Algebra and Discrete Mathematics RESEARCH ARTICLE
Volume **17** (2014). Number 2, pp. 280 – 287

Subpower Higson corona of a metric space

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Communicated by V. I. Sushchansky

Abstract. We define a subpower Higson corona of a metric space. This corona turns out to be an intermediate corona between the Higson corona and sublinear Higson corona. It is proved that the subpower compactification of an unbounded proper metric space contains a topological copy of the Stone-Čech compactification of a countable discrete space. We also provide an example of a map between geodesic spaces that is not asymptotically Lipschitz but that generates a continuous map of the corresponding subpower Higson coronas.

1. Introduction

One of the main notions of the coarse geometry is that of coarse structure (see, e.g., [1]). This is a special case of the notion of ball structure introduced and investigated by I. Protasov [9]. In particular, the ball structures allow us to combine both the macro- and micro-scale approach to geometry.

In the realm of metric spaces one usually considers the so-called bounded coarse structure generated by metric. However, it turned out that another coarse structures related to metric are of importance in the coarse geometry. The notion of sublinear coarse structure on a proper metric space was introduced in [4]. One of the goals of this note is to consider the so-called subpower coarse structure on metric spaces.

The notion of Higson corona (see, e.g., [1]) provides a link between the coarse geometry and the theory of compact metric spaces. A general definition of K-corona for the ball structures is defined in [8].

2010 MSC: 46J10, 54D35.

One of the most interesting results concerning the coronas is the relation between the asymptotic dimension of a (proper) metric space and the covering dimension of its corona [5]. Counterparts of these results for the sublinear corona are proved in [2]. The dimension theory that corresponds to the sublinear coarse structure is the so-called asymptotic Assouad-Nagata dimension theory [4].

In this note we show that the subpower Higson corona (i.e., the remainder of the compactification generated by the mentioned above subpower coarse structure) of an unbounded proper metric space contains a topological copy of the space $\beta\mathbb{N}\setminus\mathbb{N}$ (the remainder of the Stone-Čech compactification of a countable discrete space). This is a counterpart of a result obtained by J. Keesling [6].

2. Higson subpower compactification

Let (X, d_X) be a metric space and let $x_0 \in X$ be a base point. The norm of $x \in X$ is the number $|x| = d_X(x, x_0)$. By $B_r(x)$ we denote the open ball of radius $r > 0$ centered at $x \in X$. Recall that a metric space (X, d_X) is said to be proper if every its closed ball is compact. The following notion is introduced in [2].

Definition 2.1. A continuous function $s\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called asymptotically sublinear if for each non-constant linear function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ there is M such that $s(r) < f(r) + M$. Equivalently, there is $r_0 > 0$ such that $s(r) < f(r)$ for $r > r_0$.

Definition 2.2. A continuous function $s\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called asymptotically subpower if for every $\alpha > 0$ there exists $t_0 > 0$ such that $s(t) \leqslant t^\alpha$, for all $t \geqslant t_0$.

The function $s(t) = \ln(t+1)$ is an example of a subpower function.

Definition 2.3. A continuous map $f\colon (X, d_X) \to (Y, d_Y)$ is called Higson subpower (respactively Higson sublinear) if, for every asymptotically subpower (respectively asymptotically sublinear) function $s\colon \mathbb{R}_+ \to \mathbb{R}_+$ the conditions $x_n, y_n \to \infty$ and $d_X(x_n, y_n) \leqslant s(|y_n|)$ for all $n \geqslant 1$ imply $d_Y(f(x_n), f(y_n)) \to 0$.

Equivalently, for every sequence (x_n) and every asymptotically subpower (respectively asymptotically sublinear) function s we have $\lim_{n\to\infty} \operatorname{diam}(B_{s(|x_n|)}) = 0$.

The following is an immediate consequence of the definition.

