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RESEARCH ARTICLE

Non-commutative Grillet semigroups

B. V. Novikov

Abstract. Grillet semigroups are introduced. This class of semigroups contains regular semigroups and complete commutative semigroups (by Grillet's terminology). Some structural theorems are proved.

1. Introduction

For the class of finite commutative semigroups I.S.Ponizovsky [3, 4] introduced the notion of an elementary semigroup. He constructed for the finite commutative semigroups an ideal series whose factors are elementary semigroups, and used it for the study of matrix representations. Further P.-A. Grillet [7, 8] generalized this construction and used it for a class of commutative semigroups called by him *complete*. Non-commutative analogues of elementary semigroups occur in many tasks, namely, a description of homomorphisms into commutative semigroups [2], constructing of quasi-Frobenius algebras. Therefore it is rational to consider these semigroups from a unified point of view.

In Section 2 we recall the Ponizovsky's results, further we generalize them to non-commutative semigroups (Section 3). In Section 4 we offer an approach to investigation of elementary semigroups by bi-actions.

All non-defined here notions may be taken from [1].

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2. Ponizovsky's construction

Recall one Ponizovsky's result by Grillet's monograph [7].

A commutative semigroup P is called *elementary* if it is an ideal extension of a nilpotent semigroup N by a group G so that the unity of G is a unity of P also (therefore P has 0 and 1; in particular, groups and nilpotent semigroups are not elementary).

We need the following notations. By $E(S)$ we shall denote the set of all idempotents of the semigroup S. Define a partial order on the set $E(S)$ as follows: $e \leqslant f$ for $e, f \in E(S)$ if and only if $ef = e$.

Ponizovsky's Theorem. *Every finite commutative semigroup S with 0 and 1 has an ideal series*

$$S = I_0 \supset I_1 \supset \ldots \supset I_{m+1} = 0$$

whose factors I_k/I_{k+1} ($k \leqslant m$) are elementary.

Proof. It will be made by induction on $|S|$. Let e be a minimal non-zero idempotent in S. Put $I_m{=}Se$ and denote by H the subgroup of invertible elements of I_m. Obviously the complement $M{=}I_m \setminus H$ is an ideal of I_m. As $E(I_m) = \{0, e\}$ and $|I_m| < \infty$, then for every element of M some power of it is equal to zero; therefore the semigroup M is nilpotent.

It remains to use the induction to S/I_m. □

Grillet calls the semigroups I_k/I_{k+1} of proved theorem *Ponizovsky's factors*; we shall call them briefly Π-*factors*.

It is obvious that the existence the zero and the unity in S is not essential; it's only then that the group and nilpotent semigroups were not considered as Π-factors.

Ponizovsky applied this theorem to the investigations of semigroup algebras. Namely, let F be a field, S be a finite commutative semigroup, $P_k = I_k/I_{k+1}$ be its Π-factors. Then a (contracted) semigroup algebra FS decomposes into a direct sum of ideals

$$FS = FP_0 \oplus \ldots \oplus FP_m,$$

moreover, in view of the existence of unities in Π-factors, the algebra FS is quasi-Frobenius if and only if the algebras FP_k are quasi-Frobenius. From this Ponizovsky obtained the necessary and sufficient conditions of the quasi-Frobenius property of semigroup algebras.

Grillet [7] generalized the Ponizovsky's Theorem to a class of infinite semigroups which were called by him *complete* and used this result for a description of congruences of a free commutative semigroup.

Moreover, Grillet shown how to construct the commutative semigroup by means of its Π-factors.

Another application of Ponizovsky's Theorem may be noted.

Corollary ([7], Corollary IV.5.6). *A finite commutative subdirectly indecomposable semigroup is either a group, or a nilpotent semigroup, or an elementary semigroup.* □

Thus, the Ponizovsky's Theorem is an effective tool for investigation in commutative semigroups. Therefore we may think that its non-commutative variant will be also used.

Some words about a structure of elementary semigroups. It is nontrivial, and it must be investigated "modulo" groups and nilpotent semigroups, i.e. to reduce it to the problem of extension. Two approaches may take place. An elementary semigroup $P = G \cup N$ is an ideal extension of N by means of G. Such presentation are unique and visual, however it does not give an acceptable solution of the problem of synthesis P from N and G.

