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Densities, submeasures and partitions of groups

Taras Banakh, Igor Protasov, Sergiy Slobodianiuk

Abstract. In 1995 in Kourovka notebook the second author asked the following problem: *is it true that for each partition $G = A_1 \cup \dots \cup A_n$ of a group G there is a cell A_i of the partition such that $G = FA_iA_i^{-1}$ for some set $F \subset G$ of cardinality $|F| \leqslant n$?* In this paper we survey several partial solutions of this problem, in particular those involving certain canonical invariant densities and submeasures on groups. In particular, we show that for any partition $G = A_1 \cup \dots \cup A_n$ of a group G there are cells A_i, A_j of the partition such that

- $G = FA_jA_j^{-1}$ for some finite set $F \subset G$ of cardinality $|F| \leqslant \max_{0<k\leqslant n} \sum_{p=0}^{n-k} k^p \leqslant n!$;
- $G = F \cdot \bigcup_{x \in E} xA_iA_i^{-1}x^{-1}$ for some finite sets $F, E \subset G$ with $|F| \leqslant n$;
- $G = FA_iA_i^{-1}A_i$ for some finite set $F \subset G$ of cardinality $|F| \leqslant n$;
- the set $(A_iA_i^{-1})^{4^{n-1}}$ is a subgroup of index $\leqslant n$ in G.

The last three statements are derived from the corresponding density results.

1. Introduction

In this paper we survey partial solutions to the following open problem posed by I.V.Protasov in 1995 in the Kourovka notebook [14, Problem 13.44].

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Problem 1.1. *Is it true that for any finite partition $G = A_1 \cup \dots \cup A_n$ of a group G there is a cell A_i of the partition and a subset $F \subset G$ of cardinality $|F| \leqslant n$ such that $G = FA_iA_i^{-1}$?*

In [14] it was observed that this problem has simple affirmative solution for amenable groups (see Theorem 4.3 below).

Problem 1.1 is a partial case of its "idealized" G-space version. Let us recall that a G-space is a set X endowed with a left action $\alpha : G \times X \to X$, $\alpha : (g, x) \mapsto gx$, of a group G. Each group G will be considered as a G-space endowed with the left action $\alpha : G \times G \to G$, $\alpha : (g, x) \mapsto gx$.

A non-empty family $\mathcal{I}$ of subsets of a set X is called a *Boolean ideal* if for any $A, B \in \mathcal{I}$ and $C \subset X$ we get $A \cup B \in \mathcal{I}$ and $A \cap C \in \mathcal{I}$. A Boolean ideal $\mathcal{I}$ on a set X will be called *trivial* if it coincides with the Boolean ideal $\mathcal{B}(X)$ of all subsets of X. By $[X]^{<\omega}$ we shall denote the Boolean ideal consisting of all finite subsets of X. A Boolean ideal $\mathcal{I}$ on a G-space X is called *G-invariant* if for any $A \in \mathcal{I}$ and $g \in G$ the shift gA of A belongs to the ideal $\mathcal{I}$. By an *ideal G-space* we shall understand a pair $(X, \mathcal{I})$ consisting of a G-space X and a non-trivial G-invariant Boolean ideal $\mathcal{I} \subset \mathcal{B}(X)$.

For an ideal G-space $(X, \mathcal{I})$ and a subset $A \subset X$ the set

$$\Delta_{\mathcal{I}}(A) = \{x \in G : A \cap xA \notin \mathcal{I}\} \subset G$$

will be called the *$\mathcal{I}$-difference set* of A. It is not empty if and only if $A \notin \mathcal{I}$.

For a non-empty subset $A \subset G$ of a group G its *covering number* is defined as

$$\mathrm{cov}(A) = \min\{|F| : F \subset G,\ G = FA\}.$$

More generally, for a Boolean ideal $\mathcal{J} \subset \mathcal{B}(G)$ on a group G and a non-empty subset $A \subset G$ let

$$\mathrm{cov}_{\mathcal{J}}(A) = \min\{|F| : F \subset G,\ \ G \setminus FA \in \mathcal{J}\}$$

be the *$\mathcal{J}$-covering number* of A.

Observe that for the smallest Boolean ideal $\mathcal{I} = \{\varnothing\}$ on a group G and a subset $A \subset G$ the $\mathcal{I}$-difference set $\Delta_{\mathcal{I}}(A)$ is equal to AA^{-1}. That is why Problem 1.1 is a partial case of the following more general

Problem 1.2. *Is it true that for any finite partition $X = A_1 \cup \dots \cup A_n$ of an ideal G-space $(X, \mathcal{I})$ some cell A_i of the partition has*

- $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$*?*

- $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ *for some non-trivial G-invariant Boolean ideal $\mathcal{J}$ on the acting group G?*

Problems 1.1 and 1.2 can be reformulated in terms of the functions $\Phi_G(n)$, $\Phi_{(X,\mathcal{I})}(n)$ defined as follows. For an ideal G-space $\mathbf{X} = (X, \mathcal{I})$ and a (real) number $n \geqslant 1$ denote by

$$X/n = \{\mathcal{C} \subset \mathcal{B}(X) : |\mathcal{C}| \leqslant n,\ \cup\mathcal{C} = X\}$$

the family of all at most n-element covers of X and put $\Phi_{\mathbf{X}}(n) = \sup_{\mathcal{C}\in X/n} \min_{C\in\mathcal{C}} \mathrm{cov}(\Delta_{\mathcal{I}}(C))$. For each G-space X we shall write $\Phi_X(n)$ instead of $\Phi_{(X,\{\varnothing\})}(n)$ (in this case we identify X with the ideal G-space $(X, \{\varnothing\})$. In particular, for each group G we put

$$\Phi_G(n) = \sup_{\mathcal{A}\in G/n} \min_{A\in\mathcal{A}} \mathrm{cov}(AA^{-1}).$$

For every ideal G-space $\mathbf{X} = (X, \mathcal{I})$ the definition of the number $\Phi_{\mathbf{X}}(n)$ implies that for any partition $X = A_1 \cup \cdots \cup A_n$ of X there is a cell A_i of the partition with $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant \Phi_{\mathbf{X}}(n)$. This fact allows us to reformulate and extend Problem 1.2 as follows.

Problem 1.3. *Study the growth of the function $\Phi_{\mathbf{X}}(n)$ for a given ideal G-space $\mathbf{X} = (X, \mathcal{I})$. Detect ideal G-spaces $\mathbf{X}$ with $\Phi_{\mathbf{X}}(n) \leqslant n$ for all $n \in \mathbb{N}$.*

Problems 1.1–1.3 have many partial solutions, which can be divided into three categories corresponding to methods used in these solutions.

The first category contains results giving upper bounds on the function $\Phi_{\mathbf{X}}(n)$ proved by a combinatorial approach first exploited by Protasov and Banakh in [19, §12] and then refined by Erde [10], Slobodianiuk [23] and Banakh, Ravsky, Slobodianiuk [8]. These results are surveyed in Section 2. The first non-trivial result proved by this approach was the upper bound $\Phi_{\mathbf{X}}(n) \leqslant 2^{2^{n-1}-1}$ proved in Theorem 12.7 of [19] for groups G endowed with the smallest ideal $\mathcal{I} = \{\varnothing\}$ and generalized later by Slobodianiuk (see [17, 4.2]) and Erde [10] to infinite groups G endowed with the ideal $\mathcal{I}$ of finite subsets of G. Later Slobodianiuk [23] using a tricky algorithmic approach, improved this upper bound to $\Phi_{\mathbf{X}}(n) \leqslant n!$ for any ideal G-space $\mathbf{X}$. This algorithmic approach was developed by Banakh, Ravsky and Slobodianiuk [8] who proved the upper bound $\Phi_{\mathbf{X}}(n) \leqslant \varphi(n+1) := \max_{1<k<n} \sum_{i=0}^{n-k} k^i \leqslant n!$, which is the best general upper bound on the function $\Phi_{\mathbf{X}}(n)$ available at the moment. The

function $\varphi(n)$ grows faster than any exponent a^n but slower than the sequence of factorials $n!$. Unfortunately, it grows much faster than the identity function n required in Problem 1.3.

The second category of partial solutions of Problems 1.1–1.3 exploits various G-invariant submeasures $\mu : \mathcal{B}(X) \to [0,1]$ on a G-space X and is presented in Sections 3–7. Given such a submeasure μ, for any partition $X = A_1 \cup \cdots \cup A_n$ of a G-space X, we use the subadditivity of μ to select a cell A_i with submeasure $\mu(A_i) \geqslant \frac{1}{n}$ and then use some specific properties of the submeasure μ to derive certain largeness property of the $\mathcal{I}$-difference set $\Delta_{\mathcal{I}}(A_i)$ for G-invariant Boolean ideals $\mathcal{I} \subset \{A \in \mathcal{B}(X) : \mu(A) = 0\}$.

In Section 4 we use for this purpose G-invariant finitely additive measures (which exist only on amenable G-spaces), and prove that for each partition $X = A_1 \cup \cdots \cup A_n$ of G-space X with amenable acting group G, endowed with a non-trivial G-invariant Boolean ideal $\mathcal{I} \subset \mathcal{B}(G)$, some cell A_i of the partition has $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$, which is equivalent to saying that $\Phi_{(G,\mathcal{I})}(n) \leqslant n$ for all $n \in \mathbb{N}$.

In Section 5 we apply the extremal density $\mathsf{is}_{12} : \mathcal{B}(X) \to [0,1]$, $\mathsf{is}_{12} : A \mapsto \inf_{\mu \in P_\omega(G)} \sup_{\nu \in P_\omega(X)} \mu * \nu(A)$, introduced in [24] and studied in [1]. On any G-space X with amenable acting group G the density is_{12} is subadditive and coincides with the upper Banach density d^*, well-known in Combinatorics of Groups (see e.g., [12]). Using the density is_{12} we show that each subset $A \subset X$ with positive density $\mathsf{is}_{12}(A) > 0$ has $\mathrm{cov}(\Delta_{\mathcal{I}}(A)) \leqslant 1/\mathsf{is}_{12}(A)$. In general, the density is_{12} is not subadditive, which does not allow to apply it directly to partitions of groups. However, its modification sis_{123} considered in Section 7 is subadditive. Using this feature of the submeasure sis_{123} we prove that for each partition $X = A_1 \cup \cdots \cup A_n$ there is a cell A_i of the partition and a finite set $E \subset G$ such that for each G-invariant Boolean ideal $\mathcal{I} \subset \{A \in \mathcal{B}(X) : \mathsf{sis}_{123}(A) = 0\}$ the set $\Delta_{\mathcal{I}}(A_i)^{\wr E} = \bigcup_{x \in E} x^{-1} \Delta_{\mathcal{I}}(A_i) x$ has $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i^{\wr E})) \leqslant n$. This implies an affirmative answer to Problem 1.1 for partitions $G = A_1 \cup \cdots \cup A_n$ of groups into conjugation-invariant sets $A_i = A_i^{\wr G}$.

