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Characterizing semigroups with commutative superextensions

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Abstract. We characterize semigroups X whose semigroups of filters $\varphi(X)$, maximal linked systems $\lambda(X)$, linked upfamilies $N_2(X)$, and upfamilies $\upsilon(X)$ are commutative.

1. Introduction

In this paper we investigate the algebraic structure of various extensions of a semigroup X and detect semigroups whose extensions $\varphi(X)$, $\lambda(X)$, $N_2(X)$, $\upsilon(X)$ and their subsemigroups $\varphi^\bullet(X)$, $\lambda^\bullet(X)$, $N_2^\bullet(X)$, $\upsilon^\bullet(X)$ are commutative.

The thorough study of various extensions of semigroups was started in [9] and continued in [1]–[6]. The largest among these extensions is the semigroup $\upsilon(X)$ of all upfamilies on X.

A family $\mathcal{F}$ of subsets of a set X is called an *upfamily* if $\varnothing \notin \mathcal{F} \neq \varnothing$ and for each set $F \in \mathcal{F}$ any subset $E \supset F$ of X belongs to $\mathcal{F}$. Each family $\mathcal{F}$ of non-empty subsets of X generates the upfamily

$$\langle F : F \in \mathcal{F}\rangle = \{E \subset X : \exists F \in \mathcal{F}\ \ F \subset E\}.$$

The space of all upfamilies on X is denoted by $\upsilon(X)$. It is a closed subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$ endowed with the compact

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Hausdorff topology of the Tychonoff product $\{0,1\}^{\mathcal{P}(X)}$. Identifying each point $x \in X$ with the *principal ultrafilter* $\langle x \rangle = \{A \subset X : x \in A\}$, we can identify X with a subspace of $\upsilon(X)$. Because of that we call $\upsilon(X)$ an extension of X. For an upfamily $\mathcal{F} \in \upsilon(X)$ by

$$\mathcal{F}^{\perp} = \{E \subset X : \forall F \in \mathcal{F} \;\; E \cap F \neq \varnothing\}$$

we denote the *transversal* of $\mathcal{F}$. By [8], $(\mathcal{F}^{\perp})^{\perp} = \mathcal{F}$, so

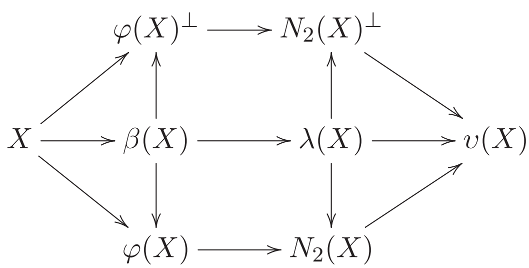
$$\perp\colon \upsilon(X) \to \upsilon(X), \;\; \perp\colon \mathcal{F} \mapsto \mathcal{F}^{\perp},$$

is an involution on $\upsilon(X)$. For a subset $S \subset \upsilon(X)$ we put $S^{\perp} = \{\mathcal{F}^{\perp} : \mathcal{F} \in S\} \subset \upsilon(X)$.

The compact Hausdorff space $\upsilon(X)$ contains many other important extensions of X as closed subspaces. In particular, it contains the spaces $N_2(X)$ of linked upfamilies, $\lambda(X)$ of maximal linked upfamilies, $\varphi(X)$ of filters, and $\beta(X)$ of ultrafilters on X; see [8]. Let us recall that an upfamily $\mathcal{F} \in \upsilon(X)$ is called

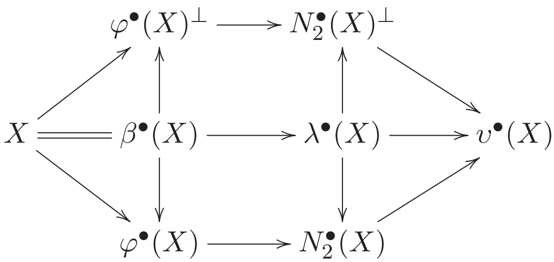
- *linked* if $A \cap B \neq \varnothing$ for any sets $A, B \in \mathcal{F}$;
- *maximal linked* if $\mathcal{F} = \mathcal{F}'$ for any linked upfamily $\mathcal{F}' \in \upsilon(X)$ that contains $\mathcal{F}$;
- a *filter* if $A \cap B \in \mathcal{F}$ for any $A, B \in \mathcal{F}$;
- an *ultrafilter* if $\mathcal{F}$ is a filter and $\mathcal{F} = \mathcal{F}'$ for any filter $\mathcal{F}' \in \upsilon(X)$ that contains $\mathcal{F}$.

The family $\beta(X)$ of all ultrafilters on X is called the *Stone-Čech extension* and the family $\lambda(X)$ of all maximal linked upfamilies is called the *superextension* of X, see [14] and [17]. It can be shown that $\lambda(X) = \{\mathcal{F} \in \upsilon(X) : \mathcal{F}^{\perp} = \mathcal{F}\}$, so $\lambda(X)^{\perp} = \lambda(X)$ and $\beta(X)^{\perp} = \beta(X)$. The arrows in the following diagram denote the identity inclusions between various extensions of a set X.



We say that an upfamily $\mathcal{U} \in \upsilon(X)$ *is finitely supported* if $\mathcal{U} = \langle F_1, \dots, F_n \rangle$ for some non-empty finite subsets $F_1, \dots, F_n \subset X$. By

$\upsilon^\bullet(X)$ we denote the subspace of $\upsilon(X)$ consisting of finitely supported upfamilies on X. Let $\varphi^\bullet(X) = \varphi(X) \cap \upsilon^\bullet(X)$, $\lambda^\bullet(X)=\lambda(X) \cap \upsilon^\bullet(X)$, $N_2^\bullet(X)=N_2(X) \cap \upsilon^\bullet(X)$. Since each finitely supported ultrafilter is principal, the set $\beta^\bullet(X) = \beta(X) \cap \upsilon^\bullet(X)$ coincides with X (identified with the subspace of all principal ultrafilters in $\upsilon(X)$). The embedding relations between these spaces of finitely supported upfamilies are indicated in the following diagram:



Any map $f : X \to Y$ induces a continuous map

$$\upsilon f : \upsilon(X) \to \upsilon(Y), \quad \upsilon f : \mathcal{F} \mapsto \{A \subset Y : f^{-1}(A) \in \mathcal{F}\}.$$

By [8], $\upsilon f(\mathcal{F}^\perp) = \big(\upsilon f(\mathcal{F})\big)^\perp$ and $\upsilon f\big(\beta(X)\big) \subset \beta(Y)$, $\upsilon f\big(\lambda(X)\big) \subset \lambda(Y)$, $\upsilon f\big(\varphi(X)\big) \subset \varphi(Y)$, $\upsilon f\big(N_2(X)\big) \subset N_2(Y)$, and $\upsilon f\big(\upsilon^\bullet(X)\big) \subset \upsilon^\bullet(X)$. If the map f is injective, then υf is a topological embedding, which allows us to identify the extensions $\beta(X)$, $\lambda(X)$, $\varphi(X)$, $N_2(X)$ and $\upsilon(X)$ with corresponding closed subspaces in $\beta(Y)$, $\lambda(Y)$, $\varphi(Y)$, $N_2(Y)$ and $\upsilon(Y)$, respectively.

If $* : X \times X \to X$, $* : (x, y) \mapsto xy$, is a binary operation on X, then there is an obvious way of extending this operation onto the space $\upsilon(X)$. Just put

$$\mathcal{A} \circledast \mathcal{B} = \langle A * B : A \in \mathcal{A},\ B \in \mathcal{B}\rangle$$

where $A * B = \{ab : a \in A,\ b \in B\}$ is the pointwise product of sets $A, B \subset X$. The upfamily $\mathcal{A} \circledast \mathcal{B}$ will be called the *pointwise product* of the upfamilies $\mathcal{A}, \mathcal{B}$. It is clear that this extension $\circledast : \upsilon(X) \times \upsilon(X) \to \upsilon(X)$ of the operation $* : X \times X \to X$ is associative and commutative if so is the operation $*$. So, for an associative binary operation $*$ on X, its extension $\upsilon(X)$ endowed with the operation $\circledast$ of pointwise product becomes a semigroup, containing the subspaces $\varphi(X)$, $\varphi^\bullet(X)$, $N_2(X)$, and $N_2^\bullet(X)$ as subsemigroups. However, the subspaces $\beta(X)$ and $\lambda(X)$ are not subsemigroups in $(\upsilon(X), \circledast)$. To make $\beta(X)$ a semigroup extension of X, Ellis [7] suggested a less obvious extension of an (associative) binary operation $*: X \times X \to X$ to an (associative) binary operation on $\beta(X)$ letting

$$\mathcal{A} * \mathcal{B} = \Big\langle \bigcup_{a \in A} a * B_a : A \in \mathcal{A},\ \{B_a\}_{a \in A} \subset \mathcal{B} \Big\rangle \tag{1}$$

for ultrafilters $\mathcal{A}, \mathcal{B} \in \beta(X)$ (see [15, §4]). It is clear that $\mathcal{A} \circledast \mathcal{B} \subset \mathcal{A} * \mathcal{B}$ but in general $\mathcal{A} \circledast \mathcal{B} \neq \mathcal{A} * \mathcal{B}$.

In [9] it was observed that the formula (1) determines an extension of the operation $*$ to an (associative) binary operation $*: \upsilon(X) \times \upsilon(X) \to \upsilon(X)$ on the extension $\upsilon(X)$ of X. So, for each semigroup $(X, *)$, its extension $\upsilon(X)$ endowed with the extended operation $*$ is a semigroup, containing the subspaces $\beta(X)$, $\varphi(X)$, $\lambda(X)$, $N_2(X)$ as closed subsemigroups. Moreover, since $\upsilon^\bullet(X)$ is a subsemigroup of $\upsilon(X)$, the subspaces $X = \beta^\bullet(X)$, $\varphi^\bullet(X)$, $\lambda^\bullet(X)$, $N_2^\bullet(X)$ also are subsemigroups of $\upsilon(X)$. Algebraic and topological properties of these semigroups have been studied in [9], [1]–[6]. In particular, in [6] and [3] we studied properties of extensions of groups, while [4] and [5] were devoted to extensions of semilattices and inverse semigroups, respectively. In contrast to the operation $\circledast$, the extended operation $*$ on the semigroup $\upsilon(X)$ and its subsemigroups rarely is commutative. For example, by [6] a group X has commutative superextension $\lambda(X)$ if and only if X is a group of cardinality $|X| \leqslant 4$. According to [4], a semilattice X has commutative extension $\upsilon(X)$ if and only if X is finite and linearly ordered.

Let X be a semigroup. A subsemigroup $S \subset \upsilon(X)$ is defined to be *supercommutative* if $\mathcal{A} * \mathcal{B} = \mathcal{A} \circledast \mathcal{B} = \mathcal{B} \circledast \mathcal{A} = \mathcal{B} * \mathcal{A}$ for any upfamilies $\mathcal{A}, \mathcal{B} \in S$. It is clear that each supercommutative subsemigroup $S \subset \upsilon(X)$ is commutative. The converse is not true as we shall see in Section 10.

In this paper we study the commutativity and supercommutativity of the semigroups $\upsilon(X)$, $N_2(X)$, $\lambda(X)$, $\varphi(X)$, $\beta(X)$, $\upsilon^\bullet(X)$, $N_2^\bullet(X)$, $\lambda^\bullet(X)$, $\varphi^\bullet(X)$ and characterize semigroups X whose various extensions are commutative or supercommutative. In the preliminary Sections 2, 3 we shall analyze the structure of periodic commutative semigroups and projective extensions of semigroup, Section 5 is devoted to square-linear semigroups which will play a crucial role in Sections 7 and 8 devoted to the study of commutativity and supercommutativity of the semigroups $\upsilon(X)$, $\upsilon^\bullet(X)$, $N_2(X)$ and $N_2^\bullet(X)$. In Section 6 we characterize semigroups X with (super)commutative extensions $\beta(X)$, $\varphi(X)$, $\varphi^\bullet(X)$, and in Section 9 we detect semigroups with commutative extensions $\lambda(X)$ and $\lambda^\bullet(X)$. In Section 10 we study the structure of semigroups X whose superextension $\lambda(X)$ is supercommutative.

2. The structure of periodic commutative semigroups

In this section we recall some known information on the structure of periodic commutative semigroups. A semigroup S is called *periodic* if each element $x \in S$ generates a finite semigroup $\{x^k\}_{k\in\mathbb{N}}$. A semigroup S generated by a single element x is called *monogenic*. If a monogenic semigroup is infinite, then it is isomorphic to the additive semigroup $\mathbb{N}$ of positive integers. A finite monogenic semigroup $S = \{x^k\}_{k\in\mathbb{N}}$ also has simple structure (cf. [13, §1.2]): there are positive integer numbers $n < m$ such that

- $S = \{x, \dots, x^{m-1}\}$, $m = |S| + 1$ and $x^n = x^m$;
- $C_x = \{x^n, \dots, x^{m-1}\}$ is a cyclic subgroup of S;
- the cyclic subgroup C_x coincides with the minimal ideal of S;
- the neutral element e_x of the group C_x is a unique idempotent of S and the cyclic group C_x is generated by the element xe_x.

Such monogenic semigroups will be denoted by $\langle x \mid x^n = x^m\rangle$.

For a semigroup S let

$$E(S) = \{e \in S : ee = e\}$$

be the *idempotent part* of S. For each idempotent $e \in E(S)$ let

$$H_e = \{x \in S : \exists y \in S \;\; xyx = x,\; yxy = y,\; xy = e = yx\}$$

be the maximal subgroup of S containing the idempotent e. The union

$$C(S) = \bigcup_{e\in E(S)} H_e$$

of all (maximal) subgroups of S is called the *Clifford part* of S. The Clifford part $C(S)$ is contained in the *regular part*

$$R(S) = \{x \in S : x \in xSx\}$$

of S. If a semigroup S is commutative, then $R(S) = C(S)$ and the subsets $E(S)$ and $R(S) = C(S)$ are subsemigroups of S.

