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On the units of integral group ring of $C_n \times C_6$

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Abstract. There are many kind of open problems with varying difficulty on units in a given integral group ring. In this note, we characterize the unit group of the integral group ring of $C_n \times C_6$ where $C_n = \langle a : a^n = 1\rangle$ and $C_6 = \langle x : x^6 = 1\rangle$. We show that $\mathcal{U}_1(\mathbb{Z}[C_n \times C_6])$ can be expressed in terms of its 4 subgroups. Furthermore, forms of units in these subgroups are described by the unit group $\mathcal{U}_1(\mathbb{Z}C_n)$. Notations mostly follow [11].

1. Introduction

Let G given as a finite group. Its integral group ring is denoted by $\mathbb{Z}G$. Invertible elements in $\mathbb{Z}G$ is called by units and the set of units forms a group according to the multiplication and is shown by $\mathcal{U}(\mathbb{Z}G)$. The group of units with augmentation 1 is displayed by $\mathcal{U}_1(\mathbb{Z}G)$. If one pay attention to the corresponding literature, that can easily see that the obtained results mostly arises from finite groups especially finite abelian groups. Fundamentals of the unit problem have come from the thesis of G. Higman in 1940. Higman stated and proved the following [4]:

Lemma 1. *If $U(\mathbb{Z}G) = \pm G$, then $U(\mathbb{Z}[G \times C_2]) = \pm[G \times C_2]$.*

Also, the following useful lemma was shown in [4] and [3].

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Lemma 2. *$\mathcal{U}(\mathbb{Z}G)$ has a torsion-free complement of finite rank $\rho = \frac{1}{2}(|G| + n_2 + 1 - 2l)$ where n_2 shows the number of elements of order 2 in G and l is the number of all distinct cyclic subgroups of G.*

On the other hand, in [7], Li considered the question: If $\mathcal{U}(\mathbb{Z}G)$ has a normal complement generated by bicyclic units, does $\mathcal{U}(\mathbb{Z}[G \times C_2])$ has also a normal complement generated by bicyclic units? Jespers showed that the answer for this question is yes while $G = D_6$ or D_8 [8–10]. Li gave a counterexample for showing this is not true in general by considering the group $D_8 \times C_2 \times C_2$ [7]. However, Li proved that if $\mathcal{U}(\mathbb{Z}G)$ is generated by unitary units, then $\mathcal{U}(\mathbb{Z}[G \times C_2])$ is also generated by unitary units [7]. Another description of $\mathcal{U}(\mathbb{Z}[G \times C_2])$ was given by Low in [6] by linearly extending some group epimorphisms to the group ring homomorphisms. He also tried to generalize the problem for $\mathcal{U}(\mathbb{Z}[G \times C_p])$ where p is a prime integer. In [6], He showed that

$$\mathcal{U}(\mathbb{Z}[G \times C_p]) = K \rtimes \mathcal{U}(\mathbb{Z}G) \cong M \rtimes \mathcal{U}(\mathbb{Z}G)$$

where K is the kernel of the natural group homomorphism: $\pi : \mathcal{U}(\mathbb{Z}[G \times C_p]) \longrightarrow \mathcal{U}(\mathbb{Z}G)$ and M is a subgroup of finite index in $\mathcal{U}(\mathbb{Z}[\zeta]G)$ such that ζ is a primitive p^{th} root of unity. Low also explicitly proved the following 4 lemmas [6]:

Lemma 3. *Let $G^* = G \times \langle x : x^2 = 1\rangle$. Then, $\mathcal{U}(\mathbb{Z}G^*)$ is obtained as*

$$\{u = 1 + (x-1)\alpha : \alpha \in \mathbb{Z}G, u \in \mathcal{U}(\mathbb{Z}G^*)\} \rtimes \mathcal{U}(\mathbb{Z}G).$$

Further, $1 + (x-1)\alpha \in \mathcal{U}(\mathbb{Z}G^*) \Leftrightarrow 1 - 2\alpha \in \mathcal{U}(\mathbb{Z}G)$.