The third group of partial solutions of Problems 1.1 and 1.2 is presented in Section 8 and exploits the compact right-topological semigroup $P(G)$ of measures on a group G. This approach is developed in a recent paper of Banakh and Frączyk [3] where they used minimal measures to prove that for each partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $(X, \mathcal{I})$ either $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ for all i or else there is a cell A_i of the partition such that $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i) \cdot \Delta_{\mathcal{I}}(A_i)) < n$ and $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$ for some G-invariant Boolean ideal $\mathcal{J} \not\ni \Delta_{\mathcal{I}}(A_i)$ on G. Using quasi-invariant idempotent measures, Banakh and Frączyk proved [3] that for any partition $G = A_1 \cup \dots \cup A_n$ of a group G either $\mathrm{cov}(A_i A_i^{-1}) \leqslant n$ for all i or else $\mathrm{cov}(A_i A_i^{-1} A_i) < n$ for some cell A_i of the partition. We use these facts to prove that for each partition $G = A_1 \cup \dots \cup A_n$ of a group G for some cell A_i of the partition, the product $(A_i A_i)^{4^{n-1}}$ is a subgroup of index $\leqslant n$ in G.

In Section 9 we apply the density is_{12} to IP*-sets and show that for each subset A of a group G with positive density $\mathsf{is}_{21}(A) > 0$ the set AA^{-1} is an IP*-set in G and hence belongs to every idempotent of the compact right-topological semigroup βG.

In the final Section 10 we pose some open problems related to Problems 1.1–1.3.

2. Some upper bounds on the function $\Phi_{\mathbf{X}}(n)$

In this section we survey some results giving upper bounds on the function $\Phi_{\mathbf{X}}(n)$, which are proved by combinatorial and algorithmic arguments. Unfortunately, the obtained upper bounds are much higher than the upper bound n required in Problem 1.3.

We recall that for an ideal G-space $\mathbf{X} = (X, \mathcal{I})$ the function $\Phi_{\mathbf{X}}(n)$ is defined by

$$\Phi_{\mathbf{X}}(n) = \sup_{\mathcal{C} \in X/n} \min_{A \in \mathcal{C}} \mathrm{cov}(\Delta_{\mathcal{I}}(A)) \quad \text{for } n \in \mathbb{N}.$$

If $\mathcal{I} = \{\varnothing\}$, then we write $\Phi_X(n)$ instead of $\Phi_{(X,\{\varnothing\})}(n)$. In particular, for a group G,

$$\Phi_G(n) = \sup_{\mathcal{C} \in G/n} \min_{A \in \mathcal{C}} \mathrm{cov}(AA^{-1}) \quad \text{for } n \in \mathbb{N}.$$

Historically, the first non-trivial upper bound on the function $\Phi_{\mathbf{X}}(n)$ appeared in Theorem 12.7 of [19].

Theorem 2.1 (Protasov-Banakh). *For any partition $G = A_1 \cup \dots \cup A_n$ of a group G some cell A_i of the partition has $\mathrm{cov}(A_i A_i^{-1}) \leqslant 2^{2^{n-1}-1}$, which implies that $\Phi_G(n) \leqslant 2^{2^{n-1}-1}$ for all $n \in \mathbb{N}$.*

In fact, the proof of Theorem 12.7 from [19] gives a bit better upper bound than $2^{2^{n-1}-1}$, namely:

Theorem 2.2 (Protasov-Banakh). *For any partition $G = A_1 \cup \dots \cup A_n$ of a group G some cell A_i of the partition has $\mathrm{cov}(A_i A_i^{-1}) \leqslant u(n)$ where*

$u(1) = 1$ and $u(n+1) = u(n)(u(n)+1)$ for $n \in \mathbb{N}$. Consequently, $\Phi_G(n) \leqslant u(n)$ for all $n \in \mathbb{N}$.

Observe that the sequence $u(n)$ has double exponential growth

$$2^{2^{n-2}} \leqslant u(n) \leqslant 2^{2^{n-1}-1} \quad \text{for} \quad n \geqslant 2.$$

The method of the proof of Theorem 2.2 works also for ideal G-spaces (see [10]) which allows us to obtain the following generalization of Theorem 2.2.

Theorem 2.3. *For any partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $\mathbf{X} = (X, \mathcal{I})$ some cell A_i of the partition has $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant u(n) \leqslant 2^{2^{n-1}-1}$ where $u(1) = 1$ and $u(n+1) = u(n)(u(n)+1)$ for $n \in \mathbb{N}$. This implies $\Phi_{\mathbf{X}}(n) \leqslant u(n)$ for all $n \in \mathbb{N}$.*

In [23] S.Slobodianiuk invented a method allowing to replace the upper bound $u(n)$ in Theorem 2.3 by a function $\hbar(n)$, which grows slower that $n!$. To define the function $\hbar(n)$ we need to introduce some notation.

Given an natural number n denote by ω^n the set of all functions defined on the set $n = \{0, \dots, n-1\}$ and taking values in the set ω of all finite ordinals. The set ω^n is endowed with a partial order in which $f \leqslant g$ if and only if $f(i) \leqslant g(i)$ for all $i \in n$. For an index $i \in n$ by $\chi_i : n \to \{0,1\}$ we denote the characteristic function of the singleton $\{i\}$, which means that $\{i\} = \chi_i^{-1}(1)$.

For subsets $A_0, \dots, A_{n-1} \subset \omega^n$ let

$$\sum_{i \in n} A_i = \Big\{ \sum_{i \in n} a_i : \forall i \in n \;\; a_i \in A_i \Big\}$$

be the pointwise sum of these sets.

Now given any function $h \in \omega^n$ we define finite subsets $h^{[m]}(i) \subset \omega^n$, $i \in n$, $m \in \omega$, by the recursive formula:

- $h^{[0]}(i) = \{\chi_i\}$;
- $h^{[m+1]}(i) = h^{[m]}(i) \cup \{x - x(i)\chi_i : x \in \sum_{i \in n} h^{[m]}(i),\; x \leqslant h\}$ for $m \in \omega$.

A function $h \in \omega^n$ is called 0-*generating* if the constant zero function $\mathbf{0} : n \to \{0\}$ belongs to the set $h^{[m]}(i)$ for some $i \in n$ and $m \in \omega$.

Now put $\hbar(n)$ be the smallest number $c \in \omega$ for which the constant function $h : n \to \{c\}$ is 0-generating. The function $\hbar(n)$ coincides with the function $s_{-\infty}(n)$ considered and evaluated in [8]. The proof of the following theorem (essentially due to Slobodianiuk) can be found in [8].

Theorem 2.4 (Slobodianiuk)**.** *For any partition $X = A_1 \cup \dots \cup A_n$ of an ideal G-space $\mathbf{X} = (X, \mathcal{I})$ some cell A_i of the partition has* $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant \hbar(n)$*. This implies that $\Phi_{\mathbf{X}}(n) \leqslant \hbar(n)$ for all $n \in \mathbb{N}$.*

The growth of the sequence $\hbar(n)$ was evaluated in [8] with help of the functions

$$\varphi(n) = \max_{0<k<n} \sum_{i=0}^{n-k-1} k^i = \max_{1<k<n} \frac{k^{n-k}-1}{k-1} \quad \text{and} \quad \phi(n) = \sup_{1<x<n} \frac{x^{n-x}-1}{x-1}.$$

Theorem 2.5 (Banakh-Ravsky-Slobodianiuk)**.** *For every $n \geqslant 2$ we get*

$$\varphi(n) \leqslant \phi(n) < \hbar(n) \leqslant \varphi(n+1) \leqslant \phi(n+1).$$

The growth of the function $\phi(n)$ was evaluated in [8] with help of the Lambert W-function, which is inverse to the function $y = xe^x$. So, $W(y)e^{W(y)} = y$ for each positive real numbers y. It is known [13] that at infinity the Lambert W-function $W(x)$ has asymptotical growth

$$W(x) = L - l + \frac{l}{L} + \frac{l(-2+l)}{2L^2} + \frac{l(6-9l+2l^2)}{6L^3} + \\ + \frac{l(-12+36l-22l^2+3l^3)}{12L^4} + O\Big[\Big(\frac{l}{L}\Big)^5\Big]$$

where $L = \ln x$ and $l = \ln\ln x$.

The growth of the sequence $\phi(n+1)$ was evaluated in [8] as follows:

Theorem 2.6 (Banakh)**.** *For every $n > 50$*

$$nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n} < \ln\phi(n+1) < \\ < nW(ne) - 2n + \frac{n}{W(ne)} + \frac{W(ne)}{n} + \frac{\ln\ln(ne)}{n}$$

and hence

$$\ln\phi(n+1) = n\ln n - n - n\big(\ln\ln(n) + O(\tfrac{\ln\ln n}{\ln n})\big).$$

It light of Theorem 2.6, it is interesting to compare the growth of the sequence $\phi(n)$ with the growth of the sequence $n!$ of factorials. Asymptotical bounds on $n!$ proved in [22] yield the following lower and upper bounds on the logarithm $\ln n!$ of $n!$:

$$n \ln n - n + \frac{1}{2}\ln n + \frac{\ln 2}{2} + \frac{1}{12n+1} < \ln n! <$$
$$< n \ln n - n + \frac{\ln n}{n} + \frac{1}{2}\ln n + \frac{\ln 2}{2} + \frac{1}{12n}.$$

Comparing these two formulas, we see that the sequence $\phi(n)$ as well as $\hbar(n)$ grows faster than any exponent a^n, $a > 1$, but slower than the sequence $n!$ of factorials.

