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# Lattice groups

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*Dedicated to Professor Efim Zelmanov in occasion of his 60th birthday*

Abstract. In this paper, we introduce some algebraic structure associated with groups and lattices. This structure is a semigroup and it appeared as the result of our new approach to the fuzzy groups and $L$-fuzzy groups where $L$ is a lattice. This approach allows us to employ more convenient language of algebraic structures instead of currently accepted language of functions.

The purpose of this work is to look with a somewhat different angle at algebraic structures related to the functions defined on a group. For every subset $M$ of a set $S$ there exists its characteristic function, that is the mapping $\chi_M \colon S \to \{0,1\}$ such that $\chi_M(y) = 1$ for all $y \in M$ and $\chi_M(y) = 0$ for all $y \notin M$. In many commonly used cases, a subset of $M$ is identified with its characteristic function. In 1965, L.A. Zadeh [6] based on his generalization of the characteristic function introduced the fuzzy mathematics. Thus, a fuzzy set on a set $S$ is a sort of generalized "characteristics function" on $S$, for whose "degrees of membership" we can use more diverse set than simple {yes, no}. In fact, we can consider the set $L$ of degrees of membership. In the optimization problems, $L$ may express the degree of optimality of the choice (in $S$); in the classification problems, it may express the degree of membership in a pattern class; in other contexts other terminologies appear. In fuzzy mathematics, a habitual step was to review the situation when $L = [0,1]$ is the usual closed interval of real numbers with its natural order. The following

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interpretation justifies this approach: we can consider a value of the generalized characteristic function as a probability of the fact that the given element belongs to the given subset. In this way, the algebraic fuzzy structures were constructed as follows. With every algebraic structure $A$, a corresponding fuzzy structure which characterized by a specific functions of $A$ on $[0,1]$ associated with this conventional algebraic structure $A$, was connected (see, for example, [4]). For instance, in fuzzy group theory the objects of study are the functions $\gamma\colon G \to [0,1]$, $G$ is a group, satisfying the following conditions:

$$\gamma(xy) \geqslant \gamma(x) \wedge \gamma(y) \text{ for all } x, y \in G; \text{ and } \gamma(x^{-1}) \geqslant \gamma(x) \text{ for every } x \in G$$

(see, for example, [5] S 1.2). Some generalizations have appeared immediately. More concretely, considerations of the function $\gamma\colon G \to \mathfrak{L}$ where $\mathfrak{L}$ is a distributive lattice [1] were initiated. The theory of fuzzy groups was developed quite rapidly. However it was upswing in breadth rather than depth development. A variety of results obtained there was not planned properly. Even in the book [5], there were no attempts to systematize these results. A large array of results on fuzzy groups just has been collected in this book with no proper arrangement. In the $L$-fuzzy groups, regardless of the most common results, there was no serious progress at all.

Perhaps the key obstacle here is in the interpretation of an algebraic structure as a function, which is not very convenient most of the time. Because of that, very often the function $\gamma$ is interpreted as an all point function $\chi(g, \gamma(g)), g \in G$. Here $\chi(g, \mathfrak{a})$ is a function such that $\chi(g, \mathfrak{a})(g) = \mathfrak{a}$, $\chi(g, \mathfrak{a})(y) = 0$ whenever $y \neq g$. However, in some cases we need to consider the function $\gamma$ as an union of all point functions $\chi(g, \mathfrak{a})$ for all $g \in G$ and $\mathfrak{a} \leqslant \gamma(g)$ (see, for example, [2], [3]). Actually speaking, the point functions $\chi(g, \mathfrak{a})$ play here the role of elements, formally the subfunctions of $\gamma$, so that each time it is necessary to implement keep in mind some special reservations.

In the current article we offer the interpretation of $L$-fuzzy groups as sets with operations. With this algebraic approach, the basic concepts and results of algebraic nature acquire its natural form, and the process of their appearance becomes more meaningful. We present the basic concepts of the theory of $L$-fuzzy groups, as well as the results in the form in which they are needed to be for our transformation. The resulting structure is formally different, and therefore the term for it to be used is different. In the article we are concerned only with the basic concepts, but nevertheless, our approach will make it possible to see the general structural picture.

As for the term $L$-fuzzy group, it seems it does not reflect the essence of the case, so we will use the term group function. We do not seek maximize generality, it seems more natural to consider the case, when lattice $L$ is distributive and finite, although the obtained results can be extended on the case of an arbitrary complete distributive lattice.