Lemma 2.4. *Every asymptotically subpower function is asymptotically sublinear.*

Proposition 2.5. *Every Higson sublinear map is Higson subpower.*

Proof. Let $f\colon X \to Y$ be a Higson sublinear map. Let $s\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an asymptotically subpower function. Let x_n, y_n be sequences in X such that $d_X(x_n, y_n) \leqslant s(|x_n|)$. Since all asymptotically subpower functions are asymptotically sublinear we obtain that f is Higson subpower. □

It is easy to verify that all bounded subpower functions form a subalgebra in the algebra of all continuous functions on a proper metric space X. The compactification that corresponds to this subalgebra is called the Higson subpower compactification of X and is denoted by $h_P(X)$.

Proposition 2.5 implies that the identity map of a space X extends to a (canonical) map $h_P(X) \to h_L(X)$. The following example demonstrates that this map is not necessarily a homeomorphism.

Example 2.6. Consider a sequence (x_n) in $\mathbb{R}$ defined by the conditions $x_1 = 1$ and $x_{n+1} = x_n + x_{n+1}^{1/4}$. Let $X = \{x_n \mid n \in \mathbb{N}\}$ and define $f\colon X \to \mathbb{R}$ as follows: $f(x_n) = 1$ for $n \in 2\mathbb{N}$ and $f(x_n) = 0$ for $n \in 2\mathbb{N} - 1$. We are going to show that f is a Higson subpower function which is not Higson sublinear.

Let $p\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a subpower function. Then there exists $N \in \mathbb{N}$ such that, for every $n > N$, we have $B_{p(|x_n|)}(x_n) = \{x_n\}$. Indeed, assume the contrary. Then $x_{n-1} \in B_{p(|x_n|)}(x_n)$, for infinite number of n. Therefore, $x_n^{1/4} = x_n - x_{n-1} < p(|x_n|) = p(x_n)$, for infinite number of n, which provides a contradiction. Thus, $\lim_{n\to\infty} \operatorname{diam}(f(B_{p(|x_n|))}(x_n)) = 0$ and therefore f is Higson subpower.

Remark that the function $l\colon \mathbb{R}_+ \to \mathbb{R}_+$, $l(x) = \sqrt{x}$, is asymptotically sublinear. For any $n > 1$, we have $\{x_n, x_{n-1}\} \subset B_{l(|x_n|)}(x_n)$, because $|x_n - x_{n-1}| = |x_n|^{1/4} < \sqrt{|x_n|}$. Thus, $\lim_{n\to\infty} \operatorname{diam}(f(B_{l(|x_n|)}(x_n))) = 1$ and the function f is not Higson sublinear.

Thus, in general, the Higson subpower compactification is strictly greater than the Higson sublinear compactification. One can construct a similar example which demonstrates that, in general, the Higson compactification is strictly greater than the Higson subpower compactification.

We therefore obtain an intermediate compactification $h_P(X)$ between the compactifications $h(X)$ and $h_L(X)$. The remainder $\upsilon_P(X)$ of this compactification is called the subpower Higson corona of X.

Theorem 2.7. *Let X be an unbounded proper metric space. Then the compactification $h_L(X)$ contains a copy of $\beta\mathbb{N}$.*

Proof. Pick any base point in X. Let (p_i) be any sequence in X with the following property: $p_j \notin B_{|p_i|/2}(p_i)$, for $j \neq i$, $i, j \in \mathbb{N}$. Note that then the family $\mathcal{B} = \{B_{|p_i|/4}(p_i) \mid i \in \mathbb{N}\}$ is disjoint.

Put $r_i = |p_i|/4$. Let $s\colon \mathbb{R}_+ \to \mathbb{R}_+$ be an asymptotically sublinear function and let $\varepsilon > 0$.

Given $M > 0$, we are going to show that, for $x \in X$ of norm large enough and such that $B_{s(|x|)}(x) \cap B_{r_i}(p_i) \neq \varnothing$, we have $r_i > \frac{4s(|x|)M}{\varepsilon}$.