In fact the upper bound $\varphi(n+1) \leqslant n!$ can be easily proved by induction:

Proposition 2.7. *Each ideal G-space $\mathbf{X} = (X, \mathcal{I})$ has $\Phi_{\mathbf{X}}(n) \leqslant \hbar(n) \leqslant \varphi(n+1) \leqslant n!$ for every $n \geqslant 2$.*

Proof. The inequalities $\Phi_{\mathbf{X}}(n) \leqslant \hbar(n) \leqslant \varphi(n+1)$ follow from Theorems 2.4 and 2.5. The inequality $\varphi(n+1) \leqslant n!$ will be proved by induction. It holds for $n = 2$ as $\varphi(3) = \sum_{i=0}^{3-1-1} 1^i = 2 = 2!$. Assume that for some $n \geqslant 1$ we have proved that $\varphi(n) \leqslant (n-1)!$. Observe that for every $0 < k < n$

$$\sum_{i=0}^{n-k} k^i = \sum_{i=0}^{n-1-k} k^i + k^{n-k} \leqslant \varphi(n) + \frac{k^{n-k}-1}{k-1}(k-1) + 1 \leqslant$$
$$\leqslant \varphi(n) + \varphi(n)(k-1) + 1 = \varphi(n)k + 1 \leqslant \varphi(n)(k+1),$$

which implies

$$\varphi(n+1) = \max_{0<k\leqslant n} \sum_{i=0}^{n-k} k^i = \max_{0<k<n} \sum_{i=0}^{n-k} k^i \leqslant$$
$$\leqslant \max_{0<k<n} \varphi(n)(k+1) = \varphi(n) \cdot n \leqslant (n-1)! \cdot n = n!. \quad \square$$

The definition of the numbers $\hbar(n)$ is algorithmic and can be calculated by computer. However the complexity of calculation grows very quickly. So, the exact values of the numbers $\hbar(n)$ are known only $n \leqslant 7$. For $n = 8$ by long computer calculations we have merely found an upper bound on $\hbar(8)$. In particular, finding the upper bound $\hbar(8) \leqslant 136$ required a year of continuous calculations on a laptop computer. The values of the sequences $\varphi(n)$, $1 + \lfloor \phi(n) \rfloor$, $\hbar(n)$, $\varphi(n+1)$, $n!$, $u(n)$, and $2^{2^{n-1}-1}$ for $n \leqslant 8$

are presented in the following table:

n	2	3	4	5	6	7	8
$\varphi(n)$	1	2	3	7	15	40	121
$1+\lfloor\phi(n)\rfloor$	2	3	4	8	17	42	122
$\hbar(n)$	2	3	5	9	19	47	$\leqslant 136$
$\varphi(n+1)$	2	3	7	15	40	121	364
$n!$	2	6	24	120	720	4320	30240
$u(n)$	2	6	42	1806	3263442	10650056950806	$---$
$2^{2^{n-1}-1}$	2	8	128	32768	2147483648	9223372036854775808	2^{127}

This table shows that the upper bound given by Theorem 2.4 is much better than those from Theorems 2.1–2.3. Since $\hbar(n) = n$ for $n \leqslant 3$, Theorem 2.4 implies a positive answer to Problem 1.3 for $n \leqslant 3$.

Corollary 2.8. *For each partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $\mathbf{X} = (X, \mathcal{I})$ into $n \leqslant 3$ pieces some cell A_i of the partition has* $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$. *Consequently, $\Phi_{\mathbf{X}}(n) \leqslant n$ for $n \leqslant 3$.*

3. Densities and submeasures on G-spaces

The other approach to solution of Problems 1.1–1.3 exploits various densities and submeasures on G-spaces. Partial solutions of Problems 1.1–1.3 obtained by this method are surveyed in Sections 3–7. In this section we recall the necessary definitions related to densities and submeasures.

Let X be a G-space and $\mathcal{B}(X)$ be the Boolean algebra of all subsets of X. A function $\mu : \mathcal{B}(X) \to [0,1]$ is called

- *G-invariant* if $\mu(gA) = \mu(A)$ for any $A \subset X$ and $g \in G$;
- *monotone* if $\mu(A) \leqslant \mu(B)$ for any subsets $A \subset B \subset X$;
- *subadditive* if $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ for any subsets $A, B \subset X$;
- *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ for any disjoint subsets $A, B \subset X$;
- a *density* if μ is monotone, $\mu(\varnothing) = 0$ and $\mu(X) = 1$;
- a *submeasure* if μ is a subadditive density;
- a *measure* if μ is an additive density.

So, all our measures are finitely additive probability measures defined on the Boolean algebra $\mathcal{B}(X)$ of all subsets of X.

The space of all densities on X will be denoted by $D(X)$ and will be considered as a closed convex subspace of the Tychonoff cube $[0,1]^{\mathcal{B}(X)}$.

The space $D(X)$ contains a closed convex subset $P(X)$ consisting of measures.

A measure μ on X is called *finitely supported* if $\mu(F) = 1$ for some finite subset $F \subset X$. In this case μ can be written as a convex combination $\mu = \sum_{i=1}^n \alpha_i \delta_{x_i}$ of Dirac measures. Let us recall that the *Dirac measure* δ_x supported at a point $x \in X$ is the $\{0,1\}$-valued measure assigning to each subset $A \subset X$ the number

$$\delta_x(A) = \begin{cases} 0 & \text{if } x \notin A, \\ 1 & \text{if } x \in A. \end{cases}$$

The family of all finitely supported measures on X will be denoted by $P_\omega(X)$. It is a convex dense subset in the space $P(X)$ of all measures on X.

A finitely supported measure $\mu \in P_\omega(X)$ is called a *uniformly distributed measure* if $\mu = \frac{1}{|F|}\sum_{x\in F}\delta_x$ for some non-empty finite subset $F \subset X$ (which coincides with the support of the measure μ). The set of uniformly distributed measures will be denoted by $P_u(X)$.

For any finitely supported measures $\mu = \sum_i \alpha_i \delta_{a_i}$ and $\nu = \sum_j \beta_j \delta_{b_j}$ on a group G we can define their convolution by the formula

$$\mu * \nu = \sum_{i,j} \alpha_i \beta_j \delta_{a_i b_j}.$$

More generally, the convolution $\mu * \nu$ can be well-defined for any measure $\mu \in P(G)$ on a group G and a density $\nu \in D(X)$ on a G-space X:

$$\mu * \nu(A) = \int_G \nu(x^{-1}A)d\mu(x) \text{ for any set } A \subset X.$$

It can be shown that the convolution operation $* : P(G) \times D(X) \to D(X)$ is *right continuous* in the sense that for every density $\nu \in D(X)$ the right shift $\rho_\nu : P(G) \to D(X)$, $\rho_\nu : \mu \mapsto \mu * \nu$, is continuous. By a standard argument (see e.g. [12, 4.4]), it can be shown that the operation of convolution is associative in the sense that

$$(\mu_1 * \mu_2) * \mu_3 = \mu_1 * (\mu_2 * \mu_3)$$

for any measures $\mu_1, \mu_2 \in P(G)$ and a density $\mu_3 \in D(X)$. The right-continuity of the convolution operation implies that for every density $\nu \in D(X)$ its $P(G)$-orbit $P(G) * \nu = \{\mu * \nu : \mu \in P(G)\}$ is a closed convex set in $D(X)$, which coincides with the closed convex hull of the G-orbit $G\nu = \{\delta_g * \nu : g \in G\}$ of ν.

A density $\nu \in D(X)$ will be called *minimal* if each density $\mu \in P(G)*\nu$ has $P(G)$-orbit $P(G)*\mu = P(G)*\nu$. Zorn's Lemma and the compactness of $P(G)$-orbits implies that for each density $\nu \in D(X)$ its $P(G)$-orbit $P(G)*\nu$ contains a minimal density.

Let $D_{\min}(X)$ be the set of all minimal densities on X and $P_{\min}(X) = P(X) \cap D_{\min}(X)$ be the set of all minimal measures on X. Observe that the set $P_{\min}(X)$ is not empty and contains the set $P_G(X)$ of all G-invariant measures (which can be empty). A G-space X is called *amenable* if $P_G(X) \neq \varnothing$, i.e., X admits a G-invariant measure $\mu : \mathcal{B}(X) \to [0,1]$. It can be shown that a G-space X is amenable if it satisfies the *Følner condition*: for every $\varepsilon > 0$ and every finite set $F \subset G$ there is a finite set $E \subset X$ such that $|FE| < (1+\varepsilon)|E|$. It is well-known [16] that a group G is amenable if and only if it satisfies the Følner condition.

Each group G will be considered as a G-space endowed with the left action $\alpha : G \times G \to G$, $\alpha : (g,x) \mapsto gx$, of G on itself. In this case the space $P(G)$ endowed with the operation of convolution is a compact right-topological semigroup. G-Invariant densities or Boolean ideals on G will be called *left-invariant*.

For a density $\mu : \mathcal{B}(X) \to [0,1]$ on a set X its *subadditivization* $\widehat{\mu} : \mathcal{B}(X) \to [0,1]$ is defined by the formula

$$\widehat{\mu}(A) = \sup_{B \subset X} \big(\mu(A \cup B) - \mu(B)\big).$$

The subadditivization $\widehat{\mu}$ is a submeasure such that $\mu \leqslant \widehat{\mu}$. A density μ is subadditive if and only if it $\mu = \widehat{\mu}$.

For a density $\mu : \mathcal{B}(X) \to [0,1]$ on a set X by $[\mu{=}0]$ we shall denote the family $\{A \subset X : \mu(A) = 0\}$. The family $[\mu{=}0]$ is a non-trivial Boolean ideal on X if μ is subadditive. The inequality $\mu \leqslant \widehat{\mu}$ implies $[\widehat{\mu}{=}0] \subset [\mu{=}0]$ for any density μ on X.

In this paper we shall meet many examples of so-called *extremal densities*. Those are densities obtained by applying infima and suprema to convolutions of measures over certain families of measures on groups or G-spaces. The simplest examples of extremal densities are the densities $\mathsf{i}_1 : \mathcal{B}(X) \to \{0,1\}$ and $\mathsf{s}_1 : \mathcal{B}(X) \to [0,1]$ defined on each set X by

$$\mathsf{i}_1(A) = \inf_{\mu_1 \in P_\omega(X)} \mu_1(A) \quad \text{and} \quad \mathsf{s}_1(A) = \sup_{\mu_1 \in P_\omega(X)} \mu_1(A).$$

It is clear that

$$\mathsf{i}_1(A) = \begin{cases} 0, & \text{if } A \neq X, \\ 1 & \text{if } A = X, \end{cases} \quad \text{and} \quad \mathsf{s}_1(A) = \begin{cases} 0, & \text{if } A = \varnothing, \\ 1 & \text{if } A \neq \varnothing, \end{cases}$$

which implies that i_1 and s_1 are the smallest and largest densities on X, respectively. The density s_1 is subadditive while i_1 is not (for a set X containing more than one point). More complicated extremal densities will appear in Sections 5–7.

Another important example of an extremal density is the *upper Banach density*

$$d^* : \mathcal{B}(X) \to [0,1], \ \ d^* : A \mapsto \sup_{\mu \in P_{\min}(X)} \mu(A)$$

defined on each G-space X. It is clear that the upper Banach density d^* is subadditive and hence is a submeasure on X.