If a semigroup S is periodic, then for each element $x \in S$ the monogenic semigroup $\{x^k\}_{k\in\mathbb{N}}$ contains a unique idempotent e_x. So, we can consider the map

$$e_* : S \to E(S), \quad e_* : x \mapsto e_x,$$

which projects the semigroup S onto its idempotent part $E(S)$. The map

$$c_* : S \to C(S), \quad c_* : x \mapsto e_x \cdot x,$$

projects the semigroup S onto its Clifford part. If a periodic semigroup S is commutative, then the projections $e_* : S \to E(S)$ and $c_* : S \to C(S)$ are semigroup homomorphisms. In this case, for every idempotent $e \in E(S)$, $S_e = \{x \in S : e_x = e\}$ is a subsemigroup of S with a unique idempotent e. So, the semigroup S decomposes into the disjoint union $S = \bigcup_{e \in E(S)} S_e$ of semigroups S_e parametrized by idempotents $e \in E(S)$.

3. Projection extensions of semigroups

A semigroup X is called a *projection extension* of a subsemigroup $Z \subset X$ if there is a function $\pi : X \to Z$ (called the *projection of X onto Z*) such that

- $\pi(z) = z$ for each $z \in Z$;
- $x \cdot y = \pi(x) \cdot \pi(y) \in Z$ for all $x, y \in X$.

It follows from $\pi(xy) = xy = \pi(x) \cdot \pi(y)$ that the projection $\pi : X \to Z$ necessarily is a homomorphism of X onto its subsemigroup Z.

If a map $\pi : X \to Z$ of semigroups X and Z is a homomorphism, then by [9] the map $\upsilon\pi : \upsilon(X) \to \upsilon(Z)$ is a homomorphism too. So, we have the following statement.

Proposition 3.1. *If a semigroup X is a projection extension of a subsemigroup $Z \subset X$, then the projection $\pi : X \to Z$ induces a homomorphism $\upsilon\pi : \upsilon(X) \to \upsilon(Z)$ witnessing that the semigroup $\upsilon(X)$ is a projection extension of the subsemigroup $\upsilon(Z)$.*

Corollary 3.2. *Assume that a semigroup X is a projection extension of a subsemigroup $Z \subset X$, $\pi : X \to Z$ is a projection of X onto Z. Then each subsemigroup $S \subset \upsilon(X)$ with $\upsilon\pi(S) \subset S$ is a projection extension of the subsemigroup $\upsilon\pi(S) = S \cap \upsilon(Z)$. Consequently, the semigroup S is (super)commutative if and only if so is its subsemigroup $S \cap \upsilon(Z)$.*

Corollary 3.3. *Assume that a semigroup X is a projection extension of a subsemigroup $Z \subset X$, and $\varepsilon \in \{\upsilon, \upsilon^\bullet, N_2, N_2^\bullet, \varphi, \varphi^\bullet, \lambda, \lambda^\bullet, \beta, \beta^\bullet\}$. The extension $\varepsilon(X)$ of X is (super)commutative if and only if the extension $\varepsilon(Z)$ of the semigroup Z is (super)commutative.*

4. Semicomplete digraphs

In this section we recall some information on digraphs. In the next section this information will be used for describing the structure of square-linear semigroups.

By an *directed graph* (briefly, a *digraph*) we shall understand a pair (X,Δ) consisting of a set X and a subset $\Delta\subset X\times X$. Elements $x\in X$ are called *vertices* and ordered pairs $(x,y)\in\Delta$ called *edges* of the digraph (X,Δ). An edge $(x,y)\in\Delta$ is called *pure* if $(y,x)\notin\Delta$. A digraph (X,Δ) is called *complete* if $\Delta=X\times X$ and *semicomplete* if $\Delta\cup\Delta^{-1}=X\times X$, where $\Delta^{-1}=\{(y,x):(x,y)\in\Delta\}$.

A sequence $x_0,\dots,x_n$ of vertices of a digraph (X,Δ) is called a (*pure*) *cycle of length* n if $x_0=x_n$ and for every $i<n$ the pair (x_i,x_{i+1}) is a (pure) edge of the digraph (X,Δ). A cycle $x_0,x_1,\dots,x_n$ in a digraph (X,D) is called *bipartite* if the number n is even and for each numbers $i,j\in\{1,\dots,n\}$ with odd difference $i-j$ we get $(x_i,x_j)\notin\Delta\cap\Delta^{-1}$. Bipartite cycles can be equivalently defined as cycles $x_0,y_1,x_1,y_2,\dots,y_n,x_n$ such that $(x_i,y_j)\notin\Delta\cap\Delta^{-1}$ for any $1\leqslant i,j\leqslant n$.

It is easy to see that a cycle of length 4 is bipartite if and only if it is pure.

Lemma 4.1. *A semicomplete digraph* (X,Δ) *contains a pure cycle of length 4 if and only if it contains a bipartite cycle.*

Proof. Let $x_0,x_1,\dots,x_n$ be a bipartite cycle in the digraph of the smallest possible length n. The length n is even and cannot be equal to 2 as otherwise $(x_1,x_2)=(x_1,x_0)\in\Delta\cap\Delta^{-1}$. So, $n\geqslant 4$. We claim that $n=4$. Assume conversely that $n>4$ and consider the pair (x_0,x_3). Since the cycle is bipartite and the digraph (X,Δ) is semicomplete, either (x_0,x_3) or (x_3,x_0) is a pure edge of the digraph. If $(x_3,x_0)\in\Delta$, then x_0,x_1,x_2,x_3,x_0 is a bipartite (and pure) cycle of length 4 in (X,Δ). If $(x_0,x_3)\in\Delta$, then $x_0,x_3,x_5,\dots,x_n$ is a bipartite cycle of length $n-2\geqslant 4$ in (X,Δ), which contradicts the minimality of n. □

5. Square-linear semigroups

A semigroup S is called *linear* if $xy\in\{x,y\}$ for any elements $x,y\in S$. It follows that each element x of a linear semigroup is an idempotent. So, linear semigroups are *bands*, i.e., semigroups of idempotents. Commutative bands are called *semilattices*. So, each linear commutative semigroup is a semilattice. Each semilattice E is endowed with a partial order $\leqslant$ defined by $x\leqslant y$ iff $xy=x$.

A semigroup S is called *square-linear* if $xy \in \{x^2, y^2\}$ for all elements $x, y \in S$.

Proposition 5.1. *Let S be a square-linear commutative semigroup and $x, y, z \in S$ be any elements. Then*

1) *S is periodic and $x^3 = x^4 = e_x \in E(S)$;*
2) *the idempotent part $E(S)$ of S is a linear semilattice;*
3) *the Clifford part $C(S)$ of S coincides with $E(S)$;*
4) *$xy = e_x e_y$ if $x^2, y^2 \in E(S)$;*
5) *$xyz = e_x e_y e_z$;*
6) *if $x^2 \notin E(S)$, then e_x is the largest element of the semilattice $E(S)$.*

Proof. 1. It follows from $x^3 = x \cdot x^2 \in \{x^2, x^4\}$ that $x^3 = x^4 = e_x$ and hence $x^3 = x^n$ for all $n \geqslant 3$. So, the monogenic semigroup $\{x^n\}_{n\in\mathbb{N}} = \{x, x^2, x^3\}$ is finite and hence S is periodic.

2. If x, y are idempotents, then $xy \in \{x^2, y^2\} = \{x, y\}$ implies that the semilattice $E(S)$ is linear.

3. The identity $x^3 = x^4$ implies that each subgroup of S is trivial and hence $C(S) = E(S)$.

4. If $x^2, y^2 \in E(S)$, then $x^4 = x^2$ and hence $x^2 = x^4 = x^3 = e_x$. Then $xy \in \{x^2, y^2\} = \{e_x, e_y\}$ implies that xy is an idempotent and hence $xy = e_{xy} = e_x \cdot e_y$ as the projection $e_* : S \to E(S)$ is a homomorphism.

5. First we show that $xyz \in E(S)$. Since S is square-linear, we get $xy \in \{x^2, y^2\}$. We lose no generality assuming that $xy = x^2$. Now consider the product $xz \in \{x^2, z^2\}$. If $xz = x^2$, then $xyz = x^2z = x(xz) = x^3 \in E(S)$. If $xz = z^2$, then $xyz = x^2z = xxz = xz^2 = xzz = z^2z = z^3 \in E(S)$. Since the projection $e_* : S \to E(S)$ is a homomorphism, we conclude that $xyz = e_{xyz} = e_x e_y e_z$.

6. Assume that $x^2 \notin E(S)$ but the idempotent e_x is not maximal in the linear semilattice $E(S)$. Then there is an idempotent $e \in E(S)$ such that $ee_x = e_x \neq e$. It follows that $xe \in \{x^2, e^2\}$. We claim that $xe \neq e$. Assuming that $xe = e$, we conclude that $xee = ee = e$. On the other hand, the preceding item guarantees that $xee = e_x e_e e_e = e_x ee = e_x \neq e$. So, $xe = x^2 \notin E(S)$, which contradicts $xe = xee \in E(S)$. □

Each square-linear semigroup S endowed with the set of directed edges

$$\Delta = \{(x, y) \in S \times S : xy = x^2\}$$

becomes a semicomplete digraph. In fact, the algebraic structure of a square-linear semigroup S is completely determined by its digraph structure Δ and the duplication map $S \to S$, $x \mapsto x^2$. The semigroup operation $S \times S \to S$, $(x,y) \mapsto xy$, can be recovered from Δ and the duplication map by the formula

$$xy = \begin{cases} x^2 & \text{if } (x,y) \in \Delta, \\ y^2 & \text{if } (y,x) \in \Delta. \end{cases}$$

6. Commutativity of the semigroups $\beta(X)$, $\varphi(X)$ and $\varphi^\bullet(X)$

The following characterization was proved in Theorem 4.27 of [12].

Theorem 6.1. *For a commutative semigroup X the following conditions are equivalent:*

1) *the semigroup $\beta(X)$ is commutative;*
2) $\{a_k b_n : k,n \in \omega,\ k < n\} \cap \{b_k a_n : k,n \in \omega,\ k < n\} \neq \varnothing$ *for any sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ in X.*

Corollary 6.2. *If the semigroup $\beta(X)$ is commutative, then*

1) *for each square-linear subsemigroup $S \subset X$ the set $\{x^2 : x \in S\}$ is finite;*
2) *each subsemigroup of X contains a finite ideal;*
3) *each monogenic subsemigroup of X is finite.*

Proof. Assume that the semigroup $\beta(X)$ is commutative.

1. Assume that X contains a square-linear subsemigroup $S \subset X$ with infinite subset $\{x^2 : x \in S\}$. Then there is a sequence $\{x_n\}_{n\in\omega}$ in S such that $x_n^2 \neq x_m^2$ for any $n \neq m$. Define a 2-coloring $\chi : [\omega]^2 \to \{0,1\}$ of the set $[\omega]^2 = \{(n,m) \in \omega^2 : n < m\}$ letting

$$\chi(n,m) = \begin{cases} 0 & \text{if } x_n x_m = x_n^2 \\ 1 & \text{if } x_n x_m = x_m^2. \end{cases}$$

By Ramsey's Theorem [16] (see also [10, Theorem 5]), there is an infinite subset $\Omega \subset \omega$ and a color $c \in \{0,1\}$ such that $\chi(n,m) = c$ for any pair $(n,m) \in [\omega]^2 \cap \Omega^2$. Let $\Omega = \{k_n : n \in \omega\}$ be the increasing enumeration of the set Ω. Then for the sequences $a_n = x_{k_{2n}}$ and $b_n = x_{k_{2n+1}}$, $n \in \omega$, we get

$$\{a_k b_n\}_{k<n} \cap \{b_k a_n\}_{k<n} \subset \{a_k^2\}_{k\in\omega} \cap \{b_k^2\}_{k\in\omega} = \\ = \{x_{k_{2n}}^2\}_{n\in\omega} \cap \{x_{k_{2n+1}}^2\}_{n\in\omega} = \varnothing,$$

which implies that the semigroup $\beta(X)$ is not commutative according to Theorem 6.1.

2. Let S be an infinite subsemigroup of X. Then the semigroup $\beta(S) \subset \beta(X)$ is commutative and hence contains at most one minimal left ideal. In this case Corollary 2.23 of [11] guarantees that the semigroup S contains a finite ideal.

3. By the preceding item, for every $x \in X$ the monogenic semigroup $\{x^k\}_{k\in\mathbb{N}}$ contains a finite ideal and hence is finite. □

Theorem 6.3. *For a commutative semigroup X and the semigroup $\varphi(X)$ of filters on X the following conditions are equivalent:*

1) *$\varphi(X)$ is commutative;*
2) *$\varphi(X)$ is supercommutative;*
3) *$\{a_k b_n\}_{k\leqslant n} \cap \{b_n a_{n+1}\}_{n\in\omega} \neq \varnothing$ for any sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ in X.*

Proof. We shall prove the implications (2) ⇒ (1) ⇒ (3) ⇒ (2). The implication (2) ⇒ (1) is trivial.