Lemma 4. *Let $P = \langle a, b : a^4 = b^4 = 1, [b, a] = a^2\rangle$ be the indecomposable group of order 16. Then,*

$$\mathcal{U}(\mathbb{Z}[P \times C_2]) = \pm[F_{65} \rtimes F_9] \rtimes (P \times C_2)$$

where F_i denotes a free group of rank i.

Lemma 5. *Let $C_5^* = \langle c : c^5 = 1\rangle \times \langle x : x^2 = 1\rangle$. Then, the unit group*

$$\mathcal{U}(\mathbb{Z}C_5^*) = \langle 1 + (x-1)P\rangle \times \langle v\rangle \times C_5^*$$

where $P = -3 - c + 3c^2 + 3c^3 - c^4$ and $v = (c+1)^2 - \widehat{c}$.

Lemma 6. *Let $C_8^* = \langle c : c^8 = 1\rangle \times \langle x : x^2 = 1\rangle$. Then, the unit group*

$$\mathcal{U}(\mathbb{Z}C_8^*) = \langle 1 + (x-1)P\rangle \times \langle v\rangle \times C_8^*$$

where $P = -4 - 3c + 3c^3 + 4c^4 + 3c^5 - 3c^7$ and $v = 3 - \widehat{c} + 2(c + c^7) + (c^2 + c^6)$.

Kelebek and Bilgin considered the finite abelian group $C_n \times K_4$ where K_4 is the Klein 4-group and characterized the unit group of its integral group ring in terms of 4 components as follows [1]:

Theorem 1. $\mathcal{U}_1(\mathbb{Z}[C_n \times K_4]) = \mathcal{U}_1(\mathbb{Z}C_n) \times (1+K^x) \times (1+K^y) \times (1+K^{xy})$ *where*

$$\begin{aligned} 1+K^x &= \{1+(x-1)P : 1-2P \in \mathcal{U}_1(\mathbb{Z}C_n)\} \\ 1+K^y &= \{1+(y-1)P : 1-2P \in \mathcal{U}_1(\mathbb{Z}C_n)\} \\ 1+K^{xy} &= \{1+(x-1)(y-1)P : 1+4P \in \mathcal{U}_1(\mathbb{Z}C_n)\} \end{aligned}$$

2. Motivation for construction of $\mathcal{U}_1(\mathbb{Z}[C_n \times C_6])$

Now, let us begin with some remarks.

Remark 1. The following maps are group epimorphisms:

$$\begin{aligned} \pi_{x^2} : C_n \times C_6 &\longrightarrow C_n \times \langle x^2 \rangle \\ a &\mapsto a \\ x &\mapsto x^2 \end{aligned}$$

$$\begin{aligned} \pi_{x^3} : C_n \times C_6 &\longrightarrow C_n \times \langle x^3 \rangle \\ a &\mapsto a \\ x &\mapsto x^3 \end{aligned}$$

Remark 2. $\mathrm{Ker}(\pi_{x^2}) = \langle x^3 \rangle$ and $\mathrm{Ker}(\pi_{x^3}) = \langle x^2 \rangle$.

Since $C_n \times \langle x^2 \rangle \hookrightarrow C_n \times C_6$, $C_n \times \langle x^3 \rangle \hookrightarrow C_n \times C_6$ and i denotes the inclusion map, we get the following short exact sequences at group level:

$$0 \longrightarrow \langle x^3 \rangle \xrightarrow{i} C_n \times C_6 \xrightarrow{\pi_{x^2}} C_n \times \langle x^2 \rangle \longrightarrow 0$$
$$0 \longrightarrow \langle x^2 \rangle \xrightarrow{i} C_n \times C_6 \xrightarrow{\pi_{x^3}} C_n \times \langle x^3 \rangle \longrightarrow 0$$
$$0 \longrightarrow \langle x \rangle \xrightarrow{i} C_n \times C_6 \xrightarrow{\pi_{x^2}\pi_{x^3}} C_n \longrightarrow 0$$