Let $\mathfrak{L}$ be a lattice and $G$ be a group. To avoid misunderstandings, the identity element of $G$ is denoted by $e$. We will consider a set $\mathfrak{L}^G$ of all functions $\lambda\colon G \to \mathfrak{L}$. On this set we define the operations $\wedge$ and $\vee$ by the following rules: if $\lambda, \mu \in \mathfrak{L}^G$, then put

$$(\lambda\wedge\mu)(x) = \lambda(x)\wedge\mu(x) \quad \text{and} \quad (\lambda\vee\mu)(x) = \lambda(x)\vee\mu(x) \quad \text{for each } x \in G.$$

Clearly the operations $\wedge$ and $\vee$ are commutative and associative,

$$(\lambda \wedge (\lambda \vee \mu))(x) = \lambda(x) \wedge (\lambda \vee \mu)(x) = \lambda(x) \wedge (\lambda(x) \vee \mu(x)) = \lambda(x)$$

and

$$(\lambda \vee (\lambda \wedge \mu))(x) = \lambda(x) \vee (\lambda \wedge \mu)(x) = \lambda(x) \vee (\lambda(x) \wedge \mu(x)) = \lambda(x),$$

so that $\lambda \wedge (\lambda \vee \mu) = \lambda$ and $\lambda \vee (\lambda \wedge \mu) = \lambda$. Clearly $\lambda \wedge \lambda = \lambda$ and $\lambda \vee \lambda = \lambda$. Hence a set $\mathfrak{L}^G$ is a lattice.

If $a, b \in \mathfrak{L}$, then $a\vee b = b$ is equivalent to $a \leqslant b$. Therefore we can define an order on $\mathfrak{L}^G$: for $\lambda, \mu \in \mathfrak{L}^G$ will put $\lambda \leqslant \mu$ if and only if $\lambda(x) \leqslant \mu(x)$ for each element $x \in G$.

Suppose now that a lattice $\mathfrak{L}$ is *distributive and finite*. Being finite, it has the greatest element $\mathfrak{m}$ and the least element $\mathfrak{o}$. For every function $f\colon G \to \mathfrak{L}$ define $\mathrm{Supp}(f)$ as a subset of all elements $x \in G$ such that $f(x) \neq \mathfrak{o}$.

Let $Y$ be a subset of $G$ and $\mathfrak{a} \in \mathfrak{L}$. We define the function $\chi(Y, \mathfrak{a})$ as follows:

$$\chi(Y, \mathfrak{a}) = \begin{cases} \mathfrak{a}, & \text{if } x \in Y \\ \mathfrak{o}, & \text{if } x \notin Y. \end{cases}$$

If $Y = \{y\}$, then instead of $\chi(\{y\}, \mathfrak{a})$ we will write $\chi(y, \mathfrak{a})$. The function $\chi(y, \mathfrak{a})$ is called the *point function* or shorter the *point*. By its definition, $\chi(y, \mathfrak{a}) \in \mathfrak{L}^G$. Furthermore, let $f \in \mathfrak{L}^G$. If $\mathrm{Supp}(f) = \{g_1, \dots, g_n\}$ is finite and $f(g_j) = \mathfrak{a}_\mathfrak{j}$, $1 \leqslant j \leqslant n$, then clearly $f = \chi(g_1, \mathfrak{a}_1) \vee \ldots \vee \chi(g_n, \mathfrak{a}_\mathfrak{n})$.

Define now the binary operation $\odot$ on $\mathfrak{L}^G$ by the following rule. Let $\mu, \nu \in \mathfrak{L}^G$, and $x$ be an arbitrary element of a group $G$. Consider the subset of the lattice $\mathfrak{L}$

$$\{\mu(y) \wedge \nu(z) | u, v \text{ are the elements of } G \text{ such that } yz = x\}.$$

Since $\mathfrak{L}$ is finite, this subset really is finite. Therefore we can define about its least upper bound. Put

$$(\mu \odot \nu)(x) = \vee_{y,z \in G, yz=x}(\mu(y) \wedge \nu(z)).$$

We remark that

$$(\mu \odot \nu)(x) = \vee_{y \in G}(\mu(y) \wedge \nu(y^{-1}x)) = \vee_{z \in G}(\mu(xz^{-1}) \wedge \nu(z)).$$

Consider now some basic properties of this product.

