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On the group of unitriangular automorphisms of the polynomial ring in two variables over a finite field

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Abstract. The group $UJ_2(\mathbb{F}_q)$ of unitriangular automorphisms of the polynomial ring in two variables over a finite field $\mathbb{F}_q$, $q = p^m$, is studied. We proved that $UJ_2(\mathbb{F}_q)$ is isomorphic to a standard wreath product of elementary Abelian p-groups. Using wreath product representation we proved that the nilpotency class of $UJ_2(\mathbb{F}_q)$ is $c = m(p-1)+1$ and the $(k+1)$th term of the lower central series of this group coincides with the $(c-k)$th term of its upper central series. Also we showed that $UJ_n(\mathbb{F}_q)$ is not nilpotent if $n \geq 3$.

1. Introduction

Denote by $UJ_n(\mathbb{F})$ the group of unitriangular automorphisms of the polynomial algebra in n variables over a field $\mathbb{F}$. This group over a field of characteristic zero was studied in [1] by V. Bardakov, M. Neshchadim and Yu. Sosnovsky. The case of $n = 2$ and a field of prime characteristic was considered by Zh. Dovhei and V. Sushchansky in [3, 4].

Given a finite field $\mathbb{F}_q$ with $q = p^m$ elements the group $UJ_2(\mathbb{F}_q)$ is proved to be nilpotent and the nilpotency class has the upper bound $(q-1)(p-1)+1$ and the lower bound $m(p-1)+1$ [3]. Some special subgroups of $UJ_2(\mathbb{F})$, where $\mathbb{F}$ is an arbitrary field of positive characteristic, were described in [4].

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In present paper we consider a wreath product representation of $UJ_2(\mathbb{F}_q)$ (section 3). Let the field $\mathbb{F}_q$ and its additive group be denoted by the same symbol (it is an elementary Abelian p-group of rank m). By $\mathbb{F}_q^\omega$ we denote the countable restricted direct power of $\mathbb{F}_q$. Considering the standard wreath product $\mathcal{W} = \mathbb{F}_q \wr \mathbb{F}_q^\omega$, where $\mathbb{F}_q$ is the active group, we prove that $UJ_2(\mathbb{F}_q) \cong \mathcal{W}$ (Lemma 2). Using results of H. Liebeck [5] about nilpotent standard wreath products we obtain the nilpotency class of $UJ_2(\mathbb{F}_q)$.

Theorem 1. *$UJ_2(\mathbb{F}_q)$ is nilpotent of class $m(p-1)+1$.*

In Section 4 we describe the central series of $UJ_2(\mathbb{F}_q)$. We prove

Theorem 2. *Let $c = m(p-1)+1$. Then the $(i+1)$th term of the lower central series and the $(c-i)$th term of the upper central series of $UJ_2(\mathbb{F}_q)$ are coincide.*

Particularly, the center $\mathcal{Z}(UJ_2(\mathbb{F}_q))$ is the group of all pairs

$$[0, \sum_{i\in\mathbb{N}} c_i(x^q - x)^i],$$

where $c_i \in \mathbb{F}_q$ and $c_i = 0$ for all but finitely many $i \in \mathbb{N}$.

In Section 5 we consider the group $UJ_2(R)$ over an integral domain R of a prime characteristic. It is shown that if R is a polynomial ring in one variable over a finite field then $UJ_2(R)$ is not nilpotent (Lemma 8) and we obtain

Theorem 3. *For all $n \geq 3$ the group $UJ_n(\mathbb{F}_q)$ is not nilpotent.*

2. Basic definitions and notations

Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements. Denote by $\mathbb{F}_q[x]$ and $\mathbb{F}_q[x, y]$ the algebras of polynomials over $\mathbb{F}_q$ in one and two variables respectively. Every automorphism of $\mathbb{F}_q[x, y]$ is uniquely determined by images of x and y, i.e. by a pair of polynomials $\langle a(x, y), b(x, y)\rangle$, $a(x, y), b(x, y) \in \mathbb{F}_q[x, y]$. An automorphism corresponding to a pair $\langle \alpha x + a, \beta y + f(x)\rangle$, where $\alpha \neq 0$ and $\beta \neq 0$, is called *triangular*. Additionally, if $\alpha = \beta = 1$ the automorphism is called *unitriangular*. Then the group $UJ_2(\mathbb{F}_q)$ of unitriangular automorphisms of $\mathbb{F}_q[x, y]$ is isomorphic to the group of all pairs $u = [a, f(x)]$, $a \in \mathbb{F}_q$, $f(x) \in \mathbb{F}_q[x]$, with multiplication