If we linearly extend π_{x^2} and π_{x^3} to integral group rings over $\mathbb{Z}$, we obtain the following ring homomorphisms:

$$\begin{aligned} \overline{\pi}_{x^2} : \mathbb{Z}[C_n \times C_6] &\longrightarrow \mathbb{Z}[C_n \times \langle x^2 \rangle] \\ \sum_{j=0}^{5} P_j x^j &\mapsto (P_0 + P_3) + (P_1 + P_4)x^2 + (P_2 + P_5)x^4 \end{aligned}$$

and

$$\begin{aligned}\overline{\pi}_{x^3} : \mathbb{Z}[C_n \times C_6] &\longrightarrow \mathbb{Z}[C_n \times \langle x^3 \rangle] \\ \sum_{j=0}^{5} P_j x^j &\mapsto (P_0 + P_2 + P_4) + (P_1 + P_3 + P_5)x^3\end{aligned}$$

Lemma 7. $K^{x^2} := \mathrm{Ker}(\overline{\pi}_{x^2}) = (x^3 - 1)\mathbb{Z}[C_n \times \langle x^2 \rangle]$

Proof.

$$\begin{aligned}\mathrm{Ker}(\overline{\pi}_{x^2}) &= \Big\{ \sum_{i=0}^{5} P_i x^i : \overline{\pi}_{x^2}(\sum_{i=0}^{5} P_i x^i) = 0, P_i \in \mathbb{Z}C_n \Big\} \\ &= \Big\{ \sum_{i=0}^{5} P_i x^i : P_0 + P_3 = P_1 + P_4 = P_2 + P_5 = 0 \Big\} \\ &= \Big\{ \sum_{i=0}^{5} P_i x^i : P_0 = -P_3, P_1 = -P_4, P_2 = -P_5 \Big\} \\ &= \{ -P_3 - P_4 x - P_5 x^2 + P_3 x^3 + P_4 x^4 + P_5 x^5 \} \\ &= \{ (x^3 - 1)P_3 + (x^4 + x)P_4 + (x^5 - x^2)P_5 \} \\ &= (x^3 - 1)[\mathbb{Z}C_n \oplus x^2 \mathbb{Z}C_n \oplus x^4 \mathbb{Z}C_n] \\ &= (x^3 - 1)\mathbb{Z}[C_n \times \langle x^2 \rangle].\end{aligned}$$

□

Lemma 8. $K^{x^3} := \mathrm{Ker}(\overline{\pi}_{x^3}) = (x^2 - 1)[\mathbb{Z}C_n \oplus x\mathbb{Z}C_n \oplus x^2\mathbb{Z}C_n \oplus x^3\mathbb{Z}C_n]$

Proof.

$$\begin{aligned}\mathrm{Ker}(\overline{\pi}_{x^3}) &= \Big\{ \sum_{i=0}^{5} P_i x^i : \overline{\pi}_{x^3}(\sum_{i=0}^{5} P_i x^i) = 0, P_i \in \mathbb{Z}C_n \Big\} \\ &= \Big\{ \sum_{i=0}^{5} P_i x^i : P_0 + P_2 + P_4 = P_1 + P_3 + P_5 = 0 \Big\} \\ &= \Big\{ \sum_{i=0}^{5} P_i x^i : P_0 = -(P_2 + P_4), P_1 = -(P_3 + P_5) \Big\} \\ &= \{ (x^2 - 1)[P_2 + xP_3 + (x^2 + 1)P_4 + (x^2 + 1)xP_5] \} \\ &= (x^2 - 1)[\mathbb{Z}C_n \oplus x\mathbb{Z}C_n \oplus x^2\mathbb{Z}C_n \oplus x^3\mathbb{Z}C_n]\end{aligned}$$

Similarly, we can write the following ring homomorphism:

$$\overline{\pi}_{x^2}\overline{\pi}_{x^3} : \mathbb{Z}[C_n \times C_6] \longrightarrow \mathbb{Z}C_n$$
$$\sum_{j=0}^{5} P_j x^j \mapsto \sum_{j=0}^{5} P_j. \qquad \square$$

Lemma 9. $K^{x^2x^3} := \mathrm{Ker}(\overline{\pi}_{x^2}\overline{\pi}_{x^3}) = \oplus_{j=1}^{5}(x^j - 1)\mathbb{Z}C_n$

Proof.