4. On partitions of G-spaces endowed with a G-invariant measure

In fact, Problems 1.1–1.3 have trivial affirmative answer for amenable G-spaces (cf. Theorem 12.8 [19]).

Theorem 4.1. *Let $(X,\mathcal{I})$ be an ideal G-space endowed with a G-invariant measure $\mu : \mathcal{B}(X) \to [0,1]$ such that $\mathcal{I} \subset [\mu{=}0]$. Each subset $A \subset X$ with $\mu(A) > 0$ has* $\mathrm{cov}(\Delta_{\mathcal{I}}(A)) \leqslant 1/\mu(A)$.

Proof. Choose a maximal subset $F \subset G$ such that $\mu(xA \cap yA) = 0$ for any distinct points $x, y \in F$. The additivity and G-invariance of the measure μ implies that $|F| \leqslant 1/\mu(A)$. By the maximality of F, for every $x \in G$ there is $y \in F$ such that $\mu(xA \cap yA) > 0$, which implies $yA \cap xA \notin \mathcal{I}$ and $A \cap y^{-1}xA \notin \mathcal{I}$. Then $y^{-1}x \in \Delta_{\mathcal{I}}(A)$ and $x \in y \cdot \Delta_{\mathcal{I}}(A)$. So, $X = F \cdot \Delta_{\mathcal{I}}(A)$ and $\mathrm{cov}(\Delta_{\mathcal{I}}(A)) \leqslant |F| \leqslant 1/\mu(A)$. □

Corollary 4.2. *Let $(X,\mathcal{I})$ be an ideal G-space admitting a G-invariant measure $\mu : \mathcal{B}(X) \to [0,1]$ such that $\mathcal{I} \subset [\mu{=}0]$. For each partition $X = A_1 \cup \dots \cup A_n$ of X some cell A_i of the partition has* $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$. *This implies $\Phi_{(X,\mathcal{I})}(n) \leqslant n$ for all $n \in \mathbb{N}$.*

Proof. The subadditivity of the measure μ guarantees that some cell A_i of the partition has measure $\mu(A_i) \geqslant 1/n$. Then $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant 1/\mu(A_i) \leqslant n$ according to Theorem 4.1. □

The following theorem resolves Problem 1.2 for G-spaces with amenable acting group G.

Theorem 4.3. *For each partition $G = A_1 \cup \dots \cup A_n$ of an ideal G-space $(X,\mathcal{I})$ endowed with an action of an amenable group G, some cell A_i of* *the partition has* $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$*, which implies* $\Phi_{(G,\mathcal{I})}(n) \leqslant n$ *for all* $n \in \mathbb{N}$.

Proof. Using the idea of the proof of Theorem 3.1 of [5], we shall construct a G-invariant measure $\mu : \mathcal{B}(X) \to [0,1]$ of X whose null ideal $[\mu{=}0]$ contains the G-invariant ideal $\mathcal{I}$.

Let $[G]^{<\omega}$ be the Boolean ideal of all finite subsets of the amenable group G. Consider the set $\mathcal{D} = ([G]^{<\omega} \setminus \{\varnothing\}) \times \mathbb{N} \times \mathcal{I}$ endowed with the partial order $(F,n,A) \leqslant (E,m,B)$ iff $F \subset E$, $n \leqslant m$, and $A \subset B$. To each triple $d = (F,n,A)$ assign a finitely supported measure μ_d on X as follows. Using the Følner condition, find a finite set $E \subset [G]^{<\omega}$ such that $|FE| < (1+\frac{1}{n})|E|$. Since $\mathcal{I}$ is a non-trivial G-invariant ideal on X, the set $E^{-1}A \in \mathcal{I}$ does not coincide with X and hence we can find a point $x_d \in X \setminus E^{-1}A$. Then $Ex_d \subset X \setminus A$ and hence $\mu_d(A) = 0$ for the finitely supported measure $\mu_d = \frac{1}{|E|}\sum_{g\in E}\delta_{gx_d}$ on X.

By the compactness of the Tychonoff cube $[0,1]^{\mathcal{B}(X)}$ the net $(\mu_d)_{d\in\mathcal{D}}$ has a limit point, which is a measure $\mu : \mathcal{B}(X) \to [0,1]$ such that for every neighborhood $O(\mu) \subset [0,1]^{\mathcal{B}(X)}$ and every $d_0 \in \mathcal{D}$ there is $d \geqslant d_0$ in $\mathcal{D}$ such that $\mu_d \in O(\mu)$. Repeating the argument of the proof of Theorem 3.1 [5] it can be shown that μ is a G-invariant measure on X such that $\mu(A) = 0$ for every $A \in \mathcal{I}$.

So, it is legal to apply Corollary 4.2 and conclude that for any partition $X = A_1 \cup \dots \cup A_n$ of X some cell A_i of the partition has $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$. □

So, Problems 1.2, 1.3 remains open only for G-spaces with non-amenable acting group G.

5. The extremal density is_{12}

In this section we consider the extremal density is_{12}, which is defined on each G-space X by the formula

$$\mathsf{is}_{12}(A) = \inf_{\mu_1\in P_\omega(G)} \sup_{\mu_2\in P_\omega(X)} \mu_1 * \mu_2(A) = \inf_{\mu\in P_\omega(G)} \sup_{x\in X} \mu * \delta_x(A)$$

for $A \subset X$. It can be shown that the density is_{12} is G-invariant. In case of groups this density (denoted by a) was introduced in [24] and later studied in [1].

The density $\mathsf{is}_{12}(A)$ can be used to give an upper bound for the packing index of a set A in G. For a subset $A \subset X$ of an ideal G-space $(X,\mathcal{I})$ its packing index $\mathrm{pack}_{\mathcal{I}}(A)$ is defined by

$$\begin{aligned}&\mathrm{pack}_{\mathcal{I}}(A) =\\&= \sup\{|E| : E \subset G,\ \ xA \cap yA \in \mathcal{I} \text{ for any distinct points } x, y \in E\}.\end{aligned}$$

If the ideal $\mathcal{I} = \{\varnothing\}$, the we shall write $\mathrm{pack}(A)$ instead of $\mathrm{pack}_{\{\varnothing\}}(A)$. Packing indices were introduced and studied in [4], [6]. The packing index $\mathrm{pack}_{\mathcal{I}}(A)$ upper bounds the covering number $\mathrm{cov}(\Delta_{\mathcal{I}}(A))$.

Proposition 5.1. *For any subset A of an ideal G-space $(X, \mathcal{I})$ we get* $\mathrm{cov}_{\mathcal{I}}(A) \leqslant \mathrm{pack}_{\mathcal{I}}(A)$.

Proof. Using Zorn's Lemma, choose a maximal subset $F \subset G$ such that $xA \cap yA \in \mathcal{I}$ for any distinct points $x, y \in F$. By the maximality of F, for any $x \in G$ there is $y \in F$ such that $yA \cap xA \notin \mathcal{I}$. By the G-invariance of the ideal $\mathcal{I}$, $A \cap y^{-1}xA \notin \mathcal{I}$ and hence $y^{-1}x \in \Delta_{\mathcal{I}}(A)$. Then $x \in y\,\Delta_{\mathcal{I}}(A) \subset F \cdot \Delta_{\mathcal{I}}(A)$ and hence

$$\mathrm{cov}(\Delta_{\mathcal{I}}(A)) \leqslant |F| \leqslant \mathrm{pack}_{\mathcal{I}}(A). \qquad \square$$

Applications of the extremal density is_{12} to Problems 1.1–1.3 are based on the following fact.

Proposition 5.2. *If a subset A of a G-space X has positive density* $\mathsf{is}_{12}(A) > 0$*, then*

$$\mathrm{cov}(\Delta_{\mathcal{I}}(A)) \leqslant \mathrm{pack}_{\mathcal{I}}(A) \leqslant 1/\mathsf{is}_{12}(A)$$

for any G-invariant Boolean ideal $\mathcal{I} \subset [\widehat{\mathsf{is}}_{12}{=}0]$.

Proof. The inequality $\mathrm{cov}(\Delta_{\mathcal{I}}(A)) \leqslant \mathrm{pack}_{\mathcal{I}}(A)$ was proved in Proposition 5.1. It remains to prove that $\mathrm{pack}_{\mathcal{I}}(A) \leqslant 1/\mathsf{is}_{12}(A)$. Assuming conversely that $\mathrm{pack}_{\mathcal{I}}(A) > 1/\mathsf{is}_{12}(A)$, we can find a finite subset $F \subset G$ such that $|F| > 1/\mathsf{is}_{12}(A)$ and $xA \cap yA \in \mathcal{I}$ for any distinct points $x, y \in F$. It follows that the set $Z = \bigcup\{xA \cap yA : x, y \in F,\ x \neq y\}$ belongs to the ideal $\mathcal{I}$ and so does the set $F^{-1}Z$. Consequently, $\widehat{\mathsf{is}}_{12}(F^{-1}Z) = 0$ and the set $A' = A \setminus F^{-1}Z$ has density $\mathsf{is}_{12}(A') = \mathsf{is}_{12}(A)$ according to the definition of the submeasure $\widehat{\mathsf{is}}_{12}$. The definition of the set Z implies that the indexed family $(xA')_{x \in F}$ is disjoint. We claim that $|F^{-1}z \cap A'| \leqslant 1$ for every point $z \in X$. Assuming conversely that for some $z \in X$ the set $F^{-1}z$ contains two distinct points $a, b \in A'$, we conclude that $a = x^{-1}z$ and $b = y^{-1}z$ for two distinct points $x, y \in F$, which implies that $xA' \cap yA' \ni z$

is not empty. But this contradicts the disjointness of the family $(xA')_{x\in F}$. So, $|F^{-1}z\cap A'|\leqslant 1$ and hence for the uniformly distributed measure $\mu_1=\frac{1}{|F|}\sum_{g\in F}\delta_{g^{-1}}$ we get

$$\mathsf{is}_{12}(A)=\mathsf{is}_{12}(A')\leqslant\sup_{\mu_2\in P_\omega(X)}\mu_1*\mu_2(A')=\sup_{x\in X}\mu_1*\delta_x(A')=$$
$$=\sup_{x\in X}\frac{1}{|F|}\sum_{g\in F}\delta_{g^{-1}}*\delta_z(A')=\sup_{x\in X}\frac{|F^{-1}z\cap A'|}{|F|}\leqslant\frac{1}{|F|}<\mathsf{is}_{12}(A),$$

which is a desired contradiction proving that $\mathrm{pack}_{\mathcal{I}}(A)\leqslant 1/\mathsf{is}_{12}(A)$. □

Corollary 5.3. *If for a G-space X the extremal density is_{12} is subadditive, then any partition $X=A_1\cup\dots\cup A_n$ of X some cell A_i of the partition has* $\mathrm{cov}(\Delta_{\mathcal{I}}(A))\leqslant n$ *for any left-invariant Boolean ideal $\mathcal{I}\subset[\mathsf{is}_{12}{=}0]$.*

Proof. The subadditivity of the density is_{12} on implies that $\widehat{\mathsf{is}}_{12}=\mathsf{is}_{12}$ and $\mathcal{I}\subset[\mathsf{is}_{12}{=}0]=[\widehat{\mathsf{is}}_{12}{=}0]$. Also the subadditivity of is_{12} guarantees that some cell A_i of the partition has density $\mathsf{is}_{12}(A_i)\geqslant 1/n$. Then $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i))\leqslant 1/\mathsf{is}_{12}(A_i)\leqslant n$ according to Proposition 5.2. □

The following fact was proved in Theorem 3.9 of [2].