**Proposition 1.** *The following assertions hold:*

(i) *The operation $\odot$ is associative.*

(ii) *The function $\chi(e, \mathfrak{m})$ is an identity element of the operation $\odot$.*

(iii) $\lambda \odot (\mu \vee \nu) = (\lambda \odot \mu) \vee (\lambda \odot \nu)$ *and* $(\mu \vee \nu) \odot \lambda = (\mu \odot \lambda) \vee (\nu \odot \lambda)$ *for all functions* $\lambda, \mu, \nu \in \mathfrak{L}^G$.

(iv) *If* $x, y \in G, \mathfrak{a} \in \mathfrak{L}$, *then* $(\chi(y, \mathfrak{a}) \odot \lambda)(x) = \mathfrak{a} \wedge \lambda(y^{-1}x)$; *in particular, if* $\mathfrak{a} = \vee_{x \in G}\lambda(x)$, *then* $((\chi(y, \mathfrak{a}) \odot \lambda)(x) = \lambda(y^{-1}x)$.

(v) $(\lambda \odot (\chi(y, \mathfrak{a})))(x) = \mathfrak{a} \wedge \lambda(xy^{-1})$; *in particular, if* $\mathfrak{a} = \vee_{x \in G}\lambda(x)$, *then* $(\lambda \odot \chi(y, \mathfrak{a}))(x) = \lambda(xy^{-1})$.

(vi) *if* $x, y, u \in G$, $\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}$ *then* $(\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}))(yu) = \mathfrak{a} \wedge \mathfrak{b}$ *and* $(\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}))(x) = \mathfrak{o}$ *if* $x \neq yu$. *In other words,* $\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}) = \chi(yu, \mathfrak{a} \wedge \mathfrak{b})$; *in particular,* $\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{a}) = \chi(yu, \mathfrak{a})$.

(vii) $(\chi(x, \mathfrak{a}) \odot \lambda \odot \chi(x^{-1}, \mathfrak{a}))(y) = \mathfrak{a} \wedge \lambda(x^{-1}yx)$.

*Proof.* (i) Let $\lambda, \mu, \nu \in \mathfrak{L}^G$. Put $\kappa = \lambda \odot \mu$ and $\eta = \mu \odot \nu$. We have

$$\begin{aligned}((\lambda \odot \mu) \odot \nu)(x) &= (\kappa \odot \nu)(x) = \vee_{y,z \in G, yz=x}(\kappa(y) \wedge \nu(z)) \\ &= \vee_{y,z \in G, yz=x}(\vee_{u,v \in G, uv=y}(\lambda(u) \wedge \mu(v)) \wedge \nu(z)) \\ &= \vee_{u,v,z \in G, uvz=x}((\lambda(u) \wedge \mu(v)) \wedge \nu(z)).\end{aligned}$$

$$\begin{aligned}(\lambda \odot (\mu \odot \nu))(x) &= (\lambda \odot \eta)(x) = \vee_{u,y \in G, uy=x}(\lambda(u) \wedge \eta(y)) \\ &= \vee_{u,y \in G, uy=x}(\lambda(u) \wedge (\vee_{v,z \in G, vz=y}(\mu(v) \wedge \nu(z)))) \\ &= \vee_{u,v,z \in G, uvz=x}(\lambda(u) \wedge (\mu(v) \wedge \nu(z))).\end{aligned}$$

Since $(\lambda(u) \wedge \mu(v)) \wedge \nu(z) = \lambda(u) \wedge (\mu(v) \wedge \nu(z))$ for all $u, v, z \in G$,

$$((\lambda \odot \mu) \odot \nu)(x) = (\lambda \odot (\mu \odot \nu))(x)$$

for each $x \in G$. It implies that $(\lambda \odot \mu) \odot \nu = \lambda \odot (\mu \odot \nu)$.

(ii) Let $\lambda \in \mathfrak{L}^G$ and consider the product $\lambda \odot \chi(e, \mathfrak{m})$. By its definition, $(\chi(e, \mathfrak{m}))(e) = \mathfrak{m}$ and $(\chi(e, \mathfrak{m}))(x) = \mathfrak{o}$ whenever $x \neq 1$. We have now

$$\begin{aligned}&\lambda(x) \wedge (\chi(e, \mathfrak{m}))(e) = \lambda(x) \wedge \mathfrak{m} = \lambda(x)\\ &\text{and } \lambda(y) \wedge (\chi(e, \mathfrak{m}))(z) = \mathfrak{o} \quad \text{if } z \neq 1,\end{aligned}$$

so that

$$\begin{aligned}(\lambda \odot \chi(e, \mathfrak{m}))(e) &= \vee_{y,z \in G, yz=1}(\lambda(y) \wedge \chi(e, \mathfrak{m})(z))\\ &= \lambda(e) \wedge \chi(e, \mathfrak{m})(e) = \lambda(e),\\ (\lambda \odot \chi(e, \mathfrak{m}))(x) &= \vee_{y,z \in G, yz=x}(\lambda(y) \wedge \chi(e, \mathfrak{m})(z))\\ &= \lambda(x) \wedge \chi(e, \mathfrak{m})(e) = \lambda(x).\end{aligned}$$