Indeed, to this end we consider two cases.

1) If $|p_i| \geqslant |x|$, then we immediately obtain

$$\frac{r_i}{s(|x|)} = \frac{|p_i|}{4s(|x|)} \geqslant \frac{|x|}{4s(|x|)} \to \infty \quad \text{as} \quad |x| \to \infty.$$

2) If $|p_i| < |x|$, then, since $B_{s(|x|)}(x) \cap B_{r_i}(p_i) \neq \varnothing$, we obtain

$$d(x, p_i) < r_i + s(|x|) = |p_i|/4 + s(|x|) < |x|/4 + |x|/4 = |x|/2$$

as $s(|x|) < |x|/4$ for $|x|$ large enough.

Moreover, for x with $|x|$ large enough, we obtain $|x| \leqslant |p_i| + d(x, p_i) < |p_i| + |x|/2$ and consequently $|p_i| > |x|/2$. At the same time,

$$\frac{r_i}{s(|x|)} = \frac{|p_i|}{4s(|x|)} > \frac{|x|}{8s(|x|)} \to \infty \quad \text{as} \quad |x| \to \infty.$$

Therefore, for every $\varepsilon > 0$ and every $M > 0$, there exists a compact subset $K \subset X$ such that $\frac{r_i}{s(|x|)} > \frac{4M}{\varepsilon}$ for any $x \notin K$.

Now let $P = \{p_i \mid i \in \mathbb{N}$ and let $f\colon P \to \mathbb{R}$ be a bounded function. We suppose that $|f(p)| < M$, for every $p \in P$.

We modify the construction from [6]. Consider $\bar{f}\colon X \to \mathbb{R}$ defined by the formula:

$$\bar{f}(x) = \begin{cases} f(p_i)\frac{r_i - d(p_i, x)}{r_i}, & \text{if } x \in B_{r_i}(p_i) \text{ for some } i, \\ 0, & \text{if } x \notin \cup_{i\in\mathbb{N}} B_{r_i}(p_i). \end{cases}$$

Note that $\bar{f}$ is well-defined, continuous and $\bar{f}|P = f$.

We are going to show that for every asymptotically sublinear function s and every $\varepsilon > 0$ there exists a compact subset $K \subset X$ such that, for any $x \notin K$, $\operatorname{diam}(\bar{f}(B_{s(|x|)}(x))) < \varepsilon$.

Let $y, z \in B_{s(|x|)}(x)$. We consider the following cases.

1) If $y, z \notin \cup_{i\in\mathbb{N}} B_{r_i}(p_i)$, then $|\bar{f}(z) - \bar{f}(y)| = 0 < \varepsilon$.

2) If $y \in B_{r_i}(p_i)$ for some $i \in \mathbb{N}$, and $z \notin \cup_{j\in\mathbb{N}} B_{r_j}(p_j)$, then

$$d(z, y) < 2s(|x|) < 2\frac{r_i\varepsilon}{4M} = \frac{r_i\varepsilon}{2M}$$

as $B_{s(|x|)}(x) \cap B_{r_i}(p_i) \neq \varnothing$.

Therefore,

$$|\bar{f}(z) - \bar{f}(y)| = |\bar{f}(y)| = \frac{r_i - d(y, p_i)}{r_i}|f(p_i)| <$$
$$< \frac{d(z, p_i) - d(y, p_i)}{r_i}M \leqslant \frac{d(z, y)}{r_i}M < \frac{M}{r_i}\frac{r_i\varepsilon}{2M} = \frac{\varepsilon}{2} < \varepsilon,$$

for $|x|$ large enough.