Proposition 5.4. *For any G-space X with amenable acting group G the extremal density is_{12} coincides with the upper Banach density d^* and hence is subadditive.*

6. The extremal density us_{12}

In this section we consider a uniform variation of the extremal density is_{12}, denoted by us_{12}. On each G-space X the extremal density $\mathsf{us}_{12}:\mathcal{B}(X)\to[0,1]$ is defined by

$$\mathsf{us}_{12}(A)=\inf_{\mu_1\in P_u(G)}\sup_{\mu_2\in P_\omega(X)}\mu_1*\mu_2(A)\quad\text{for } A\subset X.$$

Here $P_u(G)$ stands for the set of all uniformly distributed measures on G. It can be shown that on a group G the density us_{12} can be equivalently defined as

$$\mathsf{us}_{12}(A)=\inf_{\varnothing\neq F\in[G]^{<\omega}}\sup_{x\in X}\frac{|Fx\cap A|}{|F|}.$$

where $[G]^{<\omega}$ is the Boolean ideal consisting of all finite subsets of G. On groups the density us_{12} (denoted by u) was introduced by Solecki in [24] and studied in more details in [24] and [1].

It can be shown that the density us_{12} is G-invariant on each G-space X and $\mathsf{is}_{12} \leqslant \mathsf{us}_{12}$. Moreover a subset $A \subset X$ has $\mathsf{is}_{12}(A) = 1$ if and only if $\mathsf{us}_{12}(A) = 1$ if and only if A is *thick* in X in the sense that for every finite subset $F \subset G$ there is a point $x \in X$ with $Fx \subset A$.

On amenable groups the densities us_{12} and is_{12} coincide. This was shown by Solecki in [24]:

Proposition 6.1 (Solecki)**.** *For any amenable group G the densities us_{12} and is_{12} coincide and are subadditive. If a group G contains a non-Abelian free subgroup, then for every $\varepsilon > 0$ there is a set $A \subset G$ with $\mathsf{is}_{12}(A) < \varepsilon$ and $\mathsf{us}_{12}(A) > 1 - \varepsilon$.*

In general, the density us_{12} is not subadditive (as well as the density is_{12}):

Example 6.2. The free group with two generators can be written as the union $F_2 = A \cup B$ of two sets with $\mathsf{us}_{12}(A) = \mathsf{us}_{12}(B) = 0$.

Proof. Let a, b be the generators of the free group F_2. The elements of the group F_2 can be identified with irreducible words in the alphabet $\{a, b, a^{-1}, b^{-1}\}$. Let A be the set of irreducible words that start with a or a^{-1} and $B = F_2 \setminus A$. It can be shown that $F_2 = A \cup B$ is a required partition with $\mathsf{us}_{12}(A) = \mathsf{us}_{12}(B) = 0$. For details, see Example 3.2 in [1]. □

The extremal density us_{12} can be adjusted to a subadditive density $\widehat{\mathsf{us}}_{12} : \mathcal{B}(X) \to [0,1]$ defined by $\widehat{\mathsf{us}}_{12}(A) = \sup_{B \subset X}(\mathsf{us}_{12}(A \cup B) - \mathsf{us}_{12}(B))$ for $A \subset X$.

For our purposes, the density us_{12} will be helpful because of the following its property, which is a bit stronger than Proposition 5.2 and can be proved by analogy:

Proposition 6.3. *If a subset A of a G-space G has positive density $\mathsf{us}_{12}(A) > 0$, then*

$$\operatorname{cov}(\Delta_{\mathcal{I}}(A)) \leqslant \operatorname{pack}_{\mathcal{I}}(A) \leqslant 1/\mathsf{us}_{12}(A)$$

for any G-invariant Boolean ideal $\mathcal{I} \subset [\widehat{\mathsf{us}}_{12}{=}0]$.

7. The extremal submeasure sis_{123}

In this section we shall present applications of the G-invariant submeasure sis_{123} defined on each G-space X by the formula

$$\mathsf{sis}_{123}(A) = \sup_{\mu_1\in P_\omega(G)} \inf_{\mu_2\in P_\omega(G)} \sup_{\mu_3\in P_\omega(X)} \mu_1*\mu_2*\mu_3(A) = \\ = \sup_{\mu\in P_\omega(G)} \inf_{\nu\in P_\omega(G)} \sup_{x\in X} \mu*\nu*\delta_x(A) \quad \text{for} \quad A\subset X.$$

Proposition 7.1. *On each G-space X the density $\mathsf{sis}_{123}:\mathcal{B}(X)\to[0,1]$ is subadditive.*

Proof. It suffices to check that $\mathsf{sis}_{123}(A\cup B)\leqslant \mathsf{sis}_{123}(A)+\mathsf{sis}_{123}(B)+2\varepsilon$ for every subsets $A,B\subset X$ and real number $\varepsilon>0$. This will follow as soon as for any measure $\mu_1\in P_\omega(G)$ we find a measure $\mu_2\in P_\omega(G)$ such that $\sup_{\mu_3\in P_\omega(X)}\mu_1*\mu_2*\mu_3(A\cup B)) < \mathsf{sis}_{123}(A)+\mathsf{sis}_{123}(B)+2\varepsilon$.

By the definition of $\mathsf{sis}_{123}(A)$, for the measure μ_1 there is a measure $\nu_2\in P_\omega(G)$ such that

$$\sup_{\nu_3\in P_\omega(X)} \mu_1*\nu_2*\nu_3(A) < \mathsf{sis}_{123}(A)+\varepsilon.$$

By the definition of $\mathsf{sis}_{123}(B)$ for the measure $\eta_1=\mu_1*\nu_2$ there is a measure $\eta_2\in P_\omega(G)$ such that

$$\sup_{\eta_3\in P_\omega(X)} \eta_1*\eta_2*\eta_3(B) < \mathsf{sis}_{123}(B)+\varepsilon.$$

We claim that the measure $\mu_2=\nu_2*\eta_2$ has the required property. Indeed, for every measure $\mu_3\in P_\omega(X)$ we get

$$\mu_1*\mu_2*\mu_3(A\cup B)\leqslant \\ \leqslant \mu_1*\nu_2*(\eta_2*\mu_3)(A)+(\mu_1*\nu_2)*\eta_2*\mu_3(B) < \\ < \mathsf{sis}_{123}(A)+\varepsilon+\mathsf{sis}_{123}(B)+\varepsilon. \quad \square$$

The submeasure sis_{123} yields an upper bound on the extremal density is_{12}. The following fact was proved in [2].

Proposition 7.2. *For any G-space we get $\mathsf{is}_{12}\leqslant \widehat{\mathsf{is}}_{12}\leqslant \mathsf{sis}_{123}\leqslant d^*$. Moreover, if the acting group G is amenable, then $\mathsf{us}_{12}=\mathsf{is}_{12}=\widehat{\mathsf{is}}_{12}=\mathsf{sis}_{123}=d^*$.*

Proof. For convenience of the reader we present a proof of the inequality $\widehat{\mathsf{is}}_{12}\leqslant \mathsf{sis}_{123}$. It suffices to check that

$$\mathsf{is}_{12}(A\cup B) < \mathsf{is}_{12}(A)+\mathsf{sis}_{123}(B)+2\varepsilon$$

for every subsets $A,B\subset X$ and every $\varepsilon>0$. By the definition of $\mathsf{is}_{12}(A)$, there is a measure $\mu_1\in P_\omega(G)$ such that $\sup_{\mu_2\in P_\omega(X)}\mu_1*\mu_2(A) <$

$\mathsf{is}_{12}(A)+\varepsilon$. By the definition of $\mathsf{sis}_{123}(B)$, for the measure μ_1 there is a measure $\mu_2 \in P_\omega(G)$ such that $\sup_{\mu_3\in P_\omega(X)} \mu_1 * \mu_2 * \mu_3(B) \leqslant \mathsf{sis}_{123}(B)+\varepsilon$. Then for the measure $\nu_1 = \mu_1 * \mu_2 \in P_\omega(G)$ we get

$$\mathsf{is}_{12}(A\cup B) \leqslant \sup_{\nu_2\in P_\omega(X)} \nu_1 * \nu_2(A\cup B) \leqslant$$
$$\leqslant \sup_{\nu_2\in P_\omega(X)} (\mu_1*\mu_2*\nu_2(A)+\mu_1*\mu_2*\nu_2(B)) < \mathsf{is}_{12}(A)+\mathsf{sis}_{123}(B)+2\varepsilon. \quad \square$$

Proposition 7.3. *For any G-space X with finite acting group G we get*

$$\mathsf{is}_{12}(A) = \mathsf{sis}_{123}(A) = d^*(A) = \sup_{x\in X} \frac{|A\cap Gx|}{|Gx|}$$

for every set $A \subset X$.

Proof. Denote by $\lambda = \frac{1}{|G|}\sum_{x\in G}\delta_x$ the Haar measure on the group G and observe that for every $A \subset X$ we get

$$\mathsf{sis}_{123}(A) \leqslant \sup_{\mu_1\in P_\omega(G)} \sup_{x\in X} \mu_1 * \lambda * \delta_x(A) = \sup_{x\in X} \lambda * \delta_x(A) = \sup_{x\in X} \frac{|A\cap Gx|}{|Gx|}.$$

On the other hand,

$$\mathsf{is}_{12}(A) = \inf_{\mu_1\in P_\omega(X)} \sup_{\mu_2\in P_\omega(X)} \mu_1 * \mu_2(A) \geqslant$$
$$\geqslant \inf_{\mu_1\in P_\omega(G)} \sup_{x\in X} \mu_1 * \lambda * \delta_x = \sup_{x\in X} \lambda * \delta_x(A) = \sup_{x\in X} \frac{|A\cap Gx|}{|Gx|}. \quad \square$$

A subset A of a group G is called *conjugacy-invariant* if $xAx^{-1} = A$ for every $x \in G$.