(1) ⇒ (3) Assume that the semigroup $\varphi(X)$ is commutative and take any sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ in X. Consider the filter $\mathcal{A} = \langle A\rangle$ generated by the set $A = \{a_n\}_{n\in\omega}$ and the filter $\mathcal{B} = \{B \subset X : \exists n\ \forall m \geqslant n\ \ b_m \in B\}$. It follows that the set $C = \{a_k b_n\}_{k\leqslant n}$ belongs to the product $\mathcal{A} * \mathcal{B}$. Since the semigroup $\varphi(X)$ is commutative, $C \in \mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A}$ and hence there is a set $B \in \mathcal{B}$ such that $BA \subset C$. By the definition of the filter $\mathcal{B}$, the set B contains some element b_m. Then $b_m a_{m+1} \in BA = AB \subset C$ and hence the intersection $\{a_k b_n\}_{k\leqslant n} \cap \{b_n a_{n+1}\}_{n\in\omega} \ni b_m a_{m+1}$ is not empty.

(3) ⇒ (2) Assume that $\mathcal{A} * \mathcal{B} \neq \mathcal{A} \circledast \mathcal{B}$ for some filters $\mathcal{A}, \mathcal{B} \in \varphi(X)$. Then $\mathcal{A} * \mathcal{B} \not\subset \mathcal{A} \circledast \mathcal{B}$ and some set $C \in \mathcal{A} * \mathcal{B}$ does not belong to the filter $\mathcal{A} \circledast \mathcal{B}$. This means that $A * B \not\subset C$ for any sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We lose no generality assuming that the set C is of the basic form $C = \bigcup_{a\in A} a * B_a$ for some set $A \in \mathcal{A}$ and family $(B_a)_{a\in A} \in \mathcal{B}^A$. Pick any point $a_0 \in A$ and consider the set $B_0 = B_{a_0} \in \mathcal{B}$. Since $A * B_0 \not\subset C$, there are points $b_0 \in B_0$ and $a_1 \in A$ such that $a_1 b_0 \notin C$. Now consider the set $B_1 = B_0 \cap B_{a_1} \in \mathcal{B}$. Since $A * B_1 \not\subset C$, there are points $b_1 \in B_1$ and $a_2 \in A$ such that $a_2 b_1 \notin C$. Proceeding by induction, for every $n \in \omega$ we shall construct two sequences

of points $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ in X such that $a_n \in A$, $b_n \in \bigcap_{i=0}^n B_{a_i}$, and $a_{n+1}b_n \notin C$ for every $n \in \omega$.

Observe that for each $i \leqslant n$ we get $a_i b_n \in a_i B_{a_i} \subset C$ and hence $\{a_k b_n\}_{k\leqslant n} \cap \{a_{n+1}b_n\}_{n\in\omega} \subset C \cap (X \setminus C) = \varnothing$. □

Proposition 6.4. *For each commutative semigroup X the semigroup $\varphi^\bullet(X)$ is supercommutative. Moreover, $\mathcal{A}*\mathcal{B} = \mathcal{A}\circledast\mathcal{B}$ for each $\mathcal{A} \in \upsilon^\bullet(X)$, $\mathcal{B} \in \varphi(X)$.*

Proof. It is sufficient to prove that $\mathcal{A} * \mathcal{B} \subset \mathcal{A} \circledast \mathcal{B}$ for each $\mathcal{A} \in \upsilon^\bullet(X)$, $\mathcal{B} \in \varphi(X)$. Let $C \in \mathcal{A} * \mathcal{B}$. We lose no generality assuming that the set C is of the basic form $C = \bigcup_{a\in A} a * B_a$ for some finite set $A \in \mathcal{A}$ and a family $(B_a)_{a\in A} \in \mathcal{B}^A$. Since the set A is finite, by definition of a filter, the intersection $\bigcap_{a\in A} B_a$ is nonempty and belongs to $\mathcal{B}$. Hence $C \supset \bigcup_{a\in A} a * \big(\bigcap_{a\in A} B_a\big) = A * \big(\bigcap_{a\in A} B_a\big) \in \mathcal{A} \circledast \mathcal{B}$. □

Problem 6.5. *Characterize semigroups X whose Stone-Čech extension $\beta(X)$ is supercommutative.*

7. (Super)commutativity of semigroups $\upsilon(X)$ and $\upsilon^\bullet(X)$

In this section we shall characterize semigroups X whose extensions $\upsilon^\bullet(X)$ and $\upsilon(X)$ are commutative or supercommutative. The characterization will be given in terms of square-linear semigroups X endowed with the digraph structure

$$\Delta = \{(x, y) \in X \times X : xy = x^2\}.$$

Theorem 7.1. *For a commutative semigroup X the following conditions are equivalent:*

1) *the semigroup $\upsilon^\bullet(X)$ is commutative;*
2) *$\upsilon^\bullet(X)$ is supercommutative;*
3) *$\mathcal{A} * \mathcal{B}^\perp = \mathcal{B}^\perp * \mathcal{A}$ for any filters $\mathcal{A}, \mathcal{B} \in \varphi^\bullet(X) \subset \upsilon^\bullet(X)$;*
4) *$\mathcal{A} * \mathcal{B} = \mathcal{A} \circledast \mathcal{B}$ for any upfamilies $\mathcal{A} \in \upsilon^\bullet(X)$ and $\mathcal{B} \in \upsilon(X)$;*
5) *$\{xu, yv\} \cap \{xv, yu\} \neq \varnothing$ for any points $x, y, u, v \in X$;*
6) *X is a square-linear semigroup whose digraph (X, Δ) contains no bipartite cycles.*

Proof. We shall prove the implications (4) ⇒ (2) ⇒ (1) ⇒ (3) ⇒ (5) ⇒ (6) ⇒ (4) among which the implications (4) ⇒ (2) ⇒ (1) ⇒ (3) are trivial.

(3) ⇒ (5) Assume that $\{xu, yv\} \cap \{xv, yu\} = \varnothing$ for some points $x, y, u, v \in X$, and consider the filters $\mathcal{A} = \langle\{x, y\}\rangle$ and $\mathcal{B} = \langle\{u, v\}\rangle$, which belong to the semigroup $\varphi^\bullet(X)$. It is easy to see that $\mathcal{B}^\perp = \langle\{u\}, \{v\}\rangle$. Observe that $\mathcal{B}^\perp * \mathcal{A} = \langle\{ux, uy\}, \{vx, vy\}\rangle$ and $\{xu, yv\} \in \mathcal{A} * \mathcal{B}^\perp$. Since $\{xu, yv\} \notin \langle\{ux, uy\}, \{vx, vy\}\rangle$, we conclude that $\mathcal{A} * \mathcal{B}^\perp \neq \mathcal{B}^\perp * \mathcal{A}$.

(5) ⇒ (6) To show that the semigroup X is square-linear, take any two points $a, b \in X$ and put $x = v = a$ and $y = u = b$. Then $\{ab\} = \{xu, yv\} \subset \{xv, yu\} = \{a^2, b^2\}$, which means that the semigroup X is square-linear. Next, we show that its digraph (X, Δ) contains no bipartite cycle. Assuming the converse and applying Lemma 4.1, we conclude that X contains a pure cycle x_0, x_1, x_2, x_3, x_4 of length 4. For every $0 \leqslant i \leqslant 3$ the inclusion $(x_i, x_{i+1}) \in \Delta \setminus \Delta^{-1}$ implies $x_i x_{i+1} = x_i^2 \neq x_{i+1}^2$. Since $x_4 = x_0$, we get $x_4 x_1 = x_0 x_1 = x_4^2 \neq x_1^2$. Then for the points $x = x_1, y = x_3, u = x_2, v = x_4$, we get

$$\{xu, yv\} \cap \{uy, vx\} = \{x_1x_2, x_3x_4\} \cap \{x_2x_3, x_4x_1\} = \\ = \{x_1^2, x_3^2\} \cap \{x_2^2, x_4^2\} = \varnothing.$$

So, the condition (4) does not hold.

(6) ⇒ (4) Assume that the subgroup X is square-linear, but $\mathcal{A} * \mathcal{B} \neq \mathcal{A} \circledast \mathcal{B}$ for some upfamilies $\mathcal{A} \in \upsilon^\bullet(X)$ and $\mathcal{B} \in \upsilon(X)$. Then $\mathcal{A} * \mathcal{B} \not\subset \mathcal{A} \circledast \mathcal{B}$ and hence $C \notin \mathcal{A} \circledast \mathcal{B}$ for some set $C \in \mathcal{A} * \mathcal{B}$. We lose no generality assuming that C is of the basic form $C = \bigcup_{a \in A} a * B_a$ for some set $A \in \mathcal{A}$ and sets $B_a \in \mathcal{B}$, $a \in A$. Since $\mathcal{A} \in \upsilon^\bullet(X)$, we can assume that the set A is finite.

Taking into account that $C \notin \mathcal{A} \circledast \mathcal{B}$, we conclude that $A * B_a \not\subset C$ for each $a \in A$. Choose any element $a_0 \in A$. By induction, for every $k \in \omega$ we shall choose points $b_k \in B_{a_k}$ and $a_{k+1} \in A$ with $a_{k+1} * b_k \notin C$ as follows. Assume that for some $k \in \omega$ a point $a_k \in A$ has been constructed. Consider the set $B_{a_k} * A = A * B_{a_k} \not\subset C$ and find two points $a_{k+1} \in A$ and $b_k \in B_{a_k}$ such that $b_k a_{k+1} \notin C$.

Since the set $A \supset \{a_k\}_{k \in \omega}$ is finite, for some point $a \in A$ the set $\Omega = \{k \in \omega : a_k = a\}$ is infinite. Fix any three numbers $p, q, r \in \Omega$ such that $1 < p < p + 1 < q < q + 1 < r$. Since X is a square-linear semigroup, $a_q b_q \in \{a_q^2, b_q^2\}$.

Now consider two cases.

(i) $a_q b_q = b_q^2$. In this case we shall show that

$$(b_{q+i}, a_{q+i}) \in \Delta \quad \text{and} \quad (a_{q+i+1}, b_{q+i}) \in \Delta$$

for every $i \in \omega$. This will be proved by induction on $i \in \omega$. If $i = 0$, then the inclusion $(b_q, a_q) \in \Delta$ follows from the equality $a_q b_q = b_q^2$. Assume that for some $i \in \omega$ we have proved that $(b_{q+i}, a_{q+i}) \in \Delta$, which is equivalent to $a_{q+i} b_{q+i} = b_{q+i}^2$. It follows from $b_{q+i}^2 = a_{q+i} b_{q+i} \neq b_{q+i} a_{q+i+1} \in \{b_{q+i}^2, a_{q+i+1}^2\}$ that $b_{q+i} a_{q+i+1} = a_{q+i+1}^2$ and hence $(a_{q+i+1}, b_{q+i}) \in \Delta$. Taking into account that

$$a_{q+i+1}^2 = b_{q+i} a_{q+i+1} \neq a_{q+i+1} b_{q+i+1} \in \{a_{q+i+1}^2, b_{q+i+1}^2\},$$

we see that $a_{q+i+1} b_{q+i+1} = b_{q+i+1}^2$ and $(b_{q+i+1}, a_{q+i+1}) \in \Delta$, which completes the inductive step.

Taking into account that $\{b_{q+i}^2\}_{i\in\omega} = \{a_{q+i} b_{q+i}\}_{i\in\omega} \subset \{a_k b_k\}_{k\in\omega} \subset C$ and $\{a_{q+i+1}^2\}_{i\in\omega} = \{b_{q+i} a_{q+i+1}\}_{i\in\omega} \subset \{b_k a_{k+1}\}_{k\in\omega} \subset X \setminus C$, we conclude that $\{b_{q+i}^2\}_{i\in\omega} \cap \{a_{q+i+1}^2\}_{i\in\omega} = \varnothing$, which implies that $(b_{q+i}, a_{q+j+1}) \notin \Delta \cap \Delta^{-1}$ for every $i, j \in \omega$.

Now we see that $a_r, b_{r-1}, a_{r-1}, \dots, b_q, a_q$ is a bipartite cycle in the digraph (X, Δ).

(ii) $a_q b_q = a_q^2$. In this case we shall show that

$$(a_{q-i}, b_{q-i}) \in \Delta \text{ and } (b_{q-i-1}, a_{q-i}) \in \Delta$$

for every $0 \leqslant i < q$. This will be proved by induction on $i < q$. If $i = 0$, then the inclusion $(a_q, b_q) \in \Delta$ follows from the equality $a_q b_q = a_q^2$. Assume that for some non-negative number $i < q - 1$ we have proved that $(a_{q-i}, b_{q-i}) \in \Delta$, which is equivalent to $a_{q-i} b_{q-i} = a_{q-i}^2$. It follows from $a_{q-i}^2 = a_{q-i} b_{q-i} \neq b_{q-i-1} a_{q-i} \in \{b_{q-i-1}^2, a_{q-i}^2\}$ that $b_{q-i-1} a_{q-i} = b_{q-i-1}^2$ and hence $(b_{q-i-1}, a_{q-i}) \in \Delta$. Taking into account that $b_{q-i-1}^2 = b_{q-i-1} a_{q-i} \neq a_{q-i-1} b_{q-i-1} \in \{a_{q-i-1}^2, b_{q-i-1}^2\}$, we see that $a_{q-i-1} b_{q-i-1} = a_{q-i-1}^2$ and $(a_{q-i-1}, b_{q-i-1}) \in \Delta$, which completes the inductive step.

Taking into account that

$$\{a_{q-i}^2\}_{i=0}^{q-1} = \{a_{q-i} b_{q-i}\}_{i=0}^{q-1} \subset \{a_k b_k\}_{k\in\omega} \subset C \quad \text{and}$$
$$\{b_{q-i-1}^2\}_{i=0}^{q-1} = \{b_{q-i-1} a_{q-i}\}_{i=0}^{q-1} \subset \{b_k a_{k+1}\}_{k\in\omega} \subset X \setminus C,$$

we conclude that $\{b_{q-i-1}^2\}_{i=0}^{q-1} \cap \{a_{q-i}^2\}_{i=0}^{q-1} = \varnothing$, which implies that $(b_{q-i-1}, a_{q-j}) \notin \Delta \cap \Delta^{-1}$ for every $0 \leqslant i, j < q$.