3. Non-commutative situation

Recall that a semigroup S is called an *epigroup*[1] [5] if for any its element a, some degree of it lies in a subgroup. Denote the unity of this subgroup by e_a. Obviously all the finite semigroups are epigroups.

We will need some well-known properties of epigroups.

1) An ideal extension is an epigroup if and only if the ideal and the quotient semigroup are epigroups;

2) a 0-simple semigroup is an epigroup if and only if it is completely 0-simple.

Throughout this section S **will denote an epigroup with zero** (of course, this supposition does not restrict the generality of the results, since we may add a zero to the semigroup S). The set of all idempotents of the semigroup S we denote by $E(S)$.

[1] This term belongs to L.N. Shevrin and it is a synonymous with English terms "group-bounded semigroup" and "quasi-periodic semigroup".

Define a quasi-order $\preceq$ on the set $E(S)$ as follows:

$$e \preceq f \Longleftrightarrow SeS \subseteq SfS.$$

Clearly, the equivalence relation $\approx$ associated with the quasi-order $\preceq$ is the Green relation $\mathcal{J}$. A non-zero idempotent $e \in E(S)$ we call an *atom* if it is 0-minimal relatively to $\preceq$ (i.e. $0 \neq f \preceq e$ implies $f \approx e$).

An epigroup S is called *elementary* if $SeS = S$ for some atom $e \in E(S)$. The next statement yields a rough description of the structure of such semigroup.

Proposition 1. *A semigroup T is an elementary epigroup if and only if the following conditions are satisfied:*

a) $TeT = T$ for some idempotent $e \in E(T)$;

b) T contains such nil ideal $K \neq T$ that the Rees quotient semigroup T/K is completely 0-simple.

Proof. Let the conditions a) and b) are fulfilled. As K and T/K are epigroup then T is an epigroup. Further, T is elementary since T/K is completely 0-simple and therefore every its non-zero idempotent is an atom.

Conversely, let an epigroup T be elementary, e be an atom, $TeT = T$ and K be the greatest its nil ideal. Let I be an ideal of T containing K, $I \neq T$, $a \in I$. Then I is an epigroup and therefore $e_a \in I$. Thus $Te_aT \subseteq I \subset TeT$, and hence $e_a = 0$. Then I is a nil ideal which is impossible. Therefore the semigroup T/K is 0-simple. In view of the Property 2) of epigroups, it is completely 0-simple. □

The proof shows that the ideal K is the nil radical of the semigroup T. If it is nilpotent (this takes place, in particular, when T is finite), then the description of the elementary semigroup is simpler.

Proposition 2. *Let the nil radical K of the semigroup T be nilpotent. Then T is an elementary epigroup if and only if $T^2 = T$ and the semigroup T/K is completely 0-simple.*

Proof. The part "only if" is obvious. Conversely, let T/K is completely 0-simple, $T^2 = T$, $B = T \setminus K$ and $e = e^2 \in B$. Clearly, e is an atom and $B \subseteq BeB \subseteq TeT$. Show that $K \setminus K^2 \subset TeT$.

Let $x \in K \setminus K^2$. As $T^2 = T$, then $x = yz$, and either $y \in B$ or $z \in B$. If $y \in B \subseteq BeB$ then $x \in BeBT \subseteq TeT$. For $z \in B$ we obtain $x \in TeT$ too.

Finally, it follows from $K \setminus K^2 \subset TeT$ that $K^i \setminus K^{i+1} \subseteq K^{i-1}(K \setminus K^2) \subseteq TeT$ for any $i \geqslant 2$ which yields

$$K = \bigcup_{i \geqslant 1} (K^i \setminus K^{i+1}) \subseteq TeT. \qquad \square$$

A number of examples of elementary epigroups gives the next statement.

Proposition 3. *If e is an atom of an epigroup S then the semigroup SeS is elementary.*

Proof. In view of Property 1) of epigroups, the semigroup $T = SeS$ is an epigroup. As $e \in SeS$ then

$$TeT = SeSeSeS \supseteq SeS = T,$$

hence $TeT = T$.