$$\begin{aligned}
\mathrm{Ker}(\overline{\pi}_{x^2}\overline{\pi}_{x^3}) &= \Big\{\sum_{i=0}^{5} P_i x^i : \overline{\pi}_{x^2}\overline{\pi}_{x^3}(\sum_{i=0}^{5} P_i x^i) = 0, P_i \in \mathbb{Z}C_n\Big\} \\
&= \Big\{\sum_{i=0}^{5} P_i x^i : \sum_{i=0}^{5} P_i = 0, P_i \in \mathbb{Z}C_n\Big\} \\
&= \Big\{\sum_{i=0}^{5} P_i x^i : P_0 = -\sum_{i=1}^{5} P_i, P_i \in \mathbb{Z}C_n\Big\} \\
&= \Big\{-\sum_{i=1}^{5} P_i + \sum_{i=1}^{5} P_i x^i : P_i \in \mathbb{Z}C_n\Big\} \\
&= \Big\{\sum_{j=1}^{5}(x^j - 1)P_j : P_j \in \mathbb{Z}C_n\Big\} \\
&= \oplus_{j=1}^{5}(x^j - 1)\mathbb{Z}C_n.
\end{aligned}$$

By Remarks 1 and 2, we get the following short exact sequences at group ring level:

$$0 \longrightarrow K^{x^2} \xrightarrow{i} \mathbb{Z}[C_n \times C_6] \xrightarrow{\overline{\pi}_{x^2}} \mathbb{Z}[C_n \times \langle x^2\rangle] \longrightarrow 0$$
$$0 \longrightarrow K^{x^3} \xrightarrow{i} \mathbb{Z}[C_n \times C_6] \xrightarrow{\overline{\pi}_{x^3}} \mathbb{Z}[C_n \times \langle x^3\rangle] \longrightarrow 0$$
$$0 \longrightarrow K^{x^2x^3} \xrightarrow{i} \mathbb{Z}[C_n \times C_6] \xrightarrow{\overline{\pi}_{x^2}\overline{\pi}_{x^3}} \mathbb{Z}C_n \longrightarrow 0$$

If we restrict $\overline{\pi}_{x^2}$ and $\overline{\pi}_{x^3}$ to the unit level, we conclude that the followings are also short exact sequences:

$$1 \longrightarrow \mathcal{U}_1(1 + K^{x^2}) \xrightarrow{i} \mathcal{U}_1(\mathbb{Z}[C_n \times C_6]) \xrightarrow{\overline{\pi}_{x^2}} \mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^2\rangle]) \longrightarrow 1$$
$$1 \longrightarrow \mathcal{U}_1(1 + K^{x^3}) \xrightarrow{i} \mathcal{U}_1(\mathbb{Z}[C_n \times C_6]) \xrightarrow{\overline{\pi}_{x^3}} \mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^3\rangle]) \longrightarrow 1$$
$$1 \longrightarrow \mathcal{U}_1(1 + K^{x^2x^3}) \xrightarrow{i} \mathcal{U}_1(\mathbb{Z}[C_n \times C_6]) \xrightarrow{\overline{\pi}_{x^2}\overline{\pi}_{x^3}} \mathcal{U}_1(\mathbb{Z}C_n) \longrightarrow 1$$

Since we can consider embeddings $\mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^2 \rangle]) \hookrightarrow \mathcal{U}_1(\mathbb{Z}[C_n \times C_6])$ and $\mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^3 \rangle]) \hookrightarrow \mathcal{U}_1(\mathbb{Z}[C_n \times C_6])$, the following split extensions hold:

$$\begin{aligned}\mathcal{U}_1(\mathbb{Z}[C_n \times C_6]) &= \mathcal{U}_1(1 + K^{x^2}) \rtimes \mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^2 \rangle]) \\ &= \mathcal{U}_1(1 + K^{x^3}) \rtimes \mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^3 \rangle]) \\ &= \mathcal{U}_1(1 + K^{x^2 x^3}) \rtimes \mathcal{U}_1(\mathbb{Z}C_n).\end{aligned}$$ □

Remark 3. In $\mathcal{U}_1(\mathbb{Z}[C_n \times C_6])$ the normal subgroups $\mathcal{U}_1(1 + K^{x^2})$, $\mathcal{U}_1(1 + K^{x^3})$ and $\mathcal{U}_1(1 + K^{x^2 x^3})$ are determined as in the following forms respectively:

(i) $\{u = 1 + (x^3 - 1)[P_0 + P_2 x^2 + P_4 x^4] : u \text{ is a unit}\}$;

(ii) $\{u = 1 + (x^2 - 1)[P_0 + P_1 x + P_2 x^2 + P_3 x^3] : u \text{ is a unit}\}$;

(iii) $\{u = 1 + \sum_{j=1}^{5} (x^j - 1) P_j : u \text{ is a unit}\}$.