Proposition 7.4. *Each conjugacy-invariant subset A of a group G has density $\mathsf{is}_{12}(A) = \mathsf{sis}_{123}(A)$.*

Proof. The inequality $\mathsf{is}_{12}(A) \leqslant \mathsf{sis}_{123}(A)$ follows from Proposition 7.2. To prove that $\mathsf{sis}_{123}(A) \leqslant \mathsf{is}_{12}(A)$, fix any $\varepsilon > 0$ and find a measure $\nu \in P_\omega(G)$ such that $\sup_{\eta\in P_\omega(G)} \nu*\eta(A) < \mathsf{is}_{12}(A)+\varepsilon$. Given any measure $\mu_1 = \sum_i \alpha_i\delta_{a_i} \in P_\omega(G)$ put $\mu_2 = \nu$ and observe that for every $\mu_3 \in P_\omega(G)$ we get

$$\mu_1 * \mu_2 * \mu_3(A) = \sum_i \alpha_i\, \delta_{a_i} * \nu * \mu_3(A) = \sum_i \alpha_i\, \nu * \mu_3(a_i^{-1}A) =$$
$$= \sum_i \alpha_i\, \nu * \mu_3(Aa_i^{-1}) = \sum_i \alpha_i\, \nu * \mu_3 * \delta_{a_i}(A) \leqslant$$
$$\leqslant \sum_i \alpha_i \sup_{\eta\in P_\omega(G)} \nu * \eta(A) < \sum_i \alpha_i(\mathsf{is}_{12}(A)+\varepsilon) = \mathsf{is}_{12}(A)+\varepsilon.$$

This implies that $\mathsf{sis}_{123}(A) \leqslant \mathsf{is}_{12}(A) + \varepsilon$ for every $\varepsilon > 0$ and hence $\mathsf{sis}_{123}(A) \leqslant \mathsf{is}_{12}(A)$. □

For partitions of groups into conjugacy-invariant sets Propositions 7.1, 7.4 and 5.2 imply the following partial answer to Problem 1.1.

Corollary 7.5. *For any partition $G = A_1 \cup \dots \cup A_n$ of a group G into conjugacy-invariant sets some cell A_i of the partition has* $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ *for any left-invariant Boolean ideal $\mathcal{I} \subset [\widehat{\mathsf{is}}_{12}{=}0]$.*

Applications of the submeasure sis_{123} will be based on the following theorem.

Theorem 7.6. *If a subset A of a G-space X has positive submeasure $\mathsf{sis}_{123}(A) > 0$, then for some finite set $E \subset G$ the set $\Delta_{\mathcal{I}}(A)^{\wr E} = \bigcup_{x\in E} x^{-1}\Delta_{\mathcal{I}}(A)x$ has*

$$\mathrm{cov}(\Delta_{\mathcal{I}}(A)^{\wr E}) \leqslant 1/\mathsf{sis}_{123}(A)$$

for any G-invariant ideal $\mathcal{I} \subset [\mathsf{sis}_{123}{=}0]$.

Proof. Fix $\varepsilon > 0$ so small that each integer number $n \leqslant \frac{1}{\mathsf{sis}_{123}(A)-2\varepsilon}$ does not exceed $\frac{1}{\mathsf{sis}_{123}(A)}$. By the definition of the submeasure $\mathsf{sis}_{123}(A)$, there is a measure $\mu_1 \in P_\omega(G)$ such that

$$\inf_{\mu_2\in P_\omega(G)} \sup_{\mu_3\in P_\omega(X)} \mu_1 * \mu_2 * \mu_3(A) > \mathsf{sis}_{123}(A) - \varepsilon.$$

Write μ_1 as a convex combination $\mu_1 = \sum_{i=1}^n \alpha_i\delta_{a_i}$ and put $E = \{a_1, \dots, a_n\}$.

Using Zorn's Lemma, choose a maximal subset $M \subset G$ such that for every $a \in E$ and distinct $x, y \in M$ we get $xa^{-1}A \cap ya^{-1}A \in \mathcal{I}$. By the maximality of M, for every point $g \in G$ there are points $x \in M$ and $a \in E$ such that $ga^{-1}A \cap xa^{-1}A \notin \mathcal{I}$ and hence $ax^{-1}ga^{-1} \in \Delta_{\mathcal{I}}(A)$ and $g \in xa^{-1}\Delta_{\mathcal{I}}(A)a \subset M \cdot \Delta_{\mathcal{I}}(A)^{\wr E}$. So, $G = M \cdot \Delta_{\mathcal{I}}(A)^{\wr E}$ and hence $\mathrm{cov}(\Delta_{\mathcal{I}}(A)^{\wr E}) \leqslant |M|$. To complete the proof, it remains to check that the set M has cardinality $|M| \leqslant 1/(\mathsf{sis}_{123}(A) - 2\varepsilon)$.

Assuming the opposite, we could find a finite subset $F \subset M$ of cardinality $|F| > 1/(\mathsf{sis}_{123}(A) - 2\varepsilon)$. The choice of the set $M \supset F$ guarantees that the set

$$B = \bigcup_{i=1}^n \{xa_i^{-1}A \cap ya^{-1}A : x, y \in F,\ x \neq y\}$$

belongs to the ideal $\mathcal{I}$ and hence $B \in \mathcal{I} \subset [\mathsf{sis}_{123}{=}0]$. Put $A' = A \setminus B$ and observe that for every $a \in E$ the indexed family $(ga^{-1}A')_{g\in F}$ is disjoint. Consider the uniformly distributed measure $\mu_F = \frac{1}{|F|}\sum_{g\in F}\delta_{g^{-1}}$ on G. Since $\mathsf{sis}_{123}(B) = 0$, for the measure $\nu_1 = \mu_1 * \mu_F \in P_\omega(G)$ there is a measure $\nu_2 = \sum_j \beta_j \delta_{b_j} \in P_\omega(G)$ such that $\sup_{\nu_3(X)} \nu_1 * \nu_2 * \nu_3(B) < \varepsilon$.

By the choice of the measure μ_1 for the measure $\mu_2 = \mu_F * \nu_2 \in P_\omega(G)$ there is a measure $\mu_3 \in P_\omega(X)$ such that $\mu_1 * \mu_2 * \mu_3(A) > \mathsf{sis}_{123}(A) - \varepsilon$. The measure μ_3 can be assumed to be a Dirac measure $\mu_3 = \delta_x$ at some point $x \in X$. Then $\mu_1 * \mu_2 * \delta_x(A) > \mathsf{sis}_{123}(A) - \varepsilon > 1/|F| + \varepsilon$.

On the other hand, for every i, j the disjointness of the families $(ga_i^{-1}A')_{g\in F}$ and $(b_j^{-1}ga_i^{-1}A')_{g\in F}$ implies that $\sum_{g\in F}\delta_x(b_j^{-1}ga_i^{-1}A') \leqslant 1$ and then

$$\begin{aligned}\mu_1 * \mu_2 * \delta_x(A') &= \mu_1 * \mu_F * \nu_2 * \delta_x(A') = \\ &= \sum_{i,j}\alpha_i\beta_j\sum_{g\in F}\frac{1}{|F|}\delta_{a_i g^{-1} b_j x}(A') = \\ &= \frac{1}{|F|}\sum_{i,j}\alpha_i\beta_j\sum_{g\in F}\delta_x(b_j^{-1}ga_i^{-1}A') \leqslant \frac{1}{|F|}.\end{aligned}$$

Then

$$\begin{aligned}\mathsf{sis}_{123}(A) - \varepsilon < \mu_1 * \mu_2 * \mu_3(A) &\leqslant \mu_1 * \mu_2 * \mu_3(A') + \mu_1 * \mu_2 * \mu_3(B) = \\ &< \mu_1 * \mu_2 * \delta_x(A') + \mu_1 * \mu_F * \nu_2 * \delta_x(B) < \frac{1}{|F|} + \varepsilon < \\ &< \mathsf{sis}_{123}(A) - \varepsilon\end{aligned}$$

which is a desired contradiction. □

The subadditivity of the density sis_{123} and Theorem 7.6 imply the following corollary.

Corollary 7.7. *For any partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $(X, \mathcal{I})$ with $\mathcal{I} \subset [\mathsf{sis}_{123}{=}0]$ some cell A_i of the partition has $\mathrm{cov}\big(\Delta_{\mathcal{I}}(A)^{\wr E}\big) \leqslant n$ for some finite set $E \subset G$.*

Combining Theorem 7.6 with Theorem 2.1, we get:

Corollary 7.8. *If a subset A of a G-space X has positive submeasure $\mathsf{sis}_{123}(A) > 0$, then*

$$\mathrm{cov}(\Delta_{\mathcal{I}}(A) \cdot \Delta_{\mathcal{I}}(A)) < \infty$$

for any G-invariant ideal $\mathcal{I} \subset [\mathsf{sis}_{123}{=}0]$ on X.

Proof. By Theorem 7.6, there is a finite set $E \subset G$ such that $G = \bigcup_{x,y\in E} x\Delta_{\mathcal{I}}(A)y$. By Theorem 2.1, there are points $x, y \in E$ such that the set $x\Delta_{\mathcal{I}}(A)y \cdot (x\Delta_{\mathcal{I}}(A)y)^{-1} = x\Delta_{\mathcal{I}}(A) \cdot \Delta_{\mathcal{I}}(A)x^{-1}$ has finite covering number in G. Then the set $\Delta_{\mathcal{I}}(A) \cdot \Delta_{\mathcal{I}}(A)$ has finite covering number too. □

8. Applications of the minimal and idempotent measures

In this section we survey partial answers to Problem 1.2 obtained by Banakh and Frączyk [3] with help of minimal measures on G-spaces and quasi-invariant idempotent measures on groups. For any measure $\mu \in P(X)$ on a G-space X let $\bar{\mu} : \mathcal{B}(X) \to [0, 1]$ be the submeasure on X defined by $\bar{\mu}(A) = \sup_{x\in G} \mu(xA)$.

Theorem 8.1. *Let $(X, \mathcal{I})$ be an ideal G-space and $\mu \in P_{\mathcal{I}}(X)$ be a minimal measure on X. If some subset $A \subset X$ has $\bar{\mu}(A) > 0$, then the $\mathcal{I}$-difference set $\Delta_{\mathcal{I}}(A)$ has*

1) $\mathrm{cov}\big(\Delta_{\mathcal{I}}(A) \cdot \Delta_{\mathcal{I}}(A)\big) \leqslant 1/\bar{\mu}(A)$;
2) $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A)) \leqslant 1/\bar{\mu}(A)$ *for some G-invariant ideal $\mathcal{J} \subset \{B \in \mathcal{B}(G) : \mathsf{is}_{12}(B^{-1}) = 0\}$ on G with $\Delta_{\mathcal{I}}(A) \notin \mathcal{J}$.*

This theorem implies the following three results:

Corollary 8.2. *If a subset $A \subset X$ of a G-space X has upper Banach density $d^*(A) > 0$, then*

$$\mathrm{cov}\big(\Delta_{\mathcal{I}}(A) \cdot \Delta_{\mathcal{I}}(A)\big) \leqslant \frac{1}{d^*(A)}$$

for any G-invariant Boolean ideal $\mathcal{I} \subset [d^* = 0]$ on X.