Show that e is an atom of T. Let $0 \neq f \in E(T)$. As $f \in T = SeS$, then $f \preceq e$ in S, therefore $e \approx f$ in S. Thus, $SfS = SeS$ and

$$TfT = SeSfSeS = SeSeSeS = T,$$

i.e. $e \approx f$ in T. $\square$

We want to extend the Grillet's results [7, 8] to the non-commutative case.

For $e \in E(S)$ we put

$$L(e) = \bigcup_{f \in E(S),\ f \prec e} SfS.$$

We call the set $P(e) = SeS \setminus L(e)$ a Π-*factor*, and the quotient semigroup $\overline{P}(e) = SeS/L(e)$ a $\overline{\Pi}$-*factor*. Note that every Π-factor is non-empty: really, in the opposite case we will have $e \in SfS$ for some $f \prec e$ which is impossible.

Proposition 4. *Every $\overline{\Pi}$-factor is an elementary epigroup.*

Proof. In view of the Property 1) $\overline{P}(e)$ is an epigroup. As $P(e) \bigcup L(e) = SeS = (SeS)e(SeS) \subseteq P(e)eP(e) \bigcup L(e)$, then $P(e) \subseteq P(e)eP(e)$, therefore $\overline{P}(e)e\overline{P}(e) = \overline{P}(e)$. It remains to note that e is an atom in $\overline{P}(e)$. $\square$

We call an epigroup S a *Grillet semigroup* if it satisfies the next condition:

(Gr) For every $a \in S$ the set

$$M(a) = \{e \in E(S) \mid a \in SeS\}$$

is non-empty and has a least element e^a relatively the quasi-order $\preceq$ (i.e. $e^a \preceq e$ for all $e \in M(a)$).

The commutative Grillet semigroups are exactly the semigroups called complete by Grillet[2].

The elementary and regular semigroups are Grillet semigroup. However, in contrast of the commutative situation, not every finite monoid is a Grillet semigroup. For example, the monoid S^1 obtained by the external accession unity from the semigroup $S = \{0, a, e, f\}$ where $ea = af = a$, $e^2 = e$, $f^2 = f$, and other products are equal to zero. Indeed, in this case $M(a) = \{e, f\}$ but e and f are non-compatible with one another.

The next statement is a generalization of the Proposition IV.5.3 of [7] to the non-commutative case.

Proposition 5. *A Grillet semigroup is a disjoint union of its* Π*-factors.*

Proof. If S is a Grillet semigroup then every its element a lies in $P(e^a)$; Therefore S is covered by its Π-factors. Assume that $a \in P(e) \cap P(f)$ for some $e, f \in E(S)$. By the condition **(Gr)**, there is an idempotent $g \in E(S)$ such that $g \preceq e$, $g \preceq f$ and $a \in SgS$. But $a \notin L(e)$, therefore $SgS = SeS$ and similarly $SgS = SfS$, hence $P(e) = P(f)$. □

Under some additional restrictions the converse is true.

Proposition 6. *Let S satisfy the finiteness condition of the strictly decreasing chains of idempotents $e_1 \succ e_2 \succ \ldots$. If S is a disjunct union of its* Π*-factors then it satisfies the condition* **(Gr)**.

Proof. Let a be arbitrary element of S. Then $a \in P(e)$ for some $e \in M(a)$. Show that e is a least element of $M(a)$.

Really, let $e \not\approx f_1$ and $f_1 \in M(a)$, i.e. $a \in Sf_1S \cap P(e)$. As by condition the Π-factors do not intersect, then $a \in L(f_1)$. Therefore $a \in Sf_2S$ for some idempotent $f_2 \prec f_1$. By repeating these considerations we obtain an idempotent $f_3 \prec f_2 \prec f_1$ such that $a \in Sf_3S$ and so on. Since this chain is finite, we will obtain for some n a relation $a \in P(f_n)$. This yields $e \approx f_n \prec f_1$. □

[2]Warning: the original definition of complete semigroup given in the book [7] was corrected in [8].