$$[a, f(x)] \cdot [b, g(x)] = [a + b, f(x) + g(x + a)].$$

Following [3, 4] we define an elementary automorphism δ_a and a linear operator Δ_a, $a \in \mathbb{F}_q$, on $\mathbb{F}_q[x]$ as follows:

$$\delta_a(f(x)) = f(x+a) \quad \text{and} \quad \Delta_a(f(x)) = \delta_a(f(x)) - f(x).$$

Then $\Delta_0 f(x) = 0$ and $\Delta_a c = 0$ for every $f(x) \in \mathbb{F}_q[x]$, $a, c \in \mathbb{F}_q$, moreover, $\Delta_a\Delta_b = \Delta_b\Delta_a$ for all $a, b \in \mathbb{F}_q$.

The identity of $UJ_2(\mathbb{F}_q)$ is the pair $id = [0, 0]$. The inverse of u is $u^{-1} = [-a, -f(x-a)]$ and the commutator of u and $v = [b, g(x)]$ is the pair

$$[u, v] = uvu^{-1}v^{-1} = [0, \Delta_a(g(x)) - \Delta_b(f(x))]. \tag{1}$$

Lemma 1. *Let $f(x) \in \mathbb{F}_q[x]$ and $a \in \mathbb{F}_q$. Then*

1) $\delta_a(x^q - x) = x^q - x$ *and* $\Delta_a(x^q - x) = 0$;
2) $\delta_a[(x^q - x)f(x)] = (x^q - x)\delta_a(f(x))$;
3) $\Delta_a[(x^q - x)f(x)] = (x^q - x)\Delta_a(f(x))$;
4) $\bigcap_{a\in\mathbb{F}_q} \operatorname{Ker} \Delta_a = \mathbb{F}_q[x^q - x]$.

Proof. Parts 1), 2) and 3) can be obtained by direct computations. Let us prove part 4). From 1) we have $\mathbb{F}_q[x^q - x] \subseteq \operatorname{Ker} \Delta_a$ for every $a \in \mathbb{F}_q$. On the other hand, any polynomial $f(x) \in \bigcap_{a\in\mathbb{F}_q} \operatorname{Ker} \Delta_a$ can be written as

$$f(x) = \sum_{i=0}^{t} (x^q - x)^i f_i(x), \tag{2}$$

where $\deg f_i(x) < q$ for all $i = 0, 1, \ldots, t$. Then, according to part 3) of this lemma, $\Delta_a(f(x)) = \sum_{i=0}^{t}(x^q - x)^i \Delta_a(f_i)$.

Assume that there exists i such that $\deg f_i(x) > 0$. Denote $g(x) = \Delta_a(f_i(x))$. Then $g(0) = f_i(a) - f_i(0)$. Since $f_i(x)$ is not a constant, there exists $a \in \mathbb{F}_q^*$ such that $g(0) \neq 0$. Thus, we obtain a contradiction ($\Delta_a(f_i(x))$ should be 0 for every $a \in \mathbb{F}_q$). Hence, $f_i(x) = \text{const} \in \mathbb{F}_q$ for all $i = 0, 1, \ldots, t$. □

3. $UJ_2(\mathbb{F}_q)$ as a wreath product

The group $UJ_2(\mathbb{F}_q)$ can be represented as a wreath product of two elementary Abelian p-groups. We consider the standard wreath product of $\mathbb{F}_q^\omega$ by $\mathbb{F}_q$:

$$\mathcal{W} = \mathbb{F}_q \wr \mathbb{F}_q^\omega,$$

where $\mathbb{F}_q$ is the active group. Elements of $\mathcal{W}$ are pairs $[a, f(x)]$ such that $a \in \mathbb{F}_q$, $f(x)$ is a function from $\mathbb{F}_q$ into $\mathbb{F}_q^\omega$. Each such function can be uniquely determined by the almost-zero sequence $\langle f_0(x), f_1(x), \ldots\rangle$ of polynomials $f_i(x)$ reduced modulo the ideal generated by $x^q - x$, $i \in \mathbb{N}$ (in other words, each polynomial has degree less or equal to $q-1$ and $f_i(x) \equiv 0$ for all but finitely many $i \in \mathbb{N}$). The identity of $\mathcal{W}$ is the pair $[0, \langle 0, 0, \ldots\rangle]$. Now, if