Now we see that $a_p, b_p, a_{p+1}, b_{p+1}, \dots, a_{q-1}, b_{q-1}, a_q$ is a bipartite cycle in the digraph (X, Δ). □

Theorem 7.2. *For a commutative semigroup X the following conditions are equivalent:*

1) *the semigroup* $\upsilon(X)$ *is commutative;*
2) $\upsilon(X)$ *is supercommutative;*
3) *the semigroups* $\upsilon^\bullet(X)$ *and* $\beta(X)$ *are commutative;*
4) $\mathcal{A} * \mathcal{B}^\perp = \mathcal{B}^\perp * \mathcal{A}$ *for any filters* $\mathcal{A}, \mathcal{B} \in \varphi(X)$*;*
5) $\{a_n b_n\}_{n\in\omega} \cap \{b_n a_{n+1}\}_{n\in\omega} \neq \varnothing$ *for any sequences* $(a_n)_{n\in\omega}$ *and* $(b_n)_{n\in\omega}$ *in* X*.*

Proof. We shall prove the implications (2) ⇒ (1) ⇒ (4) ⇒ (5) ⇒ (2) and (1) ⇒ (3) ⇒ (5).

The implications (2) ⇒ (1) ⇒ (4) are trivial.

(4) ⇒ (5) Assume that there are sequences $A = \{a_n\}_{n\in\omega}$ and $B = \{b_n\}_{n\in\omega}$ in X such that $\{a_n b_n\}_{n\in\omega} \cap \{b_n a_{n+1}\}_{n\in\omega} = \varnothing$. Consider the filters $\mathcal{A} = \langle A\rangle$ and $\mathcal{B} = \langle B\rangle$. It follows that $\{b_n\} \in \mathcal{B}^\perp = \{C \subset X : C \cap B \neq \varnothing\}$ for every $n \in \omega$. Assume that $\mathcal{A} * \mathcal{B}^\perp = \mathcal{B}^\perp * \mathcal{A}$.

Since $\{a_n b_n\}_{n\in\omega} \in \mathcal{A} * \mathcal{B}^\perp = \mathcal{B}^\perp * \mathcal{A}$, there is $k \in \omega$ such that $b_k * A \subset \{a_n b_n\}_{n\in\omega}$, which is not possible as $b_k a_{k+1} \notin \{a_n b_n\}_{n\in\omega}$. So, $\mathcal{A} * \mathcal{B}^\perp \neq \mathcal{B}^\perp * \mathcal{A}$.

(5) ⇒ (2) Assume that $\mathcal{A} * \mathcal{B} \neq \mathcal{A} \circledast \mathcal{B}$ for some upfamilies $\mathcal{A}, \mathcal{B} \in \upsilon(X)$. Then $\mathcal{A} * \mathcal{B} \not\subset \mathcal{A} \circledast \mathcal{B}$ and hence $C \notin \mathcal{A} \circledast \mathcal{B}$ for some set $C \in \mathcal{A} * \mathcal{B}$. We lose no generality assuming that C is of basic form $C = \bigcup_{a\in A} aB_a$ for some set $A \in \mathcal{A}$ and sets $B_a \in \mathcal{B}$, $a \in A$.

Taking into account that $C \notin \mathcal{A} \circledast \mathcal{B}$, we conclude that $B_a * A = A * B_a \not\subset C$ for each $a \in A$. Choose any elements $a_0 \in A$. By induction, for every $k \in \omega$ we can choose points $b_k \in B_{a_k}$ and $a_{k+1} \in A$ such that $b_k a_{k+1} \notin C$. Then the sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ have the required property $\{a_n b_n\}_{n\in\omega} \cap \{b_n a_{n+1}\}_{n\in\omega} \subset C \cap (X \setminus C) = \varnothing$, which shows that (5) does not hold.

The implication (1) ⇒ (3) is trivial.

(3) ⇒ (5). Assume that the semigroups $\beta(X)$ and $\upsilon^\bullet(X)$ are commutative but $\{a_n b_n\}_{n\in\omega} \cap \{b_n a_{n+1}\}_{n\in\omega} = \varnothing$ for some sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$. By Theorem 7.1, the semigroup X is square-linear and its digraph (X, Δ) contains no bipartite cycles.

Two cases are possible.

(i) $a_n b_n \neq b_n^2$ for all $n \in \omega$, and then $a_n b_n = a_n^2$ for all $n \in \omega$. Then for each $n \in \omega$ we get $\{b_n^2, a_{n+1}^2\} \ni b_n a_{n+1} \notin \{a_k b_k\}_{k\in\omega} = \{a_k^2\}_{k\in\omega}$ and hence $b_n a_{n+1} = b_n^2$. Then $\{a_n^2\}_{n\in\omega} \cap \{b_n^2\}_{n\in\omega} = \{a_n b_n\}_{n\in\omega} \cap \{b_n a_{n+1}\}_{n\in\omega} = \varnothing$. If for every $i < j$ we get $a_i b_j = a_i^2$ and $b_i a_j = b_i^2$, then $\{a_i b_j\}_{i<j} \cap \{b_i a_j\}_{i<j} = \varnothing$ and the semigroup $\beta(X)$ is not commutative by Theorem 6.1. So, there are numbers $i < j$ such that $a_i b_j \neq a_i^2$ or $b_i a_j \neq b_i^2$.

If $a_ib_j \neq a_i^2$, then $a_ib_j = b_j^2$, and $a_i, b_i, a_{i+1}, b_{i+1}, \dots, a_j, b_j, a_i$ if a bipartite cycle in the digraph (X, Δ), which is not possible.

If $b_ia_j \neq b_i^2$, then $b_ia_j = a_j^2$, and then $b_i, a_{i+1}, b_{i+1}, \dots, b_{j-1}, a_j, b_i$ is a bipartite cycle in the digraph (X, Δ), which is not possible.

(ii) $a_mb_m = b_m^2$ for some $m \in \omega$. Repeating the argument of the proof of the implication (5) $\Rightarrow$ (3) of Theorem 7.1, we can check that for every $i \in \omega$ $a_{m+i}b_{m+i} = b_{m+i}^2 \neq a_{m+i+1}^2 = b_{m+i}a_{m+i+1}$ and hence $\{b_{m+i}^2\}_{i\in\omega} \cap \{a_{m+i+1}^2\}_{i\in\omega} \subset \{a_kb_k\}_{k\in\omega} \cap \{b_ka_{k+1}\}_{k\in\omega} = \varnothing$. If for every $i<j$ we get $a_{m+i}b_{m+j} = b_{m+j}^2$ and $b_{m+i}a_{m+j} = a_{m+j}^2$, then $\{a_{m+i}b_{m+j}\}_{i<j} \cap \{b_{m+i}a_{m+j}\}_{i<j} = \varnothing$ and the semigroup $\beta(X)$ is not commutative by Theorem 6.1. So, there are numbers $i < j$ such that $a_{m+i}b_{m+j} \neq b_{m+j}^2$ or $b_{m+i}a_{m+j} \neq a_{m+j}^2$.

If $a_{m+i}b_{m+j} \neq b_{m+j}^2$, then $a_{m+i}b_{m+j} = a_{m+i}^2$, and

$$a_{m+i}, b_{m+j}, a_{m+j}, \dots, b_{m+i}, a_{m+i}$$

is a bipartite cycle in the digraph (X, Δ), which is not possible.

If $b_{m+i}a_{m+j} \neq a_{m+j}^2$, then $b_{m+i}a_{m+j} = b_{m+i}^2$, and

$$b_{m+i}, a_{m+j}, \dots, b_{m+i+1}, a_{m+i+1}, b_{m+i}$$

is a bipartite cycle in the digraph (X, Δ), which is a contradiction. □

8. (Super)commutativity of semigroups $N_2^\bullet(X)$ and $N_2(X)$

In this section we detect semigroups with (super) commutative extensions $N_2(X)$ or $N_2^\bullet(X)$.

Theorem 8.1. *For a commutative semigroup X the following conditions are equivalent:*

1) *the semigroup $N_2^\bullet(X)$ is commutative;*
2) *$N_2^\bullet(X)$ is supercommutative;*
3) *$\{xu, yv\} \cap \{xv, yu, xw, yw\} \neq \varnothing$ for any points $x, y, u, v, w \in X$;*
4) *$\mathcal{A} * \mathcal{B} = \mathcal{A} \circledast \mathcal{B}$ for any upfamilies $\mathcal{A} \in N_2^\bullet(X)$ and $\mathcal{B} \in N_2(X)$;*
5) *$\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A}$ for any $\mathcal{A} \in \varphi^\bullet(X)$ and $\mathcal{B} \in N_2^\bullet(X)$;*
6) *Either X is a square-linear semigroup whose digraph (X, Δ) contains no bipartite cycles or else X contains a 2-element subgroup H such that $x^3 \in H$ and $xy = x^3y^3$ for each points $x, y \in X$.*

Proof. We shall prove the implications (4) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (5) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (4) among which (4) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (5) are trivial.

To prove that (5) $\Rightarrow$ (3), assume that $\{xu, yv\} \cap \{xv, yu, xw, yw\} = \varnothing$ for some points $x, y, u, v, w \in X$. Consider the filter $\mathcal{A} = \langle\{x, y\}\rangle$ and the linked upfamily $\mathcal{B} = \langle\{u,w\},\{v,w\}\rangle$. By (5), $\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A}$. Observe that the set $\{xv, xw, yu, yw\} = x \cdot \{v,w\} \cup y \cdot \{u,w\}$ belongs to the upfamily $\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A}$. Then either $\{u,w\} \cdot \{x,y\} \subset \{xv, xw, yu, yw\}$ or $\{v,w\} \cdot \{x,y\} \subset \{xv, xw, yu, yw\}$. None of the inclusions is possible as $xu, yv \notin \{xv, yu, xw, yw\}$.

(3) ⇒ (6) If the semigroup $\upsilon^\bullet(X)$ is commutative, then by Theorem 7.1, X is a square-linear semigroup whose digraph (X, Δ) contains no bipartite cycles. So, we assume that the semigroup $\upsilon^\bullet(X)$ is not commutative. Given any element $a \in X$, put $x = v = a$, $y = u = a^2$, and $w = a^3$. Then the condition (3) implies $xu = yv = a^3 \in \{xv, yu, xw, yw\} = \{a^2, a^4, a^5\}$, which yields $a^3 = a^5$ for each $a \in X$. So, the semigroup X is periodic and its set of idempotents $E = \{e \in X : e^2 = e\}$ is not empty. We claim that the semilattice E is linear. Assuming the converse, find two idempotents $x, y \in E$ with $xy \notin \{x,y\} = \{x^2, y^2\}$ and put $u = x$, $v = y$, $w = xy$. Then $\{xu, yv\} \cap \{xv, yu, xw, yw\} = \{x^2, y^2\} \cap \{xy\} = \varnothing$, which contradicts the condition (3).

Next, we show that the semilattice E has the smallest element. Assume the opposite. Since the semigroup $\upsilon^\bullet(X)$ is not commutative, Theorem 7.1 yields four points $x, y, u, v \in X$ such that $\{xu, yv\} \cap \{xv, yu\} = \varnothing$. Consider the projection $e_* : X \to E$, $e_* : x \mapsto e_x$, of X onto its idempotent band. Since the linear semilattice E does not have the smallest idempotent, there is an idempotent $w \in E$ such that $we_{xu} = w \neq e_{xu}$ and $we_{yv} = w \neq e_{yv}$. It follows that $e_{xw} = e_x \cdot e_w = w \neq e_{xu}$ and hence $xw \neq xu$. By analogy we can prove that $\{xu, yv\} \cap \{xw, yw\} = \varnothing$, which implies $\{xu, yv\} \cap \{xv, yu, xw, yw\} = \varnothing$ and contradicts (3).

Therefore, the semilattice E has the smallest element, which will be denoted by e. We claim that the maximal group H_e containing this idempotent is not trivial. It follows from $\{xu, yv\} \cap \{xv, yu\} = \varnothing$ and $\{xu, yv\} \cap \{xv, yu, xe, ye\} \neq \varnothing \neq \{xv, yu\} \cap \{xu, yv, xe, ye\}$ that the set $\{xe, ye\}$ contains two elements and lies in the maximal subgroup H_e of the idempotent e. So, the group H_e is not trivial. The equality $a^3 = a^5$ holding for each element $a \in X$ implies that $a^2 = e$ for each element a of the group H_e. We claim that $|H_e| = 2$. In the other case, we could find three pairwise distinct points $a, b, ab \in H_e \setminus \{e\}$. Put $x = u = a$, $y = v = b$, and $w = e$. Then $\{xu, yv\} \cap \{xv, yu, xw, yw\} = \{e\} \cap \{ab, a, b\} = \varnothing$, which contradicts (3).

So, $H_e = \{e, h\}$ for some element $h \in H_e$. Next, we show that e is the unique element of the semilattice E. Assume that E contains some idempotent $f \neq e$ and consider the points $x = f$, $y = h$, $u = e$,

$v = h$, $w = f$. Observe that $\{xu, yv\} \cap \{xv, yu, xw, yw\} = \{fe, h^2\} \cap \{fh, he, ff, hf\} = \{e, e\} \cap \{h, f\} = \varnothing$, which contradicts (3).

Next, we check that $a^2 \in H_e$ for each $a \in X$. Assume conversely that $a^2 \notin H_e$. It follows from $a^3 = a^5$ that a^4 is an idempotent which coincides with e and hence $a^3 \in H_e$. If $a^3 = e$, then we can consider the points $x = a$, $y = h$, $u = a^2$, $v = h$ and $w = a$. Then $\{xu, yv\} \cap \{xv, yu, xw, yw\} = \{a^3, h^2\} \cap \{ah, ha^2, a^2, ha\} = \{e\} \cap \{h, a^2\} = \varnothing$, which contradicts (2). So, $a^3 = h$ and then $a^{2i+1} = h$ and $a^{2i+2} = e$ for all $i \in \mathbb{N}$. Consider the points $x = a$, $y = a^2$, $u = a^3$, $v = a^2$, and $w = a$. Then $\{xu, yv\} \cap \{xv, yu, xw, yw\} = \{a^4\} \cap \{a^3, a^5, a^2, a^3\} = \varnothing$, which contradicts (3).