4. Bi-orbits and elementary semigroups

We obtained a “rough” description of elementary semigroups. The “thin” structure of these semigroups is much more complicated (it is seen even in commutative case). The classical theory of ideal extension (with using translations) works badly for nil-semigroup. In view of this we will consider another approach to this problem.

Let $T = B \cup K$ be a finite elementary semigroup (we preserve denotations of Section 3). Firstly, we restrict only with the case when $B = G$ is a group and its unity e is the unity of T. Secondly, it is easy to see that $G(K^i \setminus K^{i+1})G \subseteq K^i \setminus K^{i+1}$, therefore it seems reasonable on the first step to consider the semigroups of view $(G \cup K^i)/K^{i+1}$. Note that K^i/K^{i+1} is a semigroup with null multiplication, i.e. it has a very poor algebraic structure.

Thus we come to the consideration of a class of semigroups satisfying the following conditions.

a) $T = G \cup K$ is a finite semigroup, G is its subgroup;

b) the unity $e \in G$ is also the unity of T;

c) $K^2 = 0$, $K \neq 0$.

We can look at $K \setminus 0$ as a set on which the group G operates from two sides. So it is convenient to introduce the concept of “bi-action” and to study its properties.

Definition 1. Let G, H be groups, X be a set. A set X is called (G, H)-*set* if two actions are defined, namely, a left action G on X and a right action H on X such that $1 \cdot x = x \cdot 1 = x$ and

$$\forall\, a \in G \ \ \forall\, b \in H \ \ \forall\, x \in X \quad (ax)b = a(xb).$$

The mapping $G \times X \times H \longrightarrow X$ is called a **bi-action**[3].

(G, H)-set is called *bi-orbit* if

$$\forall\, x, y \in X \ \exists a \in G \ \exists b \in H \quad y = axb.$$

In application to elementary semigroups we will deal with one group $G = H$, however it is convenient to construct the theory for two group. Here the set X is the set $K \setminus 0$ which is, clearly, a disjunct union of bi-orbits. In view of this we will restrict with a more narrow class of elementary semigroups adding the following condition to conditions a)–c):

[3]In the monograph [10] such sets X are called the unitary (G, H)-bi-acts and the defined below bi-orbits are cyclic bi-acts.

d) $K \setminus 0$ is a bi-orbit relatively the bi-action of the pair (G, G).

Thus, let X be a (G, H)-bi-orbit. Clearly, a bi-action is equivalent to the usual transitive action of the group $G \times H^{op}$ on the set X (here H^{op} is a group anti-isomorphic to H). Of course, any group anti-isomorphic to the group H is isomorphic to H. However we are convenient to consider the group H^{op} as a group with inverse multiplication $x * y = y \cdot x$. Thus, the description of bi-orbits is reduced to the description of subgroups of the group $G \times H^{op}$.

The next statement is known as "Goursat Lemma" (algebraic).

Lemma 1. *Let G, H be groups, A, B, C, D be subgroups of groups G and H such that*

$$G > A \rhd B, \quad H > C \rhd D, \quad A/B \cong C/D.$$

Let φ be an isomorphism of A/B onto C/D. Then the subset

$$\begin{aligned} F = \{(x, y) \in G \times H \mid \exists a \in A,\ c \in C\ :\\ x \in aB,\ y \in cD,\ \varphi(aB) = cD\} = \bigcup_{\substack{a \in A,\ c \in C\\ \varphi(aB) = cD}} (aB \times cD) \end{aligned}$$

is a subgroup of the group $G \times H$. Conversely, any subgroup of the group $G \times H$ is of such view.

The proof of this lemma can be found, for example, in [11]; see also exhaustible survey and the generalizations of Goursat Lemma in [6]. Note that, in the denotations of Lemma, for given subgroup $F < G \times H$ we have

$$\begin{aligned} A &= \{g \in G \mid \exists h \in H\ :\ (g, h) \in F\},\\ B &= \{g \in G \mid (g, 1) \in F\},\\ C &= \{h \in H \mid \exists g \in G\ :\ (g, h) \in F\},\\ D &= \{h \in H \mid (1, h) \in F\}. \end{aligned}$$

Let P [resp., Q] be a system of representatives of left cosets of the subgroup B in the group G [resp., right cosets of C in H].