3. An explicit characterization of $\mathcal{U}_1(\mathbb{Z}[C_n \times C_6])$

In this section, an explicit characterization of $\mathcal{U}_1(\mathbb{Z}[C_n \times C_6])$ is given with the help of the results in the previous section. First, we should give some restrictions of the maps $\overline{\pi}_{x^2}$, $\overline{\pi}_{x^3}$ and $\overline{\pi}_{x^2}\overline{\pi}_{x^3}$. Let $\overline{\pi}_{x^3}|_{\mathcal{U}_1(1+K^{x^2})}$ denote the restriction of $\overline{\pi}_{x^3}$ on $\mathcal{U}_1(1 + K^{x^2})$.

Lemma 10. $W_1 := Im(\overline{\pi}_{x^3}|_{\mathcal{U}_1(1+K^{x^2})}) = 1 + (x^3 - 1)\mathbb{Z}C_n$.

Proof. Let us take an element from $\mathcal{U}_1(1 + K^{x^2})$ as $\gamma = 1 + (x^3 - 1)[P_0 + P_2 x^2 + P_4 x^4]$ where $P_i \in \mathbb{Z}C_n$.Then,

$$\overline{\pi}_{x^3} : \gamma \mapsto 1 + (x^3 - 1)[P_0 + P_2 + P_4].$$

Say $P_0 + P_2 + P_4 = P$. Thus, $Im(\overline{\pi}_{x^3}|_{\mathcal{U}_1(1+K^{x^2})})$ consists of elements of the form $1 + (x^3 - 1)P$. □

Lemma 11. $W_2 := \mathrm{Ker}(\overline{\pi}_{x^3}|_{\mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^2 \rangle])}) = 1 + (x^2 - 1)\mathbb{Z}C_n \oplus (x^4 - 1)\mathbb{Z}C_n$.

Proof. Let us take an element from $\mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^2 \rangle])$ as $\sigma = P_0 + P_2 x^2 + P_4 x^4$. Here, we can manipulate the parameter $P_0 = 1 + P_0'$. Then, we get

$$\overline{\pi}_{x^3} : \sigma \mapsto 1 + P_0' + P_2 + P_4 = 1 \Longleftrightarrow P_0' = -P_2 - P_4.$$

This means that the kernel consists of elements of the form

$$1 + (-P_2 - P_4) + P_2 x^2 + P_4 x^4 = 1 + (x^2 - 1)P_2 + (x^4 - 1)P_4.$$

Hence the required is obtained. □

Lemma 12.

$$W_3 := \mathrm{Ker}(\overline{\pi}_{x^3}|_{\mathcal{U}_1(1+K^{x^2})}) = 1+(x^3-1)(x^2-1)\mathbb{Z}C_n \oplus (x^3-1)(x^4-1)\mathbb{Z}C_n.$$

Proof. Again, let us consider an element from $\mathcal{U}_1(1+K^{x^2})$ as $\eta = 1 + (x^3-1)[P_0 + P_2x^2 + P_4x^4]$. Then,

$$\overline{\pi}_{x^3} : \eta \mapsto 1 + (x^3-1)[P_0 + P_2 + P_4] = 1 \Longleftrightarrow P_0 = -P_2 - P_4$$

Thus, $\mathrm{Ker}(\overline{\pi}_{x^3}|_{\mathcal{U}_1(1+K^{x^2})})$ consists of

$$\begin{aligned} 1 + (x^3-1)[P_0 + P_2x^2 + P_4x^4] &= 1 + (x^3-1)[-P_2 - P_4 + P_2x^2 + P_4x^4] \\ &= 1 + (x^3-1)[(x^2-1)P_2 + (x^4-1)P_4]. \quad \square \end{aligned}$$