Finally, we show that $ab \in H_e$ for any points $a, b \in X$. Assuming that $ab \notin H_e$ for some $a, b \in X$, consider the points $x = a$, $y = b$, $u = b$, $v = a$, and $w = e$. Then $\{xu, yv\} \cap \{xv, yu, xw, yw\} = \{ab\} \cap \{a^2, b^2, ae, be\} \subset \{ab\} \cap H_e = \varnothing$, which contradicts (2). So, $ab \in H_e$, and then $ab = (ab)^3 = a^3 b^3$.

(6) $\Rightarrow$ (4) If X is a square-linear semigroup whose digraph (X, Δ) contains no bipartite cycle, then by Theorem 7.1, $\mathcal{A} * \mathcal{B} = \mathcal{A} \circledast \mathcal{B}$ for any upfamilies $\mathcal{A} \in \upsilon^\bullet(X)$ and $\mathcal{B} \in \upsilon(X)$. Now assume that X contains a two-element subgroup $H \subset X$ such that $x^3 \in H$ and $xy = x^3 y^3$ for any points $x, y \in X$. This means that for the projection $\pi : X \to H$, $\pi : x \mapsto x^3$, the semigroup X is a projection extension of the subgroup H. Then the semigroup $N_2(X)$ is a projection extension of the subsemigroup $N_2(H)$. Since $|H| = 2$, by Proposition 6.4, the semigroup $N_2(H) = \varphi^\bullet(H)$ is supercommutative and hence for any linked upfamilies $\mathcal{A}, \mathcal{B} \in N_2(X)$ we get

$$\mathcal{A} * \mathcal{B} = \upsilon\pi(\mathcal{A}) * \upsilon\pi(\mathcal{B}) = \upsilon\pi(\mathcal{A}) \circledast \upsilon\pi(\mathcal{B}) = \mathcal{A} \circledast \mathcal{B}. \qquad \square$$

Theorem 8.2. *For a semigroup X the following conditions are equivalent:*

1) *the semigroup $N_2(X)$ is commutative;*
2) *$N_2(X)$ is supercommutative;*
3) *the semigroups $N_2^\bullet(X)$ and $\beta(X)$ are commutative;*
4) *$\mathcal{A} * \mathcal{B} = \mathcal{A} \circledast \mathcal{B}$ for any upfamilies $\mathcal{A} \in \varphi(X)$ and $\mathcal{B} \in N_2(X)$;*
5) *for every sequence $(a_i)_{i\in\omega} \in X^\omega$ and symmetric matrix $(b_{ij})_{i,j\in\omega} \subset X^{\omega\times\omega}$ we get*
   $\{a_i \cdot b_{ij}\}_{i,j\in\omega} \cap \{b_{ii} \cdot a_{i+1}\}_{i\in\omega} \neq \varnothing$.
6) *either the semigroup $\upsilon(X)$ is commutative or else X contains a 2-element subgroup H such that $x^3 \in H$ and $xy = x^3 y^3$ for each points $x, y \in X$.*

Proof. It suffices to prove the implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (2) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (2). In fact, the implications (2) $\Rightarrow$ (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) are trivial.

(3) $\Rightarrow$ (6) Assume that the semigroups $N_2^\bullet(X)$ and $\beta(X)$ are commutative but the semigroup $\upsilon(X)$ is not commutative. By Theorem 7.2, the semigroup $\upsilon^\bullet(X)$ is not commutative. Combining Theorems 7.1 and 8.1, we conclude that X contains a 2-element subgroup H such that $x^3 \in H$ and $xy = x^3y^3$ for each points $x, y \in X$.

(6) $\Rightarrow$ (2) If $\upsilon(X)$ is commutative, then by Theorem 7.2, it is supercommutative and so is its subsemigroup $N_2(X)$. If X contains a 2-element subgroup H such that $x^3 \in H$ and $xy = x^3y^3$ for each points $x, y \in X$, then for the projection $\pi : X \to H$, $\pi : x \mapsto x^3$, the semigroup X is a projection extension of the subgroup H. By Proposition 3.1, the semigroup $N_2(X)$ is a projection extension of the subsemigroup $N_2(H)$. Since $|H| = 2$, the semigroup $N_2(H) = \varphi^\bullet(H)$ is supercommutative by Proposition 6.4. Being a projection extension of the supercommutative semigroup $N_2(H)$, the semigroup $N_2(X)$ is supercommutative by Corollary 3.3.

(4) $\Rightarrow$ (5) Assume that for some sequence $(a_i)_{i\in\omega} \in X^\omega$ and some symmetric matrix $(b_{ij})_{i,j\in\omega} \subset X^{\omega\times\omega}$ we get $\{a_ib_{ij}\}_{i,j\in\omega} \cap \{b_{ii}a_{i+1}\}_{i\in\omega} = \varnothing$. Consider the filter $\mathcal{A} = \langle A\rangle \in \varphi(X) \subset N_2(X)$ generated by the set $A = \{a_i\}_{i\in\omega}$ and the linked system $\mathcal{B}$ generated by the family $\{B_i\}_{i\in\omega}$ of sets $B_i = \{b_{ij}\}_{j\in\omega}$, $i \in \omega$. Observe that the set $C = \{a_ib_{ij}\}_{i,j\in\omega}$ belongs to $\mathcal{A} * \mathcal{B}$. Assuming that $\mathcal{A} * \mathcal{B} = \mathcal{A} \circledast \mathcal{B}$, we would find a number $i \in \omega$ such that $A * B_i \subset C$, which is not possible as $a_{i+1}b_{ii} \notin C$.

(5) $\Rightarrow$ (2) Assuming that $\mathcal{A}*\mathcal{B}$ is not supercommutative, we could find two linked upfamilies $\mathcal{A}, \mathcal{B} \in N_2(X)$ such that $\mathcal{A} * \mathcal{B} \not\subset \mathcal{A} \circledast \mathcal{B}$. Then for some set $A \in \mathcal{A}$ and a family $(B_a)_{a\in A} \in \mathcal{B}^A$, we get $\bigcup_{a\in A} aB_a \notin \mathcal{A} \circledast \mathcal{B}$. It follows that for every $a \in A$ the product $A * B_a$ is not contained in the set $C = \bigcup_{a\in A} a * B_a$, which allows us to construct inductively two sequences of points $(a_i)_{i\in\omega} \subset A^\omega$ and $(b_i)_{i\in\omega} \in X^\omega$ such that $b_i \in B_{a_i}$ and $a_{i+1}b_i \notin C$ for every $i \in \omega$. For every numbers $i < j$ put $b_{ii} = b_i$ and let $b_{ij} = b_{ji}$ be some point of the intersection $B_{a_i} \cap B_{a_j}$ (which is not empty by the linkedness of the upfamily $\mathcal{B}$). Then the sequence $(a_i)_{i\in\omega}$ and the symmetric matrix $(b_{ij})_{i,j\in\omega}$ have the required property $\{a_ib_{ij}\}_{i,j\in\omega} \cap \{b_{ii}a_{i+1}\} \subset C \cap (X \setminus C) = \varnothing$. $\square$

9. Commutativity of superextensions $\lambda(X)$

In this section we characterize semigroups having commutative extensions $\lambda(X)$ and $\lambda^\bullet(X)$.

Theorem 9.1. *For a commutative semigroup X the following conditions are equivalent:*

1) *the semigroup $\lambda(X)$ is commutative;*
2) *for any symmetric matrices $(a_{ij})_{i,j\in\omega}, (b_{ij})_{i,j\in\omega} \in X^{\omega\times\omega}$ we get $\{a_{ii}\cdot b_{ij}\}_{i,j\in\omega} \cap \{b_{ii}\cdot a_{i+1,j}\}_{i,j\in\omega} \neq \varnothing$.*

Proof. (1) $\Rightarrow$ (2) Assuming that the semigroup $\lambda(X)$ is not commutative, find two maximal linked systems $\mathcal{A},\mathcal{B}\in\lambda(X)$ such that $\mathcal{A}*\mathcal{B}\neq\mathcal{B}*\mathcal{A}$. The maximal linked upfamilies $\mathcal{A}*\mathcal{B}$ and $\mathcal{B}*\mathcal{A}$ are distinct and hence contain two disjoint sets $C\in\mathcal{A}*\mathcal{B}$ and $C'\in\mathcal{B}*\mathcal{A}$. For these sets there are sets $A\in\mathcal{A}$, $B\in\mathcal{B}$ and families of sets $(B_a)_{a\in A}\in\mathcal{B}^A$, $(A_b)_{b\in B}\in\mathcal{A}^B$ such that $\bigcup_{a\in A} aB_a\subset C$ and $\bigcup_{b\in B} bA_a\subset C'$.

By induction we can construct two sequences $\{a_{ii}\}_{i\in\omega}\subset A$ and $\{b_{ii}\}_{i\in\omega}$ such that $b_{ii}\in B\cap B_{a_{ii}}$ and $a_{i+1,i+1}\in A\cap A_{b_{ii}}$ for every $i\in\omega$. Since the upfamilies $\mathcal{B}$ and $\mathcal{A}$ are linked, for every numbers $i<j$ we can choose points $b_{ij}\in B_{a_{ii}}\cap B_{a_{jj}}$ and $a_{i+1,j+1}\in A_{b_{ii}}\cap A_{b_{jj}}$, and put $b_{ji}=b_{ij}$ and $a_{j+1,i+1}=a_{i+1,j+1}$. Also put $a_{0i}=a_{i0}=a_{00}$ for all $i\in\omega$. In such way we have defined two symmetric matrices $(a_{ij})_{i,j\in\omega}$ and $(b_{ij})_{i,j\in\omega}$ with coefficients in the semigroup X. Observe that for each $i,j\in\omega$ we get $a_{ii}*b_{ij}\in a_{ii}*B_{a_{ii}}\subset C$ and $b_{ii}*a_{i+1,j}\in b_{ii}*A_{b_{ii}}\subset C'$, which implies that the sets $\{a_{ii}\cdot b_{ij}\}_{i,j\in\omega}$ and $\{b_{ii}\cdot a_{i+1,j}\}_{i,j\in\omega}$ are disjoint.

(2) $\Rightarrow$ (1) Assume that there are two symmetric matrices $(a_{ij})_{i,j\in\omega}$, $(b_{ij})_{i,j\in\omega}\in X^{\omega\times\omega}$ such that the sets $\{a_{ii}\cdot b_{ij}\}_{i,j\in\omega}$ and $\{b_{ii}\cdot a_{i+1,j}\}_{i,j\in\omega}$ are disjoint. Consider the sets $A=\{a_{ii}\}_{i\in\omega}$ and $A_i=\{a_{ij}\}_{j\in\omega}$ which form a linked system $\{A,A_i\}_{i\in\omega}$ which can be enlarged to a maximal linked system $\mathcal{A}$. On the other hand, the sets $B=\{b_{ii}\}_{i\in\omega}$ and $B_i=\{b_{ij}\}_{j\in\omega}$ form a linked upfamily, which can be enlarged to a maximal linked upfamily $\mathcal{B}$. We claim that $\mathcal{A}*\mathcal{B}\neq\mathcal{B}*\mathcal{A}$. This follows from the fact that the maximal linked upfamilies $\mathcal{A}*\mathcal{B}$ and $\mathcal{B}*\mathcal{A}$ contains the disjoint sets

$$\{a_{ii}b_{ij}\}_{i,j\in\omega}=\bigcup_{a_{ii}\in A} a_{ii}B_i\in\mathcal{A}*\mathcal{B}$$

and

$$\{b_{ii}a_{i+1,j}\}_{i,j\in\omega}=\bigcup_{b_{ii}\in B} b_{ii}A_{i+1}\in\mathcal{B}*\mathcal{A}.$$

Therefore the semigroup $\mathcal{A}$ is not commutative. □

For a set X consider the subset

$$\lambda_3^\bullet(X) = \{\mathcal{A} \in \lambda(X) : \exists Y \subset X \text{ such that } |Y| \leqslant 3 \text{ and } \mathcal{A} \in \lambda(Y) \subset \lambda(X)\}$$

in $\lambda^\bullet(X)$.