We recall that $d^* = \sup_{\mu\in P_{\min}(X)} \mu$.

Corollary 8.3. *For any partition $X = A_1 \cup \dots \cup A_n$ of an ideal G-space $(X, \mathcal{I})$ either*

- $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n = 1/d^*(A)$ *for all cells A_i or else*
- *some cell A_i of the partition has $\mathrm{cov}(\Delta_{\mathcal{I}}(A_i) \cdot \Delta_{\mathcal{I}}(A_i)) < n$ and $\mathrm{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$ for some G-invariant ideal $\mathcal{J} \subset \{B \in \mathcal{B}(G) : \mathsf{is}_{12}(B^{-1}) = 0\}$ with $\Delta_{\mathcal{I}}(A_i) \notin \mathcal{J}$.*

Corollary 8.4. *For any partition $X = A_1 \cup \cdots \cup A_n$ of an ideal G-space $(X, \mathcal{I})$ either* $\operatorname{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ *for all cells A_i or else* $\operatorname{cov}(\Delta_{\mathcal{I}}(A_i) \cdot \Delta_{\mathcal{I}}(A_i)) < n$ *for some cell A_i.*

For partitions of groups we can prove a more precise result using quasi-invariant idempotent measures. A measure $\mu \in P(G)$ on a group G will be called

- *idempotent* if $\mu * \mu$;
- *left quasi-invariant* (resp. *right quasi-invariant*) if there is a function $f : G \to [1, \infty)$ such that $f(x)\mu(xA) \leqslant \mu(A)$ (resp. $f(x)\mu(Ax) \leqslant \mu(A)$) for any $A \subset G$ and $x \in G$;
- *quasi-invariant* if there μ is left and right quasi-invariant.

A Boolean ideal $\mathcal{I} \subset \mathcal{B}(G)$ on a group G is called *invariant* if for every set $A \in \mathcal{I}$ and points $x, y \in G$ the shift $xAy \in \mathcal{I}$. The existence of quasi-invariant idempotent measures was established in [3]:

Proposition 8.5. *For any invariant ideal $\mathcal{I}$ on a countable group G there is a quasi-invariant idempotent minimal measure $\mu \in P(G)$ such that $\mu(A) = 0$ for all $A \in \mathcal{I}$.*

Using quasi-invariant idempotent measures Banakh and Frączyk [3] proved the following result.

Theorem 8.6. *Let $\mathcal{I}$ be a G-invariant ideal on a group G and $\mu \in P_{\mathcal{I}}(G)$ be a right quasi-invariant idempotent measure on G. If a subset $A \subset G$ has $\bar{\mu}(A) > 0$, then its $\mathcal{I}$-difference set $\Delta_{\mathcal{I}}(A)$ has*

1) $\operatorname{cov}(\Delta_{\mathcal{I}}(A) \cdot A) \leqslant 1/\bar{\mu}(A)$ *and*
2) $\operatorname{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A)) \leqslant 1/\bar{\mu}(A)$ *for some G-invariant Boolean ideal $\mathcal{J} \not\ni A^{-1}$ on G.*

This theorem implies the following partial answer to Problem 1.2.

Theorem 8.7. *Let G be a group and $\mathcal{I}$ be an invariant Boolean ideal on G which does not contain some countable subset of G. For any partition $G = A_1 \cup \cdots \cup A_n$ of G either*

- $\operatorname{cov}(\Delta_{\mathcal{I}}(A_i)) \leqslant n$ *for all cells A_i or else*
- *some cell A_i of the partition has* $\operatorname{cov}(\Delta_{\mathcal{I}}(A_i) \cdot A_i) < n$ *and* $\operatorname{cov}_{\mathcal{J}}(\Delta_{\mathcal{I}}(A_i)) < n$ *for some G-invariant ideal $\mathcal{J} \not\ni A_i^{-1}$ on G.*

Corollary 8.8. *For any partition $G = A_1 \cup \dots \cup A_n$ of a group G either* $\mathrm{cov}(A_iA_i) \leqslant n$ *for all cells A_i or else* $\mathrm{cov}(A_iA_i^{-1}A_i) < n$ *for some cell A_i of the partition.*

Taking into account that the ideal $\mathcal{J}$ appearing in Theorem 8.7 is G-invariant but not necessarily invariant, we can ask the following question.

Problem 8.9. *Is it true that for any partition $G = A_1 \cup \dots \cup A_n$ of a group G some cell A_i of the partition has* $\mathrm{cov}_{\mathcal{J}}(A_iA_i^{-1}) \leqslant n$ *for some invariant Boolean ideal $\mathcal{J}$ (for example, the ideal of small subsets) on G?*

Let us recall that a subset A of a G-space X is called *small* if $\mathrm{cov}(G \setminus FA) < \omega$ for any finite subset $F \subset G$.

Corollary 8.2 will help us to calculate the extremal densities of subgroups in groups. Below we assume that $1/\kappa = 0$ for any infinite cardinal κ.

Proposition 8.10. *If H is a subgroup of a group G, then* $\mathsf{is}_{12}(H) = \mathsf{sis}_{123}(H) = d^*(H) = 1/\mathrm{cov}(H)$.

Proof. Assume that the subgroup H has infinite index $\mathrm{cov}(H)$ in G. We claim that $d^*(H) = 0$. Assuming that $d^*(H) > 0$ and applying Corollary 8.2, we conclude that $\mathrm{cov}(H) = \mathrm{cov}(HH^{-1}HH^{-1})$ is finite and hence H has finite index in G. This contradiction shows that $\mathsf{is}_{12}(H) \leqslant \mathsf{sis}_{123}(H) \leqslant d^*(H) = 0 = 1/\mathrm{cov}(H)$.

Next, we assume that H has finite index in G. Then the normal subgroup $N = \bigcap_{x \in G} xHx^{-1}$ also has finite index in G. Consider the finite group G/N and the quotient homomorphism $q : G \to G/N$. It follows that the subgroup $q(H)$ has index $\mathrm{cov}(q(H)) = \mathrm{cov}(H)$ in the group G/N. By Proposition 7.3, $\mathsf{is}_{12}(q(H)) = \mathsf{sis}_{123}(q(H)) = d^*(H) = 1/\mathrm{cov}(q(H)) = 1/\mathrm{cov}(H)$. It can be shown that for each subset $A \subset G/N$ its preimage $q^{-1}(A) \subset G$ has densities $\mathsf{is}_{12}(q^{-1}(A)) = \mathsf{is}_{12}(A)$ and $d^*(q^{-1}(A)) = d^*(A)$. In particular, for the subgroup $H = q^{-1}(q(H))$ we get $\mathsf{is}_{12}(H) = \mathsf{is}_{12}(q(H)) = 1/\mathrm{cov}(H)$ and $d^*(H) = d^*(q(H)) = 1/\mathrm{cov}(H)$. □

Propositions 7.2 and 8.10 imply:

Proposition 8.11. *For any group G we get*

$$[d^* = 0] \subset [\mathsf{sis}_{123}{=}0] \subset [\widehat{\mathsf{is}}_{12}{=}0] \subset [\mathsf{is}_{12}{=}0].$$

If the group G is infinite, then the ideal $[d^* = 0]$ contains all sets of cardinality $< |G|$ in G.

Next, we show that for any subset A of a group G with positive upper Banach density $d^*(A)$ there is an integer number k dependent only on $d^*(A)$ such that the set $(A^{-1}A)^k$ is a subgroup of index $\leqslant 1/d^*(G)$ in G. Here for a subset $A \subset G$ its power $A^k \subset G$ is defined by induction: $A^1 = A$ and $A^{k+1} = \{xy : x \in A^k,\ y \in A\}$ for $k \in \mathbb{N}$. We shall need the following fact proved in Lemma 12.3 of [19].

Proposition 8.12. *If a symmetric subset $A = A^{-1}$ of a group G has finite covering number $k = \mathrm{cov}(A)$, then the set $A^{4^{k-1}}$ is a subgroup of G.*

Combining this proposition with Corollary 8.2, we get:

Corollary 8.13. *For any subset $A \subset G$ of positive upper Banach density $d^*(A)$ in a group G and the number $k = \mathrm{cov}(AA^{-1}AA^{-1}) \leqslant 1/d^*(A)$ the set $(AA^{-1})^{\frac{1}{2}4^k}$ is a subgroup of index $\leqslant k$.*

For partitions we can prove a bit more using Corollary 8.3.

Corollary 8.14. *For any partition $G = A_1 \cup \cdots \cup A_n$ of a group G there is a cell A_i of the partition such that the sets $(A_iA_i^{-1})^{4^{n-1}}$ is a subgroup of index $\leqslant n$ in G.*

Proof. By Corollary 8.3, some cell A_i of the partition has $\mathrm{cov}(A_iA_i^{-1}) \leqslant n = 1/d^*(A_i)$ or $\mathrm{cov}((A_iA_i^{-1})^4) \leqslant \mathrm{cov}((A_iA_i^{-1})^2) \leqslant 1/d^*(A_i) < n$. In the first case $H = (A_iA_i^{-1})^{4^{n-1}}$ is a subgroup of G. In the second case $((A_iA_i^{-1})^4)^{4^{n-2}} = (A_iA_i^{-1})^{4^{n-1}} = H$ also is a subgroup of G. In both cases $H = (A_iA_i)^{4^{n-1}}$ is a subgroup of finite index $\mathrm{cov}(H) = \frac{1}{d^*(H)} \leqslant \frac{1}{d^*(A_i)} \leqslant n$. $\square$

A subset A of a group G will be called a *shifted subgroup* if $A = xHy$ for some subgroup H and some points $x, y \in G$. Observe that for a shifted subgroup A the sets $AA^{-1} = xHx^{-1}$ and $A^{-1}A = y^{-1}Hy$ are subgroups conjugated to H and $A = AA^{-1}xy = xyA^{-1}A$.

Corollary 8.14 implies the following old result of Neumann [15].