Therefore, by Lemma 10, Lemma 11 and Lemma 12, we can construct the following commutative diagram:

$$\begin{tikzcd} W_3 \arrow[r, "i"] \arrow[d, "i"] & \mathcal{U}_1(1+K^{x^3}) \arrow[r, "\overline{\pi}_{x^2}"] \arrow[d, "i"] & W_2 \arrow[d, "i"] \\ \mathcal{U}_1(1+K^{x^2}) \arrow[r, "i"] \arrow[d, "\overline{\pi}_{x^3}"] & \mathcal{U}_1(\mathbb{Z}[C_n \times C_6]) \arrow[r, "\overline{\pi}_{x^2}"] \arrow[d, "\overline{\pi}_{x^3}"] & \mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^2 \rangle]) \arrow[d, "\overline{\pi}_{x^3}"] \\ W_1 \arrow[r, "i"] & \mathcal{U}_1(\mathbb{Z}[C_n \times \langle x^3 \rangle]) \arrow[r, "\overline{\pi}_{x^2}"] & \mathcal{U}_1(\mathbb{Z}C_n) \end{tikzcd}$$

Since we can take embeddings as the inverses of $\overline{\pi}_{x^2}$ and $\overline{\pi}_{x^3}$, this diagram splits as follows:

$$\mathcal{U}_1(\mathbb{Z}[C_n \times C_6]) = W_1 \times W_2 \times W_3 \times \mathcal{U}_1(\mathbb{Z}C_n).$$

Now, let us characterize explicitly W_1, W_2 and W_3.

Proposition 1. $u = 1+(x^3-1)P \in W_1$ *is a unit* $\Longleftrightarrow 1-2P \in \mathcal{U}_1(\mathbb{Z}C_n)$

Proof.

$$\begin{aligned} u = 1 + (x^3-1)P \text{ is unit } &\Longleftrightarrow \exists v = 1 + (x^3-1)Q : uv = 1 \\ &\Longleftrightarrow 1 + (x^3-1)[P + Q - 2PQ] = 1 \\ &\Longleftrightarrow P + Q - 2PQ = 0 \\ &\Longleftrightarrow 1 - 2P - 2Q + 4PQ = 1 \\ &\Longleftrightarrow (1-2P)(1-2Q) = 1 \\ &\Longleftrightarrow 1 - 2P \in \mathcal{U}_1(\mathbb{Z}C_n). \end{aligned}$$

$\square$

Proposition 2. *$u = 1 + (x^2 - 1)P + (x^4 - 1)Q \in W_2$ is a unit $\Longleftrightarrow$ $P^2 + Q^2 - PQ - P - Q = 0$*

Proof. First, we need to define a closed operation. If we define $\alpha = x^2 - 1$ and $\beta = x^4 - 1$, we get the following straightforward computations:

$$\begin{aligned} \alpha^2 &= (x^2-1)^2 = -2(x^2-1) + (x^4-1) = -2\alpha + \beta \\ \alpha\beta &= (x^2-1)(x^4-1) = -(x^2-1) - (x^4-1) = -\alpha - \beta \\ \beta^2 &= (x^4-1)^2 = -2(x^4-1) + (x^2-1) = \alpha - 2\beta \end{aligned}$$

Let us state this operation in a table as follows:

$\bullet$	α	β
α	$-2\alpha+\beta$	$-\alpha-\beta$
β	$-\alpha-\beta$	$\alpha-2\beta$

Now, we can give a necessary and sufficient condition to be a unit for the element u. $u = 1 + (x^2 - 1)P + (x^4 - 1)Q \in W_2$ is a unit if and only if $\exists v = 1 + (x^2 - 1)P^{'} + (x^4 - 1)Q^{'}$ such that $uv = 1$. Hence,

$$1 + \alpha P + \beta Q + \alpha P^{'} + \beta Q^{'} + \alpha^2 PP^{'} + \beta^2 QQ^{'} + \alpha\beta(PQ^{'} + P^{'}Q) = 1.$$