Let X be a (G, H)-bi-orbit. Fix an arbitrary element $x_0 \in X$. Then $X = Gx_0H$. Let $F = \{(g, h) | g \in G,\ h \in H,\ gx_0h = x_0\}$. It is not difficult to check that F is a subgroup of the group $G \times H^{op}$. In this denotation $A = \{g \in G | \exists h \in H\ \ gx_0h = x_0\}$, $B = \{g \in G | gx_0 = x_0\}$,

$C = \{h \in H | \exists g \in G \;\; gx_0h = x_0\}$, $D = \{h \in H | x_0h = x_0\}$. Show that the bi-orbit X can be identified with the set $P \times Q$. Really, for all $g \in G$, $h \in H$ we have $h = cq$ for some $c \in C$, $q \in Q$, $ux_0c = x_0$ for some $u \in G$, $gu^{-1} = pb$ for some $p \in P$, $b \in B$. Therefore $gx_0h = gx_0cq = gu^{-1}ux_0cq = gu^{-1}x_0q = pbx_0q = px_0q$. Further, let $px_0q = p'x_0q'$ for some $p, p' \in P$, $q, q' \in Q$. Then $x_0 = p^{-1}p'x_0q'q^{-1}$, hence $q'q^{-1} \in C$, which implies $q = q'$. This yields $p^{-1}p' \in B$, therefore $p = p'$.

Corollary 2. *In the denotations of Lemma 1,*

$$[G \times H \;:\; F] = [G \;:\; A][H \;:\; D] = [G \;:\; B][H \;:\; C]. \quad \square$$

Corollary 3. *In the introduced above denotations $P \times Q$ is a system of representatives of the group F in the group $G \times H$.* $\square$

Introduce the denotations: $G/\!\!/B$ denotes the set of the left cosets of B in G, $C\backslash\!\!\backslash H$ is the set of the right cosets of C in H. In view of Lemma 1 and its corollaries, it is not difficult to obtain the following result.

Theorem 3. *Let G and H be groups. Fix in G and H the subgroups such that*

$$G > A \rhd B, \quad H > C \rhd D, \quad A/B \cong C/D.$$

Let P and Q be such that in Corollary 3, $\lambda \colon A/B \to C/D$ be an isomorphism. Define a bi-action on the set $X = G/\!\!/B \times C\backslash\!\!\backslash H$ as follows:

$$u(xB,\ Cy)v = (uxB,\ Cyv) \quad (u, x \in G,\ y, v \in H).$$

Then X is a (G, H)-bi-orbit. Moreover, any (G, H)-bi-orbit can be obtained so. $\square$

Example. Consider the "non-proper" cases, i.e. when A, B, C, D coincide with 1 or with G, H.

a) $A = G$, $B = 1$, $C = H$, $D = 1$. Then $X = G \times 1$ and bi-action has a view $u \cdot (g, 1) \cdot v = (ug\lambda^{-1}(v), 1)$.

b) $A = B = C = D = 1$. Then $X = G \times H$, $u \cdot (x, y) \cdot v = (ux, yv)$.

c) $A = B = G, C = D = 1$. Then $X = 1 \times H, u \cdot (1, h) \cdot v = (1, \lambda(u)hv)$.

d) $A = B = 1$, $C = D = H$. Then $X = G \times 1$, $u \cdot (g, 1) \cdot v = (ug\lambda^{-1}(v), 1)$.

Thus, the elementary semigroups can be described in case when $K^2 = 0$. Really, in this case $K \setminus 0$ is a union of bi-orbits, the multiplication of elements of K by one another is trivial, and the multiplication of elements of K by elements of G (on the left and on the right) is defined from the description of bi-orbits.

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Contact information

B. V. Novikov Kharkov National University, Svobody sq., 4, 61077, Kharkov, Ukraine
E-Mail: `novikov@univer.kharkov.ua`

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