$$u = [a, \langle f_0(x), f_1(x), \ldots\rangle] \quad \text{and} \quad v = [b, \langle g_0(x), g_1(x), \ldots\rangle] \tag{3}$$

then

$$\begin{aligned} u^{-1} &= [-a, \langle -f_0(x-a), -f_1(x-a), \ldots\rangle]; \\ uv &= [a+b, \langle f_0(x) + g_0(x+a), f_1(x) + g_1(x+a), \ldots\rangle]. \end{aligned} \tag{4}$$

Lemma 2. $UJ_2(\mathbb{F}_q) \cong \mathcal{W}$

Proof. Let $u = [a, f(x)] \in UJ_2(\mathbb{F}_q)$ and $f(x) \in \mathbb{F}_q[x]$ has the decomposition (2). Consider a mapping $\varphi : UJ_2(\mathbb{F}_q) \mapsto \mathcal{W}$ which acts as follows:

$$\varphi(u) = [a, \langle f_0(x), f_1(x), \ldots\rangle] \in \mathcal{W}.$$

Clearly, φ is a bijection. Now suppose $v = [b, g(x)] \in UJ_2(\mathbb{F}_q)$. Then

$$\begin{aligned} uv &= [a, f(x)] \cdot [b, g(x)] = [a+b, f(x) + g(x+a)] = \\ &= [a+b, f(x) + \delta_a\left(\sum_{i\in\mathbb{N}} g_i(x)(x^q - x)^i\right)] = \\ &= [a+b, \sum_{i\in\mathbb{N}} f_i(x)(x^q-x)^i + \sum_{i\in\mathbb{N}} \delta_a[g_i(x)](x^q-x)^i] = \\ &= [a+b, \sum_{i\in\mathbb{N}} [f_i(x) + g_i(x+a)](x^q-x)^i]. \end{aligned}$$

Thus,

$$\begin{aligned} \varphi(uv) &= [a+b, \langle f_0(x) + g_0(x+a), f_1(x) + g_1(x+a), \ldots\rangle] = \\ &= [a, \langle f_0(x), f_1(x), \ldots\rangle] \cdot [b, \langle g_0(x), g_1(x), \ldots\rangle] = \varphi(u)\varphi(v) \end{aligned}$$

and φ is a homomorphism. □

Remark 1. Elements of $UJ_2(\mathbb{F}_q)$ can be considered as pairs of the type (3) with group operation defined as (4). In subsequent sections we use this representation.

Additionally, for every $i \in \mathbb{N}$ let the projection $\pi_i : UJ_2(\mathbb{F}_q) \mapsto \mathbb{F}_q \wr \mathbb{F}_q$ to be defined as

$$\pi_i([a, \langle f_0, f_1, \ldots, f_n, \ldots \rangle]) = [a, f_i].$$

Obviously, π_i is an epimorphism. By W_i we denote a subgroup of $UJ_2(\mathbb{F}_q)$ which consists of elements of the type $[a, \langle 0, \ldots, 0, f_i, 0, \ldots \rangle]$, where f_i is in the ith position. It is clear that $W_i \cong \mathbb{F}_q \wr \mathbb{F}_q$.

In [2] G. Baumslag proved the following simple criteria: $A \wr B$ is nilpotent if and only if both A and B are nilpotent p-groups, B has finite exponent and A is finite. Additionally, H. Liebeck in [5] showed that if B is an Abelian p-group of exponent p^k, and A is the direct product of cyclic groups of orders $p^{\beta_1}, \ldots, p^{\beta_n}$, where $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_n$ then $A \wr B$ has the nilpotency class

$$\sum_{i=1}^{n} (p^{\beta_i} - 1) + 1 + (k-1)(p-1)p^{\beta_1 - 1}. \tag{5}$$

Now, using the wreath product representation of $UJ_2(\mathbb{F}_q)$ we can prove Theorem 1.