By the above operation, we can arrange this equation as

$$\begin{aligned} 1 + \alpha(P + P^{'}) + \beta(Q + Q^{'}) + (-2\alpha + \beta)PP^{'} \\ + (\alpha - 2\beta)QQ^{'} + (-\alpha - \beta)(PQ^{'} + P^{'}Q) = 1 \end{aligned}$$

That is,

$$\begin{aligned} 1 + \alpha(P + P^{'} - 2PP^{'} + QQ^{'} - PQ^{'} - P^{'}Q) \\ + \beta(Q + Q^{'} + PP^{'} - 2QQ^{'} - PQ^{'} - P^{'}Q) = 1. \end{aligned}$$

This equation holds if and only if the following system of matrix has a unique solution:

$$\begin{bmatrix} 1-2P-Q & Q-P \\ P-Q & 1-2Q-P \end{bmatrix} \begin{bmatrix} P^{'} \\ Q^{'} \end{bmatrix} = \begin{bmatrix} -P \\ -Q \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} 1-2P-Q & Q-P \\ P-Q & 1-2Q-P \end{bmatrix} \in SL_2(\mathbb{Z}C_n)$$

A straightforward calculation shows that $P^2 + Q^2 - PQ - P - Q = 0$. $\square$

Proposition 3. *$u = 1 + (x^3-1)(x^2-1)P + (x^3-1)(x^4-1)Q \in W_3$ is a unit if and only if the following equation holds:*

$$2P^2 + 2Q^2 - 2PQ - P - Q = 0$$

Proof. First, let $\lambda = (x^3-1)(x^2-1)$ and $\mu = (x^3-1)(x^4-1)$. One can easily compute the followings:

$$\begin{aligned}
\lambda^2 &= (x^3-1)^2(x^2-1)^2 = 4\lambda - 2\mu,\\
\lambda\mu &= (x^3-1)^2(x^2-1)(x^4-1) = 2\lambda + 2\mu,\\
\mu^2 &= (x^3-1)^2(x^4-1)^2 = -2\lambda + 4\mu.
\end{aligned}$$

In a better expression, we write

$\bullet$	λ	μ
λ	$4\lambda - 2\mu$	$2\lambda + 2\mu$
μ	$2\lambda + 2\mu$	$-2\lambda + 4\mu$

Now, let us determine the necessary and sufficient condition to be a unit for an element u. $u = 1 + \lambda P + \mu Q \in W_3$ is a unit if and only if $\exists v = 1 + \lambda P' + \mu Q' : uv = 1$. Thus, a straight forward computation shows us that

$$\begin{aligned}
1 + \lambda(P + P' + 4PP' + 2P'Q + 2PQ' - 2QQ')&\\
+ \mu(Q + Q' - 2PP' + 2P'Q + 2PQ' + 4QQ') &= 1.
\end{aligned}$$

This equation holds if and only if the following system of matrix has a unique solution:

$$\begin{bmatrix} 1+4P+2Q & 2P-2Q \\ 2Q-2P & 1+2P+4Q \end{bmatrix} \begin{bmatrix} P' \\ Q' \end{bmatrix} = \begin{bmatrix} -P \\ -Q \end{bmatrix}$$

Then, the required result comes from the following:

$$\begin{bmatrix} 1+4P+2Q & 2P-2Q \\ 2Q-2P & 1+2P+4Q \end{bmatrix} \in SL_2(\mathbb{Z}C_n). \qquad \square$$

Consequently, we can summarize all the obtained results as follows:

Corollary 1. $\mathcal{U}_1(\mathbb{Z}[C_n \times C_6]) = \mathcal{U}_1(\mathbb{Z}C_n) \times U \times V \times W$ *where*

$$\begin{aligned}
U &= \{1 + (x^3-1)P : 1 - 2P \in \mathcal{U}_1(\mathbb{Z}C_n)\}\\
V &= \{1 + \alpha P + \beta Q : P^2 + Q^2 - PQ - P - Q = 0\}\\
W &= \{1 + \lambda P + \mu Q : 2P^2 + 2Q^2 - 2PQ - P - Q = 0\}
\end{aligned}$$

such that

$$\alpha = x^2 - 1,\ \ \beta = x^4 - 1,\ \ \lambda = (x^3-1)(x^2-1),\ \ \mu = (x^3-1)(x^4-1).$$

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