3) For $z, y \in B_{r_i}(p_i)$ for some $i \in \mathbb{N}$, we obtain

$$|\bar{f}(z) - \bar{f}(y)| = |\bar{f}(p_i)|\left|\frac{r_i - d(z, p_i)}{r_i} - \frac{r_i - d(y, p_i)}{r_i}\right| \leqslant$$
$$\leqslant M\frac{|d(y, p_i) - d(z, p_i)|}{r_i} \leqslant M\frac{d(z, y)}{r_i} < \frac{M}{r_i}\frac{r_i\varepsilon}{2M} = \frac{\varepsilon}{2} < \varepsilon,$$

for $|x|$ large enough.

4) If $z \in B_{r_i}(p_i)$, $y \in B_{r_j}(p_j)$, $i \neq j$, then

$$|\bar{f}(z) - \bar{f}(y)| \leqslant |\bar{f}(z)| + |\bar{f}(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for $|x|$ large enough.

Summing up we see that $\bar{f}$ is a Higson sublinear function. Therefore, there exists an extension $\tilde{f}$ of $\bar{f}$ onto the space $h_L(X)$. The restriction of $\tilde{f}$ onto the closure $\bar{P}$ of P is an extension of f onto $\bar{P}$. We conclude that the map $i \mapsto p_i$ from $\mathbb{N}$ to P induces a homeomorphism of βN and $\bar{P}$. □

Corollary 2.8. *Let X be an unbounded proper metric space. Then the sublinear corona $\upsilon_L(X)$ contains a copy of $\beta\mathbb{N} \setminus \mathbb{N}$.*

Theorem 2.9. *Let X be an unbounded proper metric space. Then the compactification $h_P(X)$ contains a copy of $\beta\mathbb{N}$.*

Proof. By Theorem 2.7, there exists a countable discrete subspace Y of X such that the closure $c_1(Y)$ of Y in $h_L(X)$ determines the compactification of Y isomorphic to the Stone-Čech compactification of $\mathbb{N}$.

Let $\gamma\colon h_P(X) \to h_L(X)$ be the (unique) extension of the identity map of X. This map induces the map $\gamma'\colon c_2(Y) \to c_1(Y)$, where $c_2(Y)$ denotes the closure of Y in $h_P(X)$. Since $\beta\mathbb{N}$ is the maximal compactification of $\mathbb{N}$, the map γ' is a homeomorphism and therefore $c_2(Y)$ is a copy of $\beta\mathbb{N}$. □

Corollary 2.10. *Let X be an unbounded proper metric space. Then the subpower corona $\upsilon_P(X)$ contains a copy of $\beta\mathbb{N} \setminus \mathbb{N}$.*

It is well-known that the construction of Higson corona is functorial on the category of proper metric spaces and asymptotically Lipschitz proper maps. Recall that a map of metric spaces $f\colon X \to Y$ is called asymptotically Lipschitz [3] if there exist $\lambda, s > 0$ such that $d_Y(f(x), f(y)) \leqslant \lambda d(x, y) + s$, for all $x, y \in X$. A map is proper if the preimage of every bounded set is bounded.

However, there are maps which are not asymptotically Lipschitz that generate the continuous maps of the Higson coronas. The very simple example is that of metric spaces $X = \{n^2 \mid n \in \mathbb{N}\}$, $Y = \{n^4 \mid n \in \mathbb{N}\}$ with the metrics endowed from $\mathbb{R}$ and the map $n \mapsto n^2\colon X \to Y$.

The situation is different for the geodesic metric spaces. Recall that an arcwise connected metric space is called geodesic if the distance between any two its points equals the minimal length of a path connecting these points.

The following example demonstrates that there exists a map of geodesic metric spaces which is not asymptotically Lipschitz and which generates a continuous map of subpower coronas.

Example 2.11. Let $X = \mathbb{R}_+$ and let $F\colon X \to \mathbb{R}$ be a Higson subpower function.

Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a map defined by the formula $f(x) = x + x\ln x$. We are going to show that the composite Ff is also a Higson subpower function. Consider any asymptotically subpower, non-decreasing function $s\colon \mathbb{R}_+ \to \mathbb{R}_+$ and any open ball $B_{s(x)}(x)$, with $x > 0$. Now let $\tilde{B} = \{y + y\ln y \mid y \in B_{s(x)}(x)\}$. For any $y \in B_{s(x)}(x)$ we have

$$|y + y\ln y - x - x\ln x| \leq |y - x| + |y\ln y - x\ln x| < s(x) + (1 + \ln c)|y - x|$$

where c is a number between x and y. So for x large enough,

$$\begin{aligned} &|y + y\ln y - x - x\ln x| < s(x) + |y - x| + |y - x|\ln c \\ &< 2s(x) + s(x)\ln(x + s(x)) < 2s(x) + s(x)\ln x + s(x)\ln(s(x)). \end{aligned}$$

Let us denote $t(x) = 2s(x) + s(x)\ln x + s(x)\ln(s(x))$. Notice that t is an asymptotically subpower function as any linear combination and multiplication of asymptotically subpower functions is an asymptotically subpower function and $\ln(s(x))$ is also an asymptotically subpower function as $\ln(s(x)) < \ln(x)$ for x large enough. It is now easy to verify that $\tilde{B} \subset B_{t(x)}(x + x\ln x)$. And since s is non-decreasing and so is t, we have $\tilde{B} \subset B_{t(x+x\ln x)}(x+x\ln x)$. Since t is an asymptotically subpower function and F is a Higson subpower map,

$$\operatorname{diam}(F(B_{t(x+x\ln x)}(x + x\ln x))) \to 0 \quad \text{as} \quad x \to \infty,$$

thus $\operatorname{diam}(F(\tilde{B})) \to 0$ as $x \to \infty$. Therefore, for $x \to \infty$ we have

$$\sup\{|F(y + y\ln y) - F(z + z\ln z)| \mid y, z \in B_{s(x)}(x)\} \to 0.$$

We see that $\sup\{|(Ff)(y) - (Ff)(z)| \mid y, z \in B_{s(x)}(x)\} \to 0$ and therefore $\operatorname{diam}((Ff)(B_{s(x)}(x))) \to 0$ as $x \to \infty$.

If a subpower function $s\colon \mathbb{R}_+ \to \mathbb{R}_+$ is not non-decreasing, then we can construct a non-decreasing asymptotically subpower function u that $s(x) \le u(x)$ for x large enough. Then the proof is obvious as $B_{s(x)}(x) \subset B_{u(x)}(x)$ for x large enough and, by what was proved above, $\operatorname{diam}((Ff)(B_{u(x)}(x))) \to 0$ as $x \to \infty$.

Thus, the map f induces a continuous map of the subpower coronas. Clearly, f is not asymptotically Lipschitz.

Since, as the above example shows, the construction of the subpower Higson corona is applicable to a wider class of maps between metric spaces, this may considered as a motivation of introducing this corona.

3. Remarks and open questions

It is known that the Higson compactification of a proper metric space X is never equivalent to the Stone-Čech compactification of X unless X satisfies the following property [7]: for every $M > 0$ there is a compact subset K of X such that $B_M(x) = \{x\}$ for every $x \notin K$.

Question 3.1. Characterize the proper metric spaces for which the Higson subpower compactification and the Higson sublinear compactification are not equivalent. The same for Higson subpower compactification and the Higson compactification.

In [2], the dimension of the sublinear Higson corona is characterized in terms of asymptotically Lipschitz extensions of maps into the Euclidean spaces.

Question 3.2. Is there a similar characterization of the dimension of the subpower Higson corona?

One may speculate whether there exists an asymptotic dimension theory of (proper) metric spaces which is related to the subpower corona in a way in which the Assouad-Nagata dimension is related to the sublinear corona (see [4]).

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Received by the editors: 28.06.2014
and in final form 28.06.2014.