Proof of Theorem 1. According to Lemma 2, the group $UJ_2(\mathbb{F}_q)$ is isomorphic to the wreath product of $\mathbb{F}_q^\omega$ by $\mathbb{F}_q$. Thus in terms of Formula (5) we obtain:

1) the exponent of $\mathbb{F}_q^\omega$ equals p (i.e. $k = 1$);

2) the group $\mathbb{F}_q$ as an elementary Abelian group is the direct product of m cyclic groups of order p (i.e. $\beta_1 = \beta_2 = \ldots = \beta_m = 1$).

Hence, $c(UJ_2(\mathbb{F}_q)) = \sum_{i=1}^{m} (p-1) + 1 = m(p-1) + 1$. □

4. Central series of $UJ_2(\mathbb{F}_q)$

Denote by $\mathrm{Sym}(\mathbb{N})$ the group of all permutations on $\mathbb{N} = \{0, 1, 2, \ldots\}$. Given $\sigma \in \mathrm{Sym}(\mathbb{N})$ the mapping $\Phi_\sigma : UJ_2(\mathbb{F}_q) \mapsto UJ_2(\mathbb{F}_q)$ is defined as follows:

$$\Phi_\sigma([a, \langle f_0, f_1, \ldots, f_n, \ldots \rangle]) = [a, \langle f_{\sigma(0)}, f_{\sigma(1)}, \ldots, f_{\sigma(n)}, \ldots \rangle];$$

in other words, Φ_σ permutes factors in $\mathbb{F}_q^\omega$. Simple calculations show that Φ_σ is an automorphism of $UJ_2(\mathbb{F}_q)$ for every $\sigma \in \mathrm{Sym}(\mathbb{N})$.

Lemma 3. *If K is a characteristic subgroup of $UJ_2(\mathbb{F}_q)$ then $\pi_0(K) = \pi_i(K)$ for every $i \in \mathbb{N}$.*

Proof. Let us fix i. Suppose $u = [a, f_0]$ is an elements of $\pi_0(K)$ and $v = [a, \langle f_0, f_1, \dots, f_i, \dots\rangle] \in K$, where $f_1, f_2, \dots, f_i, \dots$ are polynomials from $\mathbb{F}_q[x]/(x^q - x)$. Consider the transposition $(1, i) \in \mathrm{Sym}(\mathbb{N})$. Since K is characteristic, $w = \Phi_{(1,i)}(v) = [a, \langle f_i, f_1, \dots, f_{i-1}, f_0, f_{i+1}, \dots\rangle] \in K$. Hence, $\pi_i(w) = [a, f_0] = u \in \pi_i(K)$ and $\pi_0(K) \subseteq \pi_i(K)$. Analogously, one can show that $\pi_i(K) \subseteq \pi_0(K)$. □

The following lemma describes some properties of fully invariant subgroups of $UJ_2(\mathbb{F}_q)$.

Lemma 4. *If a fully invariant subgroup K of $UJ_2(\mathbb{F}_q)$ contains an element $u = [c, \langle \dots \rangle]$ with $c \neq 0$ then $K = UJ_2(\mathbb{F}_q)$.*

Proof. For every $i \in \mathbb{N}$ and $h(x) \in \mathbb{F}_q[x]/(x^q - x)$ we define the mapping $\Psi_i^{h(x)} : UJ_2(\mathbb{F}_q) \mapsto UJ_2(\mathbb{F}_q)$ as follows:

$$\Psi_i^{h(x)}([a, \langle f_0, f_1, \dots, f_i, \dots\rangle]) = [0, \langle \underbrace{0, \dots, 0}_{i-1}, ah(x), 0, \dots\rangle].$$

Direct calculations show that $\Psi_i^{h(x)}$ is an endomorphism.

Let us fix an index i and a polynomial $f(x) \in \mathbb{F}_q[x]/(x^q - x)$. Since K is fully invariant and $c \neq 0$, we obtain that K contains

$$u_i^{f(x)} = \Psi_i^{c^{-1}f(x)}(u) = [0, \langle \underbrace{0, \dots, 0}_{i-1}, f(x), 0, \dots\rangle]. \tag{6}$$

Now, suppose $f_0(x) = g_0(x) + dx^{q-1}$, where $\mathbb{F}_q \ni d \neq 0$ and $\deg g_0(x) < q - 1$. We define the endomorphism $\Theta : UJ_2(\mathbb{F}_q) \mapsto UJ_2(\mathbb{F}_q)$ as follows:

$$\Theta([a, \langle f_0, f_1, \dots, f_i, \dots\rangle]) = [d, \langle 0, 0, \dots\rangle].$$

