimal non-PC-groups

Lemma 4 ([1]). *A group G is a PC-group if and only if, for every finite subset $\varnothing \neq X \subseteq G$, its normal closure $\langle X\rangle^G$ is a polycyclic-by-finite group.*

Lemma 5. *A locally graded minimal non-PC-group G is countable.*

Proof. Since G is not a PC-group, in view of Lemma 4 there is a non-trivial element $g \in G$ such that $\langle g\rangle^G$ is not polycyclic-by-finite. Then $\langle g\rangle^G$ (and consequently $[G, \langle g\rangle]$) is not finitely generated. Therefore there exists an infinite properly ascending chain of subgroups

$$\langle g\rangle < \langle g, t_1\rangle < \cdots < \langle g, t_1, \ldots, t_n\rangle < \cdots$$

such that

$$t_n \in [G, \langle g\rangle]$$

and

$$t_n \notin \langle g, t_1, \ldots, t_{n-1}\rangle \qquad (n \in \mathbb{N}).$$

Let $H = \langle t_n \mid n \in \mathbb{N}\rangle$ and a subgroup K be generated by g and those elements in G involved in the expressions of all t_n. Then

$$H \leqslant \langle g\rangle^K \leqslant K,$$

and so K is not a PC-group. Hence $K = G$ and G is countable. □

Lemma 6. *A perfect locally graded minimal non-PC-group G is a locally finite p-group.*

Proof. a) Let H be a finitely generated subgroup of G. Then H is proper (and consequently polycyclic-by-finite) subgroup in G. Let K be a normal polycyclic subgroup of finite index in H. By Proposition 1.3.7 of [12], its normal core

$$K_G = \bigcap_{g \in G} g^{-1} K g$$

has a finite index in H. Then an image $\overline{H}$ of H in the quotient group $\overline{G} = G/K_G$ is contained in the center of $\overline{G}$. Hence G is a locally solvable group.

b) Let A/B be a chief factor of G. Without loss of generality, we can assume that $B = \langle 1 \rangle$ is trivial. Then A is an elementary abelian p-group for some prime p and $\langle x \rangle^G \leqslant A$ for any $x \in A$. Since $\langle x \rangle^G$ is finite, we conclude that $G = C_G(\langle x \rangle^G)$, and so $x \in Z(G)$. This means that every chief factor is central in G, and therefore G is a locally nilpotent group. This yields that G is a locally finite p-group. □

Lemma 7. *A perfect locally graded minimal non-PC-group G is indecomposable.*

Proof. Let us $1 \neq g \in G$. Since G is a countable locally finite p-group, it is non-simple and $\langle g \rangle^G$ is a proper normal subgroup in G. By Lemma 4, $\langle g \rangle^G$ is polycyclic-by-finite.

a) If S is a proper subgroup of G and $\langle S, g \rangle = G$, then $G = \langle g \rangle^G S$ is a PC-group by Lemma 4, a contradiction. Hence $\langle S, g \rangle \neq G$.

b) Now we assume that S_1, S_2 are proper subgroups in G. Since

$$S_i/(S_i \bigcap C_G(\langle g \rangle^G))$$

is a polycyclic-by-finite group ($i = 1, 2$), there exist $t_1, \ldots, t_n \in G$ such that

$$\langle S_1, S_2 \rangle \leqslant \langle C_G(\langle g \rangle^G), t_1, \ldots, t_n \rangle.$$

In view of the part a), we deduce that $\langle S_1, S_2 \rangle$ is a proper subgroup in the group G. □

Proof of Theorem 2. The assertion holds from Lemmas 5, 6 and 7. □

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