Proposition 8.15 (Neumann)**.** *For any cover $G = A_1 \cup \cdots \cup A_n$ of a group G by shifted subgroups some shifted subgroup A_i has $\mathrm{cov}(A_i) \leqslant n$.*

Proof. By Corollary 8.14, for some shifted subgroup A_i the subgroup $A_i^{-1}A_i$ has index $\mathrm{cov}(A_i^{-1}A_i) \leqslant n$. Since $A_i = xA_i^{-1}A_i$ for some $x \in G$, we conclude that $\mathrm{cov}(A_i) = \mathrm{cov}(xA_i^{-1}A_i) = \mathrm{cov}(A_i^{-1}A_i) \leqslant n$. $\square$

9. Applications of the density is_{12} to IP*-sets

In this section we present an application of the density is_{12} to IP* sets. Following [11, 16.5], we call a subset A of a group G an IP*-*set* if for any sequence $(x_n)_{n\in\omega}$ in G there are indices $i_1 < i_2 < \cdots < i_n$ such that $x_{i_1}x_{i_2}\dots x_{i_k} \in A$. By Theorem 5.12 of [11], any IP*-set $A \subset G$ belongs to every idempotent of the compact right-topological semigroup βG, and hence has a rich combinatorial structure, see [11, §14]. The following theorem can be considered as a "non-amenable" generalization of Theorem 3.1 [9].

Proposition 9.1. *Let G be a group endowed with a left-invariant Boolean ideal $\mathcal{I} \subset [\widehat{\mathsf{is}}_{12}{=}0]$. If a set A of a group G has positive density $\mathsf{is}_{12}(A) > 0$, then for every sequence $(x_i)_{i=1}^n$ of length $n > 1/\mathsf{is}_{12}(A)$ in G there are two numbers $k < m \leqslant n$ such that $x_{k+1}\cdots x_m \in \Delta_{\mathcal{I}}(A)$. Consequently, $\Delta_{\mathcal{I}}(A)$ is an* IP*-*set.*

Proof. Consider the set $P = \{x_1\cdots x_k : 1 \leqslant k \leqslant n\}$. By Proposition 5.2, $\mathrm{pack}_{\mathcal{I}}(A) \leqslant 1/\mathsf{is}_{12}(A) < n$. Consequently there are two numbers $k < m \leqslant n$ such that $x_1\cdots x_k A \cap x_1\cdots x_m A \notin \mathcal{I}$. The left invariance of the Boolean ideal $\mathcal{I}$ implies that $A \cap x_{k+1}\dots x_m A \notin \mathcal{I}$ and hence $x_{k+1}\dots x_m \in \Delta_{\mathcal{I}}(A)$. □

By analogy, we can use Proposition 6.3 to prove:

Proposition 9.2. *Let G be a group endowed with a left-invariant Boolean ideal $\mathcal{I} \subset [\widehat{\mathsf{us}}_{12}{=}0]$. If a set A of a group G has positive density $\mathsf{us}_{12}(A) > 0$, then for every sequence $(x_i)_{i=1}^n$ of length $n > 1/\mathsf{us}_{12}(A)$ in G there are two numbers $k < m \leqslant n$ such that $x_{k+1}\cdots x_m \in \Delta_{\mathcal{I}}(A)$. Consequently, $\Delta_{\mathcal{I}}(A)$ is an* IP*-*set.*

Since any conjugacy-invariant set $A = A^{\wr G} = \bigcup_{x\in G} x^{-1}Ax$ has submeasure $\mathsf{sis}_{123}(A) = \mathsf{is}_{12}(A)$, Proposition 9.1 implies:

Corollary 9.3. *Let G be a group endowed with a left-invariant Boolean ideal $\mathcal{I} \subset [\widehat{\mathsf{is}}_{12}{=}0]$. If a set A of a group G has positive density $\mathsf{sis}_{123}(A) > 0$, then for every sequence $(x_i)_{i=1}^n$ of length $n > 1/\mathsf{sis}_{123}(A)$ in G there are two numbers $k < m \leqslant n$ such that $x_{k+1}\cdots x_m \in \Delta_{\mathcal{I}}(A^{\wr G})$. Consequently, $\Delta_{\mathcal{I}}(A^{\wr G})$ is an* IP*-*set.*

The subadditivity of the submeasure sis_{123} and Corollary 9.3 implies:

Corollary 9.4. *For any partition $G = A_1 \cup \cdots \cup A_n$ of a group G endowed with a non-trivial left-invariant ideal $\mathcal{I} \subset [\widehat{\mathsf{is}}_{12}{=}0]$, there is a cell A_i of the partition such that $\Delta_{\mathcal{I}}(A^{\wr G})$ is an* IP**-set.*

Remark 9.5. By Theorem 3.8 of [9], the free group with two generators F_2 can be covered by two sets A, B such that neither AA^{-1} not BB^{-1} is an IP*-set. This example shows that the free group F_2 admits no subadditive density $\mu : \mathcal{B}(G) \to [0,1]$ such that AA^{-1} is an IP*-set for any set $A \subset G$ of positive density $\mu(A) > 0$.

10. Some open problems with comments

In this section we collect some problems related to Problems 1.1–1.3.

Motivated by Theorem 2.1, in [10, Question F], J. Erde asked whether, given a partition $\mathcal{B}$ of an infinite group G with $|\mathcal{B}| < |G|$, there is $A \in \mathcal{B}$ such that $\mathrm{cov}(AA^{-1})$ is finite. The following extremely negatively answer to this question was obtained in [20]: Any infinite group G admits a countable partition $G = \bigcup_{n\in\omega} A_n$ such that $\mathrm{cov}(A_nA_n^{-1}) \geqslant \mathrm{cf}(|G|)$ for each n.

Problem 10.1. *Does each infinite group G admit a countable partition $G = \bigcup_{n\in\omega} A_n$ such that $\mathrm{cov}(A_nA_n^{-1}) = |G|$ for all $n \in \omega$?*

The answer to this problem is affirmative if the group G is residually finite (in particular, Abelian or free), see [20]. A stronger version of Problem 10.1 was considered in [21].

Problem 10.2. *Does every infinite group G admit a countable partition $G = \bigcup_{n<\omega} A_n$ such that $\mathrm{cov}(A_n) = |G|$ for each $n \in \omega$?*

A subset $A \subset X$ of a G-space X is called *m-thick* for a natural number m if for each set $F \subset G$ of cardinality $|F| \leqslant m$ there is a point $x \in X$ such that $Fx \subset A$. A subset $A \subset G$ is *thick* if it is m-thick for every $m \in \mathbb{N}$. Observe that a set $A \subset X$ is 2-thick if and only if $\Delta_{\mathcal{I}}(A) = G$ for the smallest ideal $\mathcal{I} = \{\varnothing\}$. The following proposition was proved in [7, 1.3].

Proposition 10.3. *For any partition $X = A_1 \cup \cdots \cup A_n$ of a G-space X and any $m \in \mathbb{N}$ there are a cell A_i of the partition and a subset $F \subset G$ of cardinality $|F| \leqslant m^{n-1}$ such that the set FA_i is m-thick.*

Corollary 10.4. *For any partition $X = A_1 \cup \dots \cup A_n$ of a G-space X there are a cell A_i of the partition and a subset $F \subset G$ of cardinality $|F| \leqslant 2^{n-1}$ such that $\Delta_{\mathcal{I}}(FA) = G$ for the smallest Boolean ideal $\mathcal{I} = \{\varnothing\}$ on X.*

Corollary 10.5. *For any partition $G = A_1 \cup \dots \cup A_n$ of a group G there is a cell A_i of the partition such that $G = FA_iA_i^{-1}F^{-1}$ for some set $F \subset G$ of cardinality $|F| \leqslant 2^{n-1}$.*

Problem 10.6. *Is it true that for each partition $G = A_1 \cup \dots \cup A_n$ of a group G there is a cell A_i of the partition such that $G = FA_iA_i^{-1}F^{-1}$ for some set $F \subset G$ of cardinality $|F| \leqslant n$.*

The following problem is stronger than Problem 10.6 but weaker than Problem 1.1.

Problem 10.7. *Is it true that for any finite partition $G = A_1 \cup \dots \cup A_n$ of a group G there exist a cell A_i of the partition and a subset $F \subset G \times G$ of cardinality $|F| \leqslant n$ such that $G = \bigcup_{(x,y)\in F} xA_iA_i^{-1}y$?*

Another weaker version of Problem 1.1 also remains open:

Problem 10.8. *Let $G = A_1 \cup \dots \cup A_n$ be a partition of a group G such that $A_iA_j = A_jA_i$ for all indices $1 \leqslant i, j \leqslant n$. Is there a cell A_i of the partition with $\mathrm{cov}(A_iA_i^{-1}) \leqslant n$?*

Proposition 10.3 contrasts with the following theorem proved in [7].

Theorem 10.9. *For every $k \in \mathbb{N}$, any countable infinite group G admits a partition $G = A \cup B$ such that for every k-element subset $K \subset G$ the sets KA and KB are not thick.*

This theorem was proved with help of syndetic submeasures. A density $\mu : \mathcal{B}(G) \to [0, 1]$ on a group G is called *syndetic* if for each subset $A \subset G$ with $\mu(A) < 1$ and each $\varepsilon > \frac{1}{|G|}$ there is a subset $B \subset G \setminus A$ such that $\mu(B) < \varepsilon$ and $\mathrm{cov}(B) < \infty$. It can be shown that the density is_{12} is syndetic. According to Theorem 5.1 of [7] (deduced from [25]), each countable group admits a left-invariant syndetic submeasure. This fact was crucial in the proof of Theorem 10.9.

Problem 10.10. *Does each group G admit a left-invariant syndetic submeasure? Is the submeasure sis_{123} syndetic on each group G? Is the upper Banach density d^* syndetic on each group G?*

Also we do not know if amenability of groups can be characterized via extremal densities or packing indices.

Problem 10.11. *Is a group G amenable if for each partition $G = A_1 \cup \dots \cup A_n$ there is a cell A_i of the partition satisfying one of the conditions:* (a) $\mathsf{is}_{12}(A_i) \geqslant \frac{1}{n}$, (b) $\mathrm{pack}(A_i) \leqslant n$, (c) $\mathrm{cov}(A_i A_i^{-1}) \leqslant n$, (d) $\mathsf{is}_{12}(A_i) > 0$, (e) $\mathrm{pack}(A_i) < \omega$*?*

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Contact information

T. Banakh — Ivan Franko National University of Lviv, Ukraine and Jan Kochanowski University in Kielce, Poland
E-Mail: t.o.banakh@gmail.com

I. Protasov, S. Slobodianiuk — Taras Shevchenko National University, Kyiv, Ukraine
E-Mail: i.v.protasov@gmail.com, slobodianiuk@yandex.ru

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