Then for any given $d \in \mathbb{F}_q$ the subgroup K contains

$$v_d = \Theta([0, \langle dx^{q-1}, 0, 0, \dots\rangle]) = [d, \langle 0, 0, \dots\rangle]. \tag{7}$$

Finally, elements of types (6) and (7) generate $UJ_2(\mathbb{F}_q)$. □

Corollary 1. *If a fully invariant subgroup K of $\mathbb{F}_q \wr \mathbb{F}_q$ contains an element $u = [a, f(x)]$ with $a \neq 0$ then $K = \mathbb{F}_q \wr \mathbb{F}_q$.*

Lemma 5. *Let V be a proper verbal subgroup of $UJ_2(\mathbb{F}_q)$ generated by a collection of words $\mathcal{V}$ and V_i be verbal subgroups of W_i with respect to the same collection $\mathcal{V}$. Then $V = \prod_{i \in \mathbb{N}} V_i$.*

Proof. Given a word $w(x_1, x_2, \ldots, x_n) \in \mathcal{V}$ and $u_1, u_2, \ldots, u_n \in UJ_2(\mathbb{F}_q)$ consider $u = w(u_1, u_2, \ldots, u_n) \in V$. Then, according to the group operation (4), we obtain $\pi_i(u) = w(\pi_i(u_1), \pi_i(u_2), \ldots, \pi_i(u_n))$ for every $i \in \mathbb{N}$, i.e. u is contained in the direct product $\prod_{i\in\mathbb{N}} V_i$.

On the other hand, given $v = [0, \langle f_0, f_1, \ldots, f_i, \ldots\rangle] \in V$ (by Lemma 4, since V is a proper fully invariant subgroup of $UJ_2(\mathbb{F}_q)$, the first component of v equals 0) we consider the element $v_i = [0, \langle 0, \ldots, 0, f_i, 0, \ldots\rangle] \in V_i$. Assume $v_i = id$ for all $i \geq n$. Then $v = v_0 v_1 \ldots v_n$ and $v_i = w_{i,1} w_{i,2} \ldots w_{i,m_i}$, where $w_{i,j}$, $j = 1, 2, \ldots, m_i$, is a value of a word (from $\mathcal{V}$) in group W_i. The latter implies $\prod_{i\in\mathbb{N}} V_i \subseteq V$. □

By $\gamma_i(G)$ and $\zeta_j(G)$ we denote the ith and jth terms of the lower central series and upper central series of G respectively (note that $i = 1, 2, \ldots$, while $j = 0, 1, \ldots$). Given a word $w = w(x_1, x_2, \ldots, x_k)$ and $g \in G$ let $w_i^g = w(\ldots, x_{i-1}, x_i g, x_{i+1}, \ldots)$ and ${}^g w_i = w(\ldots, x_{i-1}, g x_i, x_{i+1}, \ldots)$. Recall that the marginal subgroup of G for the word w is the set of all $g \in G$ such that $w = w_i^g = {}^g w_i$ for all $x_1, x_2 \ldots, x_k \in G$ and all $i \in \{1, 2, \ldots, k\}$. Particularly, terms of the upper central series are marginal subgroups corresponding to simple commutators.

Proof of Theorem 2. Consider the $(c - i)$th member of the upper central series as a marginal subgroup of $UJ_2(\mathbb{F}_q)$ corresponding to the word $[\ldots[x_1, x_2], x_3], \ldots, x_{c-i+1}]$. Suppose $u \in \zeta_{c-i}(UJ_2(\mathbb{F}_q))$. Then, according to (4), we obtain $\pi_j(u) \in \zeta_{c-i}(W_j)$ for all $j \in \mathbb{N}$. Thus,

$$\zeta_{c-i}(UJ_2(\mathbb{F}_q)) \leq \prod_{j\in\mathbb{N}} \zeta_{c-i}(W_j) = \prod_{j\in\mathbb{N}} \gamma_{i+1}(W_j).$$

The last equality follows from the fact that the lower central series and upper central series of $\mathbb{F}_q \wr \mathbb{F}_q$ are coincide (for details see [7]). Since members of the lower central series are verbal subgroups, by Lemma 5 we have $\prod_{j\in\mathbb{N}} \gamma_{i+1}(W_j) = \gamma_{i+1}(UJ_2(\mathbb{F}_q))$. Thus, $\zeta_{c-i}(UJ_2(\mathbb{F}_q)) \leq \gamma_{i+1}(UJ_2(\mathbb{F}_q))$.