Since it is valid for all $x \in G$, $\lambda \odot \chi(e, \mathfrak{m}) = \lambda$. In a similar way we can prove that $\chi(e, \mathfrak{m}) \odot \lambda = \lambda$.

(iii) We have

$$\begin{aligned}\lambda \odot (\mu \vee \nu)(x) &= \vee_{y \in G}(\lambda(y) \wedge ((\mu \vee \nu)(y^{-1}x)))\\ &= \vee_{y \in G}(\lambda(y) \wedge (\mu(y^{-1}x) \vee \nu(y^{-1}x)))\\ &= \vee_{y \in G}(\lambda(y) \wedge \mu(y^{-1}x)) \vee (\lambda(y) \wedge \nu(y^{-1}x))\\ &= (\vee_{y \in G}(\lambda(y) \wedge \mu(y^{-1}x))) \vee (\vee_{y \in G}(\lambda(y) \wedge \nu(y^{-1}x)))\\ &= (\lambda \odot \mu)(x) \vee (\lambda \odot \nu)(x)\\ &= ((\lambda \odot \mu) \vee (\lambda \odot \nu))(x).\end{aligned}$$

It proves that

$$\lambda \odot (\mu \vee \nu) = (\lambda \odot \mu) \vee (\lambda \odot \nu).$$

Using similar arguments, we obtain that and

$$(\mu \vee \nu) \odot \lambda = (\mu \odot \lambda) \vee (\nu \odot \lambda).$$

(iv) Let $x$ be an arbitrary element of $G$. If $z \neq y$, then $\chi(y, \mathfrak{a})(z) = \mathfrak{o}$, so we have

$$\begin{aligned}(\chi(y, \mathfrak{a}) \odot \lambda)(x) &= \vee_{z \in G}(\chi(y, \mathfrak{a})(z) \wedge \lambda(z^{-1}x))\\ &= \chi(y, \mathfrak{a})(y) \wedge \lambda(y^{-1}x)) = \mathfrak{a} \wedge \lambda(y^{-1}x).\end{aligned}$$

The proof of (v) is similar.

(vi) If $u \in G$, $\mathfrak{b} \in \mathfrak{L}$, then $(\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}))(x) = \mathfrak{a} \wedge \chi(u, \mathfrak{b})(y^{-1}x)$. Recall that $\chi(u, \mathfrak{b})(y^{-1}x) = \mathfrak{b}$ if $y^{-1}x = u$ or $x = yu$ and $\chi(u, \mathfrak{b})(y^{-1}x) = \mathfrak{o}$ if $y^{-1}x \neq u$ or $x \neq yu$. Thus

$$(\chi(y, \mathfrak{a}) \odot \chi(u, \mathfrak{b}))(x) = \begin{cases} \mathfrak{a} \wedge \mathfrak{b}, & \text{if } x = yu \\ \mathfrak{o}, & \text{if } x \neq yu. \end{cases}$$

Hence we obtain (vi).

(vi) Using the above arguments we obtain

$$\begin{aligned} &(\chi(x, \mathfrak{a}) \odot (\gamma \odot \chi(x^{-1}, \mathfrak{a})))(y) \\ &\qquad = \vee_{u,v,z \in G, uvz = y} \chi(x, \mathfrak{a})(u) \wedge (\gamma(v) \wedge \chi(x^{-1}, \mathfrak{a}))(z) \\ &\qquad = \chi(x, \mathfrak{a})(x) \wedge \gamma(x^{-1}yx) \wedge \chi(x^{-1}, \mathfrak{a})(x^{-1}) \\ &\qquad = \mathfrak{a} \wedge \gamma(x^{-1}yx) \wedge \mathfrak{a} = \mathfrak{a} \wedge \gamma(x^{-1}yx). \end{aligned}$$ □

Let $G$ be a group and $\gamma \in \mathfrak{L}^G$. Then a surjective function $\gamma$ is said to be a *group function on* $G$ if it satisfies the following conditions:

(GF 1) $\gamma(xy) \geqslant \gamma(x) \wedge \gamma(y)$ for all $x, y \in G$,

(GF 2) $\gamma(x^{-1}) \geqslant \gamma(x)$ for every $x \in G$.