Theorem 9.2. *For a commutative semigroup X the following conditions are equivalent:*

1) *the semigroup $\lambda^\bullet(X)$ is commutative;*
2) *any two maximal linked systems $\mathcal{A}, \mathcal{B} \in \lambda_3^\bullet(X)$ commute;*
3) *any two maximal linked systems $\mathcal{A} \in \lambda^\bullet(X)$ and $\mathcal{B} \in \lambda(X)$ commute;*
4) *for any elements $a, b, c, x, y, z \in X$ the sets $\{ax, ay, cy, cz\}$ and $\{xc, xb, za, zb\}$ are not disjoint;*
5) *for any elements $x_0, x_1, x_2, x_3, x_4, x_5 \in X$ the sets $\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\}$ and $\{x_1x_4, x_2x_5, x_0x_1, x_0x_5\}$ are not disjoint.*

Proof. It suffices to prove the implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (1). In fact, the implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) are trivial while the equivalence (4) $\Leftrightarrow$ (5) follows from the observation that for any points $b = x_0$, $x = x_1$, $a = x_2$, $y = x_3$, $c = x_4$, $z = x_5$ in X we get

$$\{ax, ay, cy, cz\} \cap \{xc, xb, za, zb\} = \\ = \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_1x_4, x_1x_0, x_5x_2, x_5x_0\}.$$

(2) $\Rightarrow$ (4) Assume that for some elements $a, b, c, x, y, z \in X$ the sets $\{ax, ay, cy, cz\}$ and $\{xc, xb, za, zb\}$ are disjoint. Consider the maximal linked systems $\mathcal{A} = \{A \subset X : |A \cap \{a, b, c\}| \geqslant 2\}$ and $\mathcal{X} = \{A \subset X : |A \cap \{x, y, z\}| \geqslant 2\}$ and observe that $\mathcal{A}, \mathcal{X} \in \lambda_3^\bullet(X)$ and the products $\mathcal{A} * \mathcal{X}$ and $\mathcal{X} * \mathcal{A}$ are distinct since they contain disjoint sets

$$a\{x, y\} \cup c\{y, z\} \in \mathcal{A} * \mathcal{X} \text{ and } x\{c, b\} \cup z\{a, b\} \in \mathcal{X} * \mathcal{A}.$$

(4) $\Rightarrow$ (3) The proof of this implication is the most difficult part of the proof. Assume that (4) holds but there are two non-commuting maximal linked systems $\mathcal{A} \in \lambda^\bullet(X)$ and $\mathcal{B} \in \lambda(X)$. Then the maximal linked systems $\mathcal{A} * \mathcal{B}$ and $\mathcal{B} * \mathcal{A}$ contain disjoint sets. Consequently, we can find sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ and families $(B_a)_{a \in A} \in \mathcal{B}^A$ and $(A_b)_{b \in B} \in \mathcal{A}^B$ such that the sets $U_{\mathcal{AB}} = \bigcup_{a \in A} a * B_a \in \mathcal{A} * \mathcal{B}$ and $U_{\mathcal{BA}} = \bigcup_{b \in B} b * A_b \in \mathcal{B} * \mathcal{A}$ are disjoint. Since $\mathcal{A} \in \lambda^\bullet(X)$, we can additionally assume that the set A is finite.

By analogy with the proof of Theorem 9.1, construct inductively two sequences $(a_i)_{i\in\omega} \in A^\omega$ and $(b_i)_{i\in\omega} \in B^\omega$ such that $b_i \in B \cap B_{a_i}$ and $a_{i+1} \in A \cap A_{b_i}$. Since the set A is finite, there are two numbers k, m such that $0 < k < m-1$ and $a_k = a_m$.

Let $n = m - k \geqslant 2$ and consider the group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ endowed with the group operation of addition modulo n, which will be denoted by the symbol $\oplus$. So, $1 \oplus (n-1) = 0$. For each $i \in \mathbb{Z}_n$ let $a_{ii} = a_{k+i}$ and $b_{ii} = b_{k+i}$. For every numbers $i < j$ in $\mathbb{Z}_n$ choose points $b_{ij} = b_{ji} \in B_{a_{ii}} \cap B_{a_{jj}}$ and $a_{ij} = a_{ji} \in A_{b_{i',i'}} \cap A_{b_{j',j'}}$ where $i', j' \in \mathbb{Z}_n$ are unique numbers such that $i = i' \oplus 1$ and $j' = j \oplus 1$. It follows that $a_{ii}b_{ij} \in a_{ii}B_{a_{ii}} \subset U_{\mathcal{AB}}$ and $b_{ii}a_{i\oplus 1,j} \in b_{ii}A_{b_{ii}} \in U_{\mathcal{BA}}$. So,

$$\{a_{ii} * b_{ij}\}_{i,j\in\mathbb{Z}_n} \cap \{b_{ii} * a_{i\oplus 1,j}\}_{i,j\in\mathbb{Z}_n} \subset U_{\mathcal{AB}} \cap U_{\mathcal{BA}} = \varnothing.$$

By induction on $i \in \mathbb{Z}_n$ we shall prove that $a_{00} * b_{ii} \in U_{\mathcal{AB}}$. This is trivial for $i = 0$. Assume that for some positive number $i < n-1$ we have proved that $a_{00} * b_{ii} \in U_{\mathcal{AB}}$. Let

$$\begin{aligned} &x_0 = a_{i+1,i\oplus 2}, && x_1 = b_{i,i}, && x_2 = a_{00}, \\ &x_3 = b_{0,i+1}, && x_4 = a_{i+1,i+1}, && x_5 = b_{i+1,i+1}. \end{aligned}$$

It follows that

$$\begin{aligned} &\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} = \\ &= \{b_{i,i} * a_{00}, a_{00} * b_{0,i+1}, b_{0,i+1} * a_{i+1,i+1}, a_{i+1,i+1} * b_{i+1,i+1}\} \subset \\ &\subset U_{\mathcal{AB}} \cup a_{00} * B_{a_{00}} \cup a_{i+1,i+1} * B_{a_{i+1,i+1}} \cup a_{i+1,i+1} * B_{a_{i+1,i+1}} \subset U_{\mathcal{AB}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \{x_0x_1, x_0x_5, x_1x_4\} = \{a_{i+1,i\oplus 2} * b_{i,i}, a_{i+1,i\oplus 2} * b_{i+1,i+1}, b_{i,i} * a_{i+1,i+1}\} \subset \\ \subset b_{i,i} * A_{b_{i,i}} \cup b_{i+1,i+1} * A_{b_{i+1,i+1}} \cup b_{i,i} * A_{b_{i,i}} \subset U_{\mathcal{BA}}. \end{aligned}$$

Then $\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_0x_1, x_0x_5, x_1x_4\} \subset U_{\mathcal{AB}} \cap U_{\mathcal{BA}} = \varnothing$. By the condition (4), the intersection

$$\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_0x_1, x_0x_5, x_1x_4, x_2x_5\}$$

is not empty, which implies that

$$a_{00} * b_{i+1,i+1} = x_2x_5 \in \{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \subset U_{\mathcal{AB}}.$$

After completing the inductive construction, we conclude that $a_{00} * b_{n-1,n-1} \in U_{\mathcal{AB}}$ which is impossible as

$$a_{00} * b_{n-1,n-1} = a_k * b_{k+n-1} = a_m * b_{m-1} = b_{m-1} * a_m \in U_{\mathcal{BA}}. \qquad \square$$

We shall apply Theorem 9.2 to detecting monogenic semigroups that have commutative superextensions.

Theorem 9.3. *For a monogenic semigroup $X = \{x^k\}_{k\in\mathbb{N}}$ the following conditions are equivalent*

1) $\lambda(X)$ *is commutative;*
2) $\lambda^\bullet(X)$ *is commutative;*
3) $x^n = x^m$ *for some pair* (n,m) *in the set* $\{(1,2),\ (1,3),(2,3),\ (1,4),(2,4),(3,4),\ (1,5),(2,5),(3,5),(4,5),\ (2,6)\}$.

Proof. We shall prove the implications (3) ⇒ (1) ⇒ (2) ⇒ (3), among which the implication (1) ⇒ (2) is trivial.

(3) ⇒ (1) Assume that $x^n = x^m$ for some pair (n,m) in the set

$$\{(1,2),(1,3),(2,3),(1,4),(2,4),(3,4),(1,5),(2,5),(3,5),(4,5),(2,6)\}.$$

If $(n,m) \in \{(1,2),(1,3),(1,4),(1,5)\}$ then X is isomorphic to a cyclic group of order ≤ 4 and $\lambda(X)$ is commutative by Theorem 5.1 of [6].

If $(n,m) = (2,3)$, then the semigroup $\lambda(X) = X$ is commutative.

If $(n,m) \in \{(2,4),(3,4)\}$, then $|X| = 3$ and $\lambda(X) = X \cup \{\triangle\}$ where $\triangle = \{A \subset X : |A| \geqslant 2\}$. Taking into account that $xy = yx$ and $\triangle \cdot x = x \cdot \triangle$ for all $x, y \in X$, we see that the semigroup $\lambda(X)$ is commutative.

If $(n,m) = (2,5)$, then $xa = x^4 a$ for every $a \in X$ and hence $X = \{x, x^2, x^3, x^2\}$ is a projective extension of the cyclic subgroup $\{x^2, x^3, x^4\}$. In this case the commutativity of $\lambda(X)$ follows from the commutativity of $\lambda(C_3)$ according to Proposition 3.3.

By analogy, for $(n,m) = (2,6)$ the commutativity of the semigroup $\lambda(X)$ follows from the commutativity of the semigroup $\lambda(C_4)$.

Now consider the case $(n,m) = (3,5)$. In this case $X = \{x, x^2, x^3, x^4\}$ and the semigroup $\lambda(X)$ contains 12 elements:

$$\begin{aligned}
&k = \langle\{x^k\}\rangle,\\
&\triangle_k = \langle\{A \subset X : |A| = 2,\ x^k \notin A\}\rangle \ \text{ and}\\
&\square_k = \langle\{X \setminus \{x^k\}, A : A \subset X,\ |A| = 2,\ x^k \in A\}\rangle,
\end{aligned}$$

where $k \in \{1,2,3,4\}$. The following Cayley table of multiplication in the semigroup $\lambda(X)$ implies the commutativity of $\lambda(X)$:

$*$	$\triangle_1$	$\triangle_2$	$\triangle_3$	$\triangle_4$	$\square_1$	$\square_2$	$\square_3$	$\square_4$
$\triangle_1$	4	3	4	3	3	4	3	4
$\triangle_2$	3	$\triangle_1$	3	$\triangle_1$	$\triangle_1$	3	$\triangle_1$	3
$\triangle_3$	4	3	4	3	3	4	3	4
$\triangle_4$	3	$\triangle_1$	3	$\triangle_1$	$\triangle_1$	3	$\triangle_1$	3
$\square_1$	3	$\triangle_1$	3	$\triangle_1$	$\triangle_1$	3	$\triangle_1$	3
$\square_2$	4	3	4	3	3	4	3	4
$\square_3$	3	$\triangle_1$	3	$\triangle_1$	$\triangle_1$	3	$\triangle_1$	3
$\square_4$	4	3	4	3	3	4	3	4

In the final case $(n,m)=(4,5)$, the product of any two nonprincipal maximal linked upfamilies is equal to the principal ultrafilter $\langle\{x^4\}\rangle$, which implies that the semigroup $\lambda(X)$ is commutative.

(2) $\Rightarrow$ (3) Let $X=\{x^k\}_{k\in\mathbb{N}}$ be a monogenic semigroup with commutative extension $\lambda^\bullet(X)$. If $|X|\leqslant 4$, then $x^n=x^m$ for some $(n,m)\in\{(1,2),(1,3),(2,3),(1,4),(2,4),(3,4),(1,5),(2,5),(3,5),(4,5)\}$. If $x^6=x^2$, then we are done. So, we assume that $x^6\neq x^2$ and $|X|\geqslant 5$. In this case the elements x,x^2,x^3,x^4,x^5 are pairwise distinct.

We claim that $x^7\in\{x^3,x^4\}$. In the opposite case we can put $x_0=x^4$, $x_1=x^3$, $x_2=x$, $x_3=x^2$, $x_4=x^2$, $x_5=x$ and observe that

$$\{x_1x_2,x_2x_3,x_3x_4,x_4x_5\}\cap\{x_1x_4,x_2x_5,x_0x_1,x_0x_5\}=\\ \{x^3,x^4\}\cap\{x^2,x^5,x^7\}=\varnothing,$$

which implies that the semigroup $\lambda^\bullet(X)$ is not commutative according to Theorem 9.2. This contradiction shows that $x^7\in\{x^3,x^4\}$ and hence the monogenic semigroup X is finite.

If $x^7=x^3$, then we can put $x_0=x^5$, $x_1=x_2=x$, $x_3=x^3$, $x_4=x_5=x^2$ and observe that

$$\{x_1x_2,x_2x_3,x_3x_4,x_4x_5\}\cap\{x_1x_4,x_2x_5,x_0x_1,x_0x_5\}=\\ =\{x^2,x^4,x^5\}\cap\{x^3,x^6,x^7\}=\varnothing$$

since $x^6\neq x^2$. By Theorem 9.2, the semigroup $\lambda^\bullet(X)$ is not commutative.

If $x^7=x^4$, then we can put $x_0=x_1=x$, $x_2=x^4$, $x_3=x^3$, $x_4=x_5=x^2$ and observe that

$$\{x_1x_2, x_2x_3, x_3x_4, x_4x_5\} \cap \{x_1x_4, x_2x_5, x_0x_1, x_0x_5\} = \\ = \{x^5, x^7, x^4\} \cap \{x^3, x^6, x^2, x^3\} = \varnothing,$$

which implies that the semigroup $\lambda^\bullet(X)$ is not commutative according to Theorem 9.2. □

Now we establish some structural properties of semigroups X having commutative superextensions $\lambda(X)$. A semigroup X is called a 0-*bouquet* of its subsemigroups X_α, $\alpha \in I$, if

- $X = \bigcup_{\alpha\in A} X_\alpha$;
- X has two-sided zero 0;
- $X_\alpha \cap X_\beta = X_\alpha * X_\beta = \{0\}$ for any distinct indices $\alpha, \beta \in I$.

In this case we write $X = \bigvee_{\alpha\in I} X_\alpha$.

Proposition 9.4. *Assume that a semigroup $X = \bigvee_{\alpha\in I} X_\alpha$ is a 0-bouquet of its subsemigroups X_α, $\alpha \in I$. The superextension $\lambda(X)$ is commutative if and only if for each $\alpha \in I$ the semigroup $\lambda(X_\alpha)$ is commutative.*

Proof. The "only if" part is trivial. To prove the "if" part, assume that the semigroup $\lambda(X)$ is not commutative. By Theorem 9.1, there are two symmetric matrices $(a_{ij})_{i,j\in\omega}$ and $(b_{ij})_{i,j\in\omega}$ with the coefficients in X such that the sets $A = \{a_{ii} * b_{ij}\}_{i,j\in\omega}$ and $B = \{b_{ii} * a_{i+1,j}\}_{i,j\in\omega}$ are disjoint. Then $0 \notin A$ or $0 \notin B$.