On the other hand, $\gamma_{i+1}(UJ_2(\mathbb{F}_q)) \leq \zeta_{c-i}(UJ_2(\mathbb{F}_q))$, $i \in \{0, 1, \ldots, c\}$ (see, for example, [6], Theorem 5.31) and we obtain the result. □

In particular, the center $\mathcal{Z}(UJ_2(\mathbb{F}_q))$ of $UJ_2(\mathbb{F}_q)$ is the subgroup of all pairs

$$[0, \sum_{i\in\mathbb{N}} c_i (x^q - x)^i],$$

where $c_i \in \mathbb{F}_q$ and $c_i = 0$ for all but finitely many $i \in \mathbb{N}$. In terms of the wreath product representation $\mathcal{Z}(UJ_2(\mathbb{F}_q))$ is the group of pairs

$$[0, \langle c_0, c_1, \ldots, c_i, \ldots\rangle].$$

Hence, $\mathcal{Z}(UJ_2(\mathbb{F}_q)) \cong \prod_{i\in\mathbb{N}} \mathbb{F}_q$.

5. $UJ_2(R)$, where R is an integral domain

Let R be an integral domain (a non-trivial commutative ring with no non-zero zero divisors). Denote by $\mathrm{char}(R)$ the characteristic of R. In this section we assume that $1 \in R$ and $\mathrm{char}(R) = p$ (p is prime). Elements of $UJ_2(R)$ are represented by pairs $[a, b(x)]$, where $a \in R$ and $b(x) \in R[x]$.

Lemma 6. *If $u = [0, f(x)]$ and $v = [b, g(x)]$ are elements of $UJ_2(R)$ then $[u, v] = uvu^{-1}v^{-1} = [0, -\Delta_b(f(x))]$.*

The latter is a special case of Formula (1) that also holds for arbitrary integral domain.

Assume $R = \mathbb{F}_q[\xi]$ is the polynomial ring in variable ξ over $\mathbb{F}_q$. Now, elements of $R[x]$ can be considered as polynomials in variables ξ, x over $\mathbb{F}_q$. If $\xi^k x^l$ is a monomial then k is called the ξ-degree of the monomial and l is called the x-degree of that monomial. Also denote

$$s_m^n = p^m + p^{m+1} + \ldots + p^n,$$

where $m, n \in \mathbb{N}$ ($m \le n$). We need the following technical lemma.

Lemma 7. *Suppose a polynomial $f(\xi, x) \in \mathbb{F}_q[\xi, x]$ contains the monomial $\xi^d x^{s_m^n}$ such that no other term of $f(\xi, x)$ has ξ-degree d. Then there exists $r \in \mathbb{N}$ such that the polynomial $f(\xi, x + \xi^r) - f(\xi, x)$ contains the monomial $\xi^{d+rp^m} x^{s_{m+1}^n}$ and no other term of $f(\xi, x + \xi^r) - f(\xi, x)$ has ξ-degree $d + rp^m$.*

Proof. Let r denotes some fixed positive integer and $\xi^d x^{s_m^n}$, $c_1\xi^{a_1}x^{b_1}$, $c_2\xi^{a_2}x^{b_2}$, …, $c_t\xi^{a_t}x^{b_t}$ are all terms of $f(\xi, x)$; here $a_i, b_i \in \mathbb{N}$, $c_i \in \mathbb{F}_q$. Then

$$\begin{aligned}
\xi^d(x+\xi^r)^{s_m^n} - \xi^d x^{s_m^n} &= \xi^d(x+\xi^r)^{p^m s_0^{n-m}} - \xi^d x^{s_m^n} = \\
&= \xi^d(x^{p^m} + \xi^{rp^m})^{s_0^{n-m}} - \xi^d x^{s_m^n} = \\
&= \xi^d x^{p^m s_1^{n-m}} \xi^{rp^m} + h(\xi, x) = \\
&= \xi^{d+rp^m} x^{s_{m+1}^n} + h(\xi, x),
\end{aligned}$$

where $h(\xi, x)$ does not contain monomials of ξ-degree $d + rp^m$.