Let $\gamma, \kappa$ group functions on $G$. If $\gamma \leqslant \kappa$, then we will say that $\gamma$ *is a subgroup function of* $\kappa$. This fact we will denote $\gamma \preceq \kappa$.

**Proposition 2.** *Let $G$ be a group, $\mathfrak{L}$ be a finite distributive lattice, $\gamma \in \mathfrak{L}^G$, and suppose that $\gamma$ is a group function on $G$. Then the following assertions hold:*

(i) *$\gamma(x^{-1}) = \gamma(x)$ for every $x \in G$ (in order words, a function $\gamma$ is even).*

(ii) *$\gamma(xy^{-1}) \geqslant \gamma(x) \wedge \gamma(y)$ for all $x, y \in G$.*

(iii) *$\gamma(x^n) \geqslant \gamma(x)$ for every $x \in G$ and every integer $n$.*

(iv) *$\gamma(e) \geqslant \gamma(x)$ for every $x \in G$.*

(v) *Let $\lambda$, $\kappa \leqslant \gamma$, then $\lambda \odot \kappa \leqslant \gamma$, in particular, $\gamma \odot \gamma \leqslant \gamma$.*

*Proof.* (i) We have $x = (x^{-1})^{-1}$, so (GF 2) implies that $\gamma(x) \geqslant \gamma(x^{-1})$, which together with $\gamma(x^{-1}) \geqslant \gamma(x)$ gives $\gamma(x) = \gamma(x^{-1})$ for every element $x \in G$.

(ii) Let $x$, $y$ be arbitrary elements of $G$. By (GF 1) $\gamma(xy^{-1}) \geqslant \gamma(x) \wedge \gamma(y^{-1})$, and by (i) $\gamma(y^{-1}) = \gamma(y)$, so that $\gamma(xy^{-1}) \geqslant \gamma(x) \wedge \gamma(y)$.

(iii) Let $x \in G$. By (GF 1) $\gamma(x^2) = \gamma(xx) \geqslant \gamma(x) \wedge \gamma(y) = \gamma(x)$. Using ordinary induction, we obtain that $\gamma(x^n) \geqslant \gamma(x)$ for every $n \in \mathbb{N}$. Suppose now that $n = -k$ where $k \in \mathbb{N}$. Then $x^n = (x^{-1})^k$. By proved above

$$\gamma(x^n) = \gamma((x^{-1})^k) \geqslant \gamma(x^{-1}) = \gamma(x).$$

(iv) Let $x \in G$. By (GF 1) we have

$$\gamma(e) = \gamma(xx^{-1}) \geqslant \gamma(x) \wedge \gamma(x^{-1}) = \gamma(x) \wedge \gamma(x) = \gamma(x).$$

(v) Let $x$ be an arbitrary element of $G$. The inclusions $\lambda$, $\kappa \leqslant \gamma$ imply $\lambda(y) \wedge \kappa(z) \leqslant \gamma(y) \wedge \gamma(z)$. Since $\gamma$ is a group function, $\gamma(y) \wedge \gamma(z) \leqslant \gamma(yz)$, thus

$$(\lambda \odot \kappa)(x) = \vee_{y,z \in G, yz=x}(\gamma(y) \wedge \kappa(z)) \leqslant \vee_{y,z \in G, yz=x}\gamma(yz) = \gamma(x). \quad \square$$

**Proposition 3** (A criterion of group function)**.** *Let $G$ be a group, $\mathfrak{L}$ be a finite distributive lattice and $\gamma \in \mathfrak{L}^G$. Then $\gamma$ is a group function on $G$ if and only if the following assertions hold:*

(GF 3) $\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)) \subseteq \gamma$ *for all $x, y \in G$.*

(GF 4) $\chi(x^{-1}, \gamma(x)) \subseteq \gamma$ *for every $x \in G$.*

*Proof.* Suppose first that $\gamma$ is a group function. Clearly $\chi(x, \gamma(x)) \subseteq \gamma$ and $\chi(y, \gamma(y)) \subseteq \gamma$ for all elements $x, y \in G$. Using Proposition 2 (v) we obtain that

$$\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)) \subseteq \gamma.$$

Let $x$ be an arbitrary element of $G$. We have $(\chi(x^{-1}, \gamma(x)))(x^{-1}) = \gamma(x)$. Since $\gamma$ is a group function, $\gamma(x) \leqslant \gamma(x^{-1})$. We note that if $y \neq x^{-1}$, then $(\chi(x, \gamma(x)))(y) = \mathfrak{o}$, so that $(\chi(x^{-1}), \gamma(x)))(y) \leqslant \gamma(y)$ for every $y \in G$. This means that $\chi(x^{-1}, \gamma(x)) \subseteq \gamma$.