First assume that $0 \notin A$. Find an index $\alpha \in I$ such that $a_{00} \in X_\alpha$. It follows from $0 \notin \{a_{00}b_{0j}\}_{j\in\omega}$ that $b_{0j} \in X_\alpha$ for all $j \in \omega$. Observe that for every $i \in \omega$ we get $a_{ii}b_{i0} = a_{ii}b_{0i} \neq 0$ and hence $a_{ii} \in X_\alpha$. Finally, for each $i, j \in \omega$, the inequality $a_{ii}b_{ij} \neq 0$ implies that $b_{ij} \in X_\alpha$. So, $\{a_{ii}\}_{i\in\omega} \cup \{b_{ij}\}_{i,j\in A} \subset X_\alpha$. Now for every $i, j \in \omega$ put

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} \in X_\alpha, \\ 0 & \text{otherwise} \end{cases}$$

and observe that $(a_{ij})_{i,j\in\omega}$ is a symmetric matrix with coefficients in X_α. It follows that $\{a'_{ii}b_{ij}\}_{i,j\in\omega} = \{a_{ii}b_{ij}\}_{i,j\in\omega} = A$ and $\{b_{ii}a_{i+1,j}\}_{i,j\in\omega} \subset (B \cap X_\alpha) \cup \{0\}$. Since $A \cap (B \cup \{0\}) = \varnothing$, Theorem 9.1 implies that the semigroup $\lambda(X_\alpha)$ is not commutative.

By analogy, we can treat the case $0 \notin B = \{b_{ii} * a_{i+1,j}\}_{i,j\in\omega}$. In this case there is $\alpha \in I$ such that $\{b_{ii}\}_{i\in\omega} \cup \{a_{i+1,j}\}_{i,j\in\omega} \subset X_\alpha \setminus \{0\}$. Changing the element a_{00} by 0, if necessary, we get $\{a_{ij}\}_{i,j\in\omega} \subset X_\alpha$. Now for every

$i, j \in \omega$ put

$$b'_{ij} = \begin{cases} b_{ij} & \text{if } b_{ij} \in X_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $(a_{ij})_{i,j\in\omega}$ and $(b'_{ij})_{i,j\in\omega}$ are symmetric matrices with coefficients in X_α such that $\{a_{ii}b'_{ij}\}_{i,j\in\omega} \subset A \cup \{0\}$ and $\{b'_{ii}a_{i+1,j}\}_{i,j\in\omega} = (b_{ii}a_{i+1,j}\}_{i,j\in\omega} = B$. Since $(A \cup \{0\}) \cap B = \varnothing$, Theorem 9.1 implies that the semigroup $\lambda(X_\alpha)$ is not commutative. □

Now we detect regular semigroups X whose superextensions $\lambda(X)$ are commutative.

In the following theorem for a natural number $n \in \mathbb{N}$ by

$$C_n = \{z \in \mathbb{C} : z^n = 1\}$$

we denote the cyclic group of order n and by

$$L_n = \{0, \dots, n-1\}$$

the linear semilattice endowed with the operation of minimum.

For two semigroups $(X, *)$ and $(Y, \star)$ by $X \sqcup Y$ we denote the semigroup $X \times \{0\} \cup Y \times \{1\}$ endowed with the semigroup operation

$$(a, i) \circ (b, j) = \begin{cases} (a * b, 0) & \text{if } i = 0 \text{ and } j = 0, \\ (a, 0) & \text{if } i = 0 \text{ and } j = 1, \\ (b, 0) & \text{if } i = 1 \text{ and } j = 0, \\ (a \star b, 1) & \text{if } i = 1 \text{ and } j = 1. \end{cases}$$

The semigroup $X \sqcup Y$ is called the *ordered union* of the semigroups X and Y. For example, the ordered union $L_1 \sqcup C_2$ is isomorphic to the multiplicative semigroup $\{-1, 0, 1\}$.

Theorem 9.5. *The superextension $\lambda(X)$ of a regular semigroup X is commutative if and only if one of the following conditions holds:*

- *X is isomorphic to one of the semigroups: C_2, C_3, C_4, $C_2 \times C_2$, $C_2 \times L_2$, $L_1 \sqcup C_2$, $C_2 \bigsqcup L_n$ for some $n \in \mathbb{N}$;*
- *$X = \bigvee_{\alpha\in A} X_\alpha$ for some subsemigroups X_α, $\alpha \in A$, isomorphic to $L_1 \sqcup C_2$ or L_n for $n \in \mathbb{N}$.*

Proof. To prove the "if" part, assume that a semigroup X satisfies conditions (1) or (2). If X is isomorphic to one of the groups C_2, C_3, C_4, or $C_2 \times C_2$, then its superextension $\lambda(X)$ is commutative according to

Theorem 5.1 of [6]. If X is isomorphic to $C_2 \times L_2$ or $C_2 \bigsqcup L_n$ for some $n \in \mathbb{N}$, then $\lambda(X)$ is commutative by Theorem 1.1 of [5].

Next, assume that $X = \bigvee_{\alpha \in A} X_\alpha$ is a 0-bouquet of its subsemigroups X_α, $\alpha \in A$, isomorphic to $L_1 \sqcup C_2$ or L_n, $n \in \mathbb{N}$. By Theorem 1.1 of [5], the superextension of the semigroups $L_1 \sqcup C_2$ and L_n, $n \in \mathbb{N}$, are commutative. Consequently, for every $\alpha \in X_\alpha$ the superextension $\lambda(X_\alpha)$ is commutative and by Proposition 9.4, the superextension $\lambda(X)$ is commutative too. This completes the proof of the "if" part.

The prove the "only if" part we shall use the following:

Lemma 9.6. *The superextension $\lambda(X)$ of a semigroup X is not commutative if X is isomorphic to one of the semigroups:*

1) $L_1 \sqcup C_n$ *for* $n \geqslant 3$;
2) $C_n \sqcup L_1$ *for* $n \geqslant 3$;
3) $L_1 \sqcup C_2 \sqcup L_1$;
4) $L_2 \sqcup C_2$;
5) $(C_2 \times C_2) \sqcup L_1$;
6) $L_1 \sqcup (C_2 \times C_2)$;
7) $C_2 \sqcup C_2$.

Proof. 1. If $X = L_1 \sqcup C_n = \{e_1\} \sqcup \{a^i\}_{i=0}^{n-1}$ for some $n \geqslant 3$, then the maximal linked upfamilies $\square = \big\langle \{e_1, a^0\}, \{e_1, a\}, \{e_1, a^{-1}\}, \{a^0, a, a^{-1}\}\big\rangle$ and $\triangle = \big\langle \{a^0, a\}, \{a^0, a^{-1}\}, \{a, a^{-1}\}\big\rangle$ do not commute, since $\{e_1, a^0\} = a^0\{e_1, a^0\} \cup a\{e_1, a^{-1}\} \in \triangle * \square$ while $\{e_1, a^0\} \notin \square * \triangle$.

2. If $X = C_n \sqcup L_1 = \{a^i\}_{i=0}^{n-1} \sqcup \{e_2\}$ for some $n \geqslant 3$, then the maximal linked upfamilies $\square = \big\langle \{a^2, a\}, \{a^2, a^0\}, \{a^2, e_2\}, \{a^0, a, e_2\}\big\rangle$ and $\triangle = \big\langle \{a^0, e_2\}, \{a^0, a^2\}, \{e_2, a^2\}\big\rangle$ do not commute, since $\{a^2, e_2\} = a^2\{a^0, e_2\} \cup e_2\{a^2, e_2\} \in \square * \triangle$ while $\{e_2, a^2\} \notin \triangle * \square$.

3. If $X = L_1 \sqcup C_2 \sqcup L_1 = \{e_1\} \sqcup \{e_2, a\} \sqcup \{e_3\}$ where $a \neq a^2 = e_2$, then the maximal linked upfamilies $\square_3 = \langle \{e_1, e_3\}, \{e_2, e_3\}, \{a, e_3\}, \{e_1, e_2, a\}\rangle$ and $\square_a = \langle \{a, e_1\}, \{a, e_2\}, \{a, e_3\}, \{e_1, e_2, e_3\}\rangle$ do not commute, since $\{e_1, e_2\} = e_1\{e_1, e_2, e_3\} \cup e_2\{e_1, e_2, e_3\} \cup a\{a, e_1\} \in \square_3 * \square_a$ while $\{e_1, e_2\} \notin \square_a * \square_3$.

4. If $X = L_2 \sqcup C_2 = \{e_1, e_2\} \sqcup \{e_3, a\}$ where $a \neq a^2 = e_3$, then the maximal linked upfamilies $\square = \langle \{e_1, e_2\}, \{e_1, e_3\}, \{e_1, a\}, \{e_2, e_3, a\}\rangle$ and $\triangle = \langle \{e_2, a\}, \{e_2, e_3\}, \{a, e_3\}\rangle$ do not commute, since $\{e_2, e_3\} = e_2\{e_2, e_3\} \cup e_3\{e_2, e_3\} \cup a\{e_2, a\} \in \square * \triangle$ while $\{e_2, e_3\} \notin \triangle * \square$.

5. If $X = (C_2 \times C_2) \sqcup \{e_2\}$ where $C_2 \times C_2 = \{e_1, a, b, ab\}$ and $a^2 = b^2 = (ab)^2 = e_1$, then the maximal linked upfamilies
$\square = \langle \{a, b\}, \{a, e_1\}, \{a, e_2\}, \{e_1, e_2, b\}\rangle$ and $\triangle = \langle \{e_1, e_2\}, \{e_1, a\}, \{e_2, a\}\rangle$

do not commute, since $\{a,e_2\}=a\{e_1,e_2\}\cup e_2\{e_2,a\}\in\square*\triangle$ and $\{a,e_2\}\notin\triangle*\square$.

6. If $X=\{e_1\}\sqcup(C_2\times C_2)$ where $C_2\times C_2=\{e_2,a,b,ab\}$ and $a^2=b^2=(ab)^2=e_2$, then the maximal linked upfamilies $\square=\langle\{e_1,e_2\},\{e_1,a\},\{e_1,b\},\{e_2,a,b\}\rangle$ and $\triangle=\langle\{e_2,a\},\{e_2,b\},\{a,b\}\rangle$ do not commute, since $\{e_1,e_2\}=e_2\{e_1,e_2\}\cup a\{e_1,a\}\in\triangle*\square$ and $\{e_1,e_2\}\notin\square*\triangle$.

7. Finally assume that $X=C_2\sqcup C_2=\{e_1,a_1\}\cup\{e_2,a_2\}$ where $e_1<e_2$ are idempotents of X, $a_1^2=e_1$, $a_2^2=e_2$, and $e_1*a_2=e_1$. In this case the maximal linked upfamilies

$$\begin{aligned}\square_e&=\big\langle\{e_1,a_1\},\{e_1,a_2\},\{e_1,e_2\},\{a_1,a_2,e_2\}\big\rangle\quad\text{and}\\ \square_a&=\big\langle\{a_1,e_1\},\{a_1,e_2\},\{a_1,a_2\},\{e_1,e_2,a_2\}\big\rangle\end{aligned}$$

do not commute as $\{e_1,e_2\}=e_1\{e_1,e_2\}\cup e_2\{e_1,e_2\}\cup a_2\{e_1,a_2\}\in\square_a*\square_e$ while $\{e_1,e_2\}\notin\square_e*\square_a$. □

Now we are ready to prove the "only if" part of Theorem 9.5. Assume that the superextension $\lambda(X)$ is commutative. In this case the regular semigroup X is commutative and consequently X is a Clifford inverse semigroup. By Theorem 5.1 of [6], the commutativity of $\lambda(X)$ implies that each subgroup of X has cardinality $\leqslant 4$. By Theorem 2.7 [5], the idempotent band $E(X)=\{x\in X:xx=x\}$ of X is a 0-bouquet of finite linear semilattices.

First we assume that $E(X)$ is a finite linear semilattice, which can be written as $E(X)=\{e_1,\dots,e_n\}$ for some idempotents $e_1<\dots<e_n$. For every $i\in\{1,\dots,n\}$ by H_{e_i} we denote the maximal subgroup of X containing the idempotent e_i. As we have shown the group H_{e_i} has cardinality $|H_{e_i}|\leqslant 4$.

If $n=1$, then the Clifford inverse semigroup X coincides with the group H_{e_1} and hence is isomorphic to $C_1=L_1$, C_2, C_3, C_4 or $C_2\times C_2$.

So, we assume that $n\geqslant 2$. Lemma 9.6(2,5) implies that for every $i<n$ the maximal subgroup H_{e_i} has cardinality $|H_{e_i}|\leqslant 2$. For the maximal idempotent e_n of $E(X)$ the complement $I=X\setminus H_{e_n}$ is an ideal in X. So, we can consider the quotient semigroup X/I, which is isomorphic to $L_1\sqcup H_{e_n}$. The commutativity of $\lambda(X)$ implies the commutativity of the semigroup $\lambda(X/I)$. Now Lemma 9.6(1,6) implies that $|H_{e_n}|\leqslant 2$.

If $|E(X)|\geqslant 3$, then for any $1<i<n$, the maximal subgroup H_{e_i} is trivial according to Lemma 9.6(3) and then for the maximal idempotent e_n, the subgroup H_{e_n} is trivial according to Lemma 9.6(4). Therefore, all maximal groups H_{e_i}, $1<i\leqslant n$, are trivial. If the group H_{e_1} is trivial,

then $X = E(X)$ is isomorphic to the linear semilattice L_n. If H_{e_1} is not trivial, then H_{e_1} is isomorphic to C_2 and X is isomorphic to $C_2 \sqcup L_{n-1}$.