Let us also fix $i \in \{1, 2, \ldots, t\}$. If $b_i = p^{m_i} d_i$, where $p \nmid d_i$, then by direct computations (as in the previous case) one can show that the polynomial $c_i\xi^{a_i}(x+\xi^r)^{b_i} - c_i\xi^{a_i}x^{b_i}$ contains only terms of the form

$$\xi^{a_i + jrp^{m_i}} x^{p^{m_i}(d_i - j)}, \qquad j = 1, 2, \ldots, d_i;$$

here we omit coefficients of respective monomials. Now, suppose that some of those monomials has ξ-degree $d + rp^m$. In other words, $a_i + jrp^{m_i} = d + rp^m$ or

$$r(p^m - jp^{m_i}) = a_i - d. \tag{8}$$

If $p^m - jp^{m_i} = 0$ then Equality (8) is false for all $r \in \mathbb{N}$, since $a_i - d \neq 0$. Otherwise, if for some i and j we have $p^m - jp^{m_i} \neq 0$ then (8) can be rewritten as

$$r = \frac{a_i - d}{p^m - jp^{m_i}} \tag{9}$$

Since t and all a_i's, b_i's are finite we can choose r such that (9) does not hold for all possible i and j and, hence, $\xi^{d+rp^m} x^{s^n_{m+1}}$ is the unique monomial of ξ-degree $d + rp^m$ in $f(\xi, x + \xi^r) - f(\xi, x)$. □

Lemma 8. *If $R = \mathbb{F}_q[\xi]$ then $UJ_2(R)$ is not nilpotent.*

Proof. Let us fix $n \in \mathbb{N}$ and $u = [0, x^{s^n_1}] \in UJ_2(R)$. We'll prove that there exist elements $v_1, \ldots, v_n \in UJ_2(R)$ such that $[u, v_1, \ldots, v_n] \neq id$.

Assume $v_1 = [\xi^{r_1}, 0]$ for some $r_1 \in \mathbb{N}$. According to Lemma 6 we obtain $[u, v_1] = [0, f_1(\xi, x)]$, where $f_1(\xi, x) = -\Delta_{\xi^{r_1}}(x^{s^n_1})$. Here $x^{s^n_1}$ satisfies the conditions of Lemma 7, thus there exists r_1 such that $f_1(\xi, x)$ contains the monomial $\xi^{r_1 p} x^{s^n_2}$ (without considering the coefficient) which has the unique ξ-degree among all terms of $f_1(\xi, x)$. In general, there exist $r_1, r_2, \ldots, r_i \in \mathbb{N}$ such that after i steps we obtain $[u, v_1, \ldots, v_i] = [0, f_i(\xi, x)]$, where $f_i(\xi, x)$ has a monomial of x-degree s^n_{i+1} satisfying the conditions of Lemma 7. Finally, after n steps we obtain $[u, v_1, \ldots, v_n] \neq id$ and the lemma is proved. □

Regarding the latter lemma, it might be interesting to investigate the necessary and sufficient conditions for $UJ_2(R)$ to be nilpotent.

Finally, we consider $UJ_n(\mathbb{F}_q)$ for $n \geq 3$.

Proof of Theorem 3. Elements of the group $UJ_n(\mathbb{F}_q)$ are represented by tuples

$$[a_1, a_2(x_1), \ldots, a_n(x_1, \ldots, x_{n-1})],$$

where $a_1 \in \mathbb{F}_q$ and $a_i(x_1, \ldots, x_{i-1}) \in \mathbb{F}_q[x_1, \ldots, x_{i-1}]$, $i \in \{2, 3, \ldots, n\}$. Let H be the subgroup of $UJ_n(\mathbb{F}_q)$ that consists of all tuples

$$[0, a_1(x_1), a_2(x_1, x_2), 0, \ldots],$$

where $a_1(x_1) \in \mathbb{F}_q[x_1]$ and $a_2(x_1, x_2) \in \mathbb{F}_q[x_1, x_2]$. It is obvious that $H \cong UJ_2(\mathbb{F}_q[x_1])$. Using Lemma 8 we obtain the result. □

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