It remains to consider the case $|E(X)| = 2$. In this case the groups H_{e_1}, H_{e_2} have cardinality $\leqslant 2$ and then X is isomorphic to L_2, $C_2 \sqcup L_1$, $L_1 \sqcup C_2$, $C_2 \times L_2$ or $C_2 \sqcup C_2$. However the case $X \cong C_2 \sqcup C_2$ is excluded by Lemma 9.6(7). This completes the proof of the case of linear semilattice $E(X)$.

Now we consider the case of non-linear semilattice $E(X)$. Write $E(X)$ as a 0-bouquet $E(X) = \bigvee_{\alpha\in I} E_\alpha$ of finite linear semilattices E_α. Let e_0 be the minimal idempotent of the semilattice $E(X)$. Since $E(X)$ is not linear, there are two idempotents $e_1, e_2 \in E(X) \setminus \{e_0\}$ such that $e_1 e_2 = e_0$. We claim that the maximal subgroup H_{e_0} containing the idempotent e_0 is trivial. It follows from the "linear" case, that $|H_{e_0}| \leqslant 2$. Assuming that H_{e_0} is not trivial, write $H_{e_0} = \{a, e_0\}$ and consider the maximal linked upfamilies $\triangle_0 = \langle\{a, e_1\}, \{e_1, e_2\}, \{a, e_2\}\rangle$ and $\triangle_a = \langle\{e_0, e_1\}, \{e_1, e_2\}, \{e_0, e_2\}\rangle$ which do not commute since $\triangle_0 * \triangle_a = \langle\{e_0\}\rangle \neq \langle\{a\}\rangle = \triangle_a * \triangle_0$. Consequently, the maximal subgroup H_{e_0} is trivial and hence for every $\alpha \in A$ the subsemigroup $X_\alpha = \bigcup_{e\in E_\alpha} H_e$ is isomorphic to $L_1 \sqcup C_2$ or L_n, $n \in \mathbb{N}$, by the preceding "linear" case. □

Theorems 9.3 and 9.5 imply:

Corollary 9.7. *If a semigroup X has commutative superextension $\lambda(X)$, then*

1) *for each $x \in X$ there is a pair $(n, m) \in \{(2,5), (2,6), (3,5), (4,5)\}$ such that $x^n = x^m$;*
2) *the idempotent semilattice $E(X) = \{x \in X : xx = x\}$ of X is a 0-bouquet of finite linear semilattices;*
3) *the regular part $R(X) = \{x \in X : x \in xXx\}$ of X is isomorphic to one of the following semigroups:*
   - *L_1, C_2, C_3, C_4, $C_2 \times C_2$, $C_2 \times L_2$, $C_2 \bigsqcup L_n$ for some $n \in \mathbb{N}$;*
   - *a 0-bouquet $\bigvee_{\alpha\in A} X_\alpha$ of subsemigroups X_α, $\alpha \in I$, isomorphic to $L_1 \sqcup C_2$ or L_n for $n \geqslant 2$.*

10. Supercommutativity of superextensions $\lambda(X)$

By Theorems 6.3, 7.1, 7.2, 8.1, 8.2, for any semigroup X, the semigroups $\upsilon(X)$, $\upsilon^\bullet(X)$, $\varphi(X)$, $\varphi^\bullet(X)$, $N_2(X)$, $N_2^\bullet(X)$ are supercommutative if and only if they are commutative. In contrast, the supercommutativity of the superextension $\lambda(X)$ is not equivalent to its commutativity.

Theorem 10.1. *For a monogenic semigroup $X = \{x^k\}_{k\in\mathbb{N}}$ the following conditions are equivalent:*

1) *the semigroup $\lambda(X)$ is supercommutative;*
2) *the semigroup $\lambda^\bullet(X)$ is supercommutative;*
3) *$x^n = x^m$ for some $(n,m) \in \{(1,2),(1,3),(2,3),(2,4),(3,4),(4,5)\}$.*

Proof. We shall prove the implications (3) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) among which the implication (1) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (1). Assume that $x^n = x^m$ for some pair (n,m) from the set $\{(1,2),(1,3),(2,3),(2,4),(3,4),(4,5)\}$. For $(n,m) \in \{(1,2),(1,3),(2,3)\}$ the monogenic semigroup X has cardinality $|X| \leqslant 2$ and then the semigroup $\lambda(X) = X$ is supercommutative.

If $(n,m) = (2,4)$, then the monogenic semigroup X has cardinality $|X| = 3$ and for the unique non-principal maximal linked system $\triangle = \{A \subset X : |A| \geqslant 2\}$ in $\lambda(X)$ the product $\triangle \circledast \triangle$ is equal to the principal ultrafilter $\langle x^2\rangle = \triangle * \triangle$, which implies that the semigroup $\lambda(X)$ is supercommutative.

If $(n,m) \in \{(3,4),(4,5)\}$, then any two nonprincipal maximal linked systems $\mathcal{A},\mathcal{B}$ contain sets $A \in \mathcal{A}, B \in \mathcal{B}$ such that $x \notin A, x \notin B$. Then AB is a singleton, which implies $\mathcal{A} \circledast \mathcal{B} = \mathcal{A} * \mathcal{B}$. Consequently, the semigroup $\lambda(X)$ is supercommutative.

(2) $\Rightarrow$ (3) Assume that for a monogenic semigroup $X = \{x^k\}_{k\in\mathbb{N}}$ the superextension $\lambda^\bullet(X)$ is supercommutative. Then it is commutative and by Theorem 9.3, $x^n = x^m$ for some pair (n,m) from the set

$$\{(1,2),(1,3),(2,3),(1,4),(2,4),(3,4),(1,5),(2,5),(3,5),(4,5),(2,6)\}.$$

We claim that $|m-n| \leqslant 2$. In the opposite case X contains a cyclic subgroup C of cardinality $|C| \geqslant 3$. The subgroup C contains an element $x \in C$ such that the points x^{-1}, x^0, x^1 are pairwise distinct. Then for the maximal linked system $\triangle = \langle\{x^{-1},x^0\},\{x^0,x^1\},\{x^{-1},x^1\}\rangle \in \lambda^\bullet(C) \subset \lambda^\bullet(X)$ the product

$$\triangle \circledast \triangle = \langle\{x^{-2},x^{-1},x^0\},\{x^{-1},x^0,x^1\},\{x^0,x^1,x^2\}\rangle$$

does not belong to $\lambda(C)$, which implies that $\triangle \circledast \triangle \neq \triangle * \triangle$ and contradicts the supercommutativity of $\lambda(X)$. So, $|m-n| \leqslant 2$, which implies that $(n,m) \in \{(1,2),(1,3),(2,3),(2,4),(3,4),(3,5),(4,5)\}$. It remains to exclude the case $(n,m) = (3,5)$. In this case $X = \{x, x^2, x^3, x^4\}$ and for the maximal linked upfamilies $\square = \langle\{x^2,x^3,x^4\},\{x,x^2\},\{x,x^3\},\{x,x^4\}\rangle$ and $\triangle = \langle\{x,x^2\},\{x,x^3\},\{x^2,x^3\}\rangle$ we get

$$\square \circledast \triangle = \langle\{x^2,x^4\},\{x^3,x^4\}\rangle \neq \square * \triangle,$$

which contradicts the supercommutativity of the semigroup $\lambda(X)$. $\square$

In the following theorem by V_3 we denote the semilattice $\{0,1\}^2 \setminus \{(1,1)\}$ endowed with the operation of coordinatewise minimum. Observe that a semilattice X is isomorphic to V_3 if and only if $|X| = 3$ and X is not linear.

Theorem 10.2. *The superextension $\lambda(X)$ of a regular semigroup X is supercommutative if and only if X is isomorphic to one of the semigroups: C_2, $L_1 \sqcup C_2$, V_3 or L_n for $n \in \mathbb{N}$.*

Proof. First we prove the "if" part of the theorem. If $X = C_2$, then its superextension $\lambda(X) = X$ is supercommutative as all maximal linked upfamilies on X are principal ultrafilters.

If $X = L_1 \sqcup C_2$, then $\lambda(X)$ is supercommutative since for the unique non-principal maximal linked system $\triangle = \{A \subset X : |A| \geqslant 2\}$ we get $\triangle \circledast \triangle = \triangle = \triangle * \triangle$.

If $X = V_3$, then $\lambda(X)$ is supercommutative since for the unique non-principal maximal linked system $\triangle = \{A \subset X : |A| \geqslant 2\}$ the products $\triangle \circledast \triangle = \langle \min V_3 \rangle = \triangle * \triangle$ coincide with the principal ultrafilter generated by the minimal element $(0,0) = \min V_3$ of the semilattice V_3.

If $X = L_n$ for some $n \in \mathbb{N}$, then the supercommutativity of the semigroup $\lambda(X)$ follows from Theorem 2.5 of [4].

To prove the "only if" part, assume that X is a regular semigroup with supercommutative superextension $\lambda(X)$. First observe that every subgroup G of X has cardinality $|G| \leqslant 2$. In the opposite case the group G contains an element $x \in X$ such that $|\{x^1,x^0,x^{-1}\}| = 3$ where x^0 is the idempotent of the group G. Then for the maximal linked system $\triangle = \langle\{x^{-1},x^0\},\{x^0,x^1\},\{x^{-1},x^1\}\rangle$ the product $\triangle \circledast \triangle = \langle\{x^{-2},x^{-1},x^0\},\{x^{-1},x^0,x^1\},\{x^0,x^1,x^2\}\rangle$ does not belong to $\lambda(X)$ and hence is not equal to $\triangle * \triangle$. This contradiction shows that all subgroups of X has cardinality $\leqslant 2$. This fact combined with Theorem 9.5 yields that X is isomorphic to one of the semigroups:

- L_1, C_2, $C_2 \bigsqcup L_n$ for some $n \in \mathbb{N}$;
- a 0-bouquet $\bigvee_{\alpha \in I} X_\alpha$ of subsemigroups X_α, $\alpha \in I$, isomorphic to $L_1 \sqcup C_2$ or L_n for $n \geqslant 2$.

It remains to exclude the semigroups from this list, whose superextensions are not supercommutative.

If $X = C_2 \bigsqcup L_n$, then X contains the semigroup $C_2 \sqcup L_1 = \{e_1, a\} \cup \{e_2\}$ where $a^2 = e_1 \neq a$ and $e_1 < e_2$ are idempotents. In this case for the maximal linked system $\triangle = \langle\{a, e_1\}, \{e_1, e_2\}, \{a, e_2\}\rangle$ we get $\triangle \circledast \triangle = \langle\{a, e_1\}, \{e_1, e_2\}\rangle \notin \lambda(X)$ and hence $\triangle \circledast \triangle \neq \triangle * \triangle$, which means that $\lambda(X)$ is not supercommutative.

If $X = L_1 \sqcup C_2 = \{e_1\} \cup \{e_2, a\}$ where $a^2 = e_2 > e_1$, then for the maximal linked system $\triangle = \langle\{a, e_1\}, \{e_1, e_2\}, \{a, e_2\}\rangle$ we get $\triangle \circledast \triangle = \langle\{e_1, e_2\}, \{e_2, a\}\rangle \notin \lambda(X)$ and hence $\triangle \circledast \triangle \neq \triangle * \triangle$, which means that $\lambda(X)$ is not supercommutative.

It remains to consider the case when $X = \bigcup_{\alpha \in I} X_\alpha$ is a 0-bouquet of subsemigroups X_α, $\alpha \in I$, isomorphic to L_n for $n \geqslant 2$. If $|I| = 1$, then X is isomorphic to L_n for some $n \geqslant 2$ and $\lambda(X)$ is supercommutative according to the “if”part.

If $|I| = 2$, then $X = X_i \vee X_j$ for some non-trivial linear subsemilattices $X_i, X_j \subset X$ such that $X_j * X_j = X_i \cap X_j = \{\min X\}$. If $|X_i| = |X_j| = 2$, then the semilattice X is isomorphic to the semilattice V_3 and its superextension $\lambda(X)$ is supercommutative as proved in the “if” part. So, we assume that $|X_i| \geqslant 3$ or $|X_j| \geqslant 3$. We loss no generality assuming that $|X_i| \geqslant 3$. Then we can find idempotents $e_0 < e_1 < e_2$ in X_i and $e_3 \in X_j \setminus X_i$ such that $e_1 e_3 = e_2 e_3 = e_0 = \min X$. In this case for the maximal linked system $\triangle = \langle\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\rangle$ the product $\triangle \circledast \triangle = \langle\{e_0, e_1\}, \{e_1, e_2\}\rangle \notin \lambda(X)$ and hence $\triangle \circledast \triangle \neq \triangle * \triangle$, which means that $\lambda(X)$ is not supercommutative.

If $|I| \geqslant 3$, then the semigroup X contains a 4-element semilattice $V_4 = \{e_0, e_1, e_2, e_3\}$ where $e_i e_j = e_0 = \min X$ for any distinct number $i, j \in \{1, 2, 3\}$. In this we can consider the maximal linked system $\triangle = \langle\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\rangle \in \lambda(V_4) \subset \lambda(X)$ and observe that $\triangle \circledast \triangle = \langle\{e_0, e_1\}, \{e_0, e_2\}, \{e_0, e_3\}\rangle \notin \lambda(X)$. Consequently, $\triangle \circledast \triangle \neq \triangle * \triangle$ and the semigroup $\lambda(X)$ is not supercommutative. □

Theorems 10.1 and 10.2 imply:

Corollary 10.3. *If a semigroup X has supercommutative superextension $\lambda(X)$, then*

1) *for each $x \in X$ we get $x^4 \in \{x^2, x^5\}$;*
2) *the regular part $R(X) = \{x \in X : x \in xXx\}$ of X is isomorphic to C_2, $L_1 \sqcup C_2$, V_3 or L_n for some $n \in \mathbb{N}$.*

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