Conversely, suppose that $\gamma$ satisfies both conditions (GF 3) and (GF 4). Let $x$, $y$ be arbitrary elements of $G$. Then (GF 3) shows that $\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)) \subseteq \gamma$. By Proposition 1 (vi),

$$(\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)))(xy) = \gamma(x) \wedge \gamma(y).$$

The inclusion $\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)) \subseteq \gamma$ implies that $(\chi(x, \gamma(x)) \odot \chi(y, \gamma(y)))(xy) \leqslant \gamma(xy)$, thus we obtain $\gamma(x) \wedge \gamma(y) \leqslant \gamma(xy)$, and $\gamma$ satisfies (GF 1).

Let $x \in G$. Since $\chi(x^{-1}, \gamma(x)) \subseteq \gamma$, $(\chi(x^{-1}, \gamma(x)))(y) \leqslant \gamma(y)$ for every $y \in G$. In particular, $(\chi(x^{-1}, \gamma(x)))(x^{-1}) = \gamma(x) \leqslant \gamma(x^{-1})$, so that $\gamma$ satisfies (GF 2). $\square$

Let $G$ be a group and $\mathfrak{L}$ be a finite distributive lattice. Consider the Cartesian product $A = G \times \mathfrak{L}$. Define the operation (multiplication) on $A$ by the following rule: $(u, \mathfrak{a})(v, \mathfrak{b}) = (uv, \mathfrak{a} \wedge \mathfrak{b})$ for all $u, v \in G$, $\mathfrak{a}, \mathfrak{b} \in \mathfrak{L}$. This operation is associative because multiplication in $G$ and the operation $\wedge$ in $\mathfrak{L}$ are associative. The pair $(e, \mathfrak{m})$ is the identity element for this operation. The obtained above criterion allows us to transform the definition of the group function in the following.

A nonempty subset $\Lambda$ of $G \times \mathfrak{L}$ is called a *lattice group over* $\mathfrak{L}$ if it satisfies the following conditions:

(LG 1) if $(x, \mathfrak{a}) \in \Lambda$ and $\mathfrak{b} \leqslant \mathfrak{a}$, then $(x, \mathfrak{b}) \in \Lambda$;

(LG 2) if $(x, \mathfrak{a})$, $(y, \mathfrak{b}) \in \Lambda$, then $(x, \mathfrak{a})(y, \mathfrak{b}) \in \Lambda$;

(LG 3) if $(x, \mathfrak{a}) \in \Lambda$, then $(x^{-1}, \mathfrak{a}) \in \Lambda$.

For every element $x \in \mathrm{pr}_G(\Lambda)$ put $\mathfrak{C}_\Lambda(x) = \{\mathfrak{a} \in \mathfrak{L} | (x, \mathfrak{a}) \in \Lambda\}$.

Observe at once that a lattice group $\Lambda$ defines a group function on $G$. Indeed, for every element $x \in \mathrm{pr}_G(\Lambda)$ the set $\mathfrak{C}_\Lambda(x)$ is not empty. Put $\lambda(x) = \vee \mathfrak{C}_\Lambda(x)$. If $x \notin \mathrm{pr}_G(\Lambda)$, then put $\lambda(x) = \mathfrak{o}$. Then $\lambda$ is a function. If $u, v \in G$ and $\lambda(u) = \mathfrak{a}$, $\lambda(v) = \mathfrak{b}$, then $(uv, \mathfrak{a} \wedge \mathfrak{b}) \in \Lambda$ by condition (LG 2). It follows that $\lambda(uv) \geqslant \mathfrak{a} \wedge \mathfrak{b} = \lambda(u) \wedge \lambda(v)$, so that $\lambda$ satisfies (GF 1). Similarly, let $\lambda(u) = \mathfrak{a}$, then $(u^{-1}, \mathfrak{a}) \in \Lambda$ by condition (LG 3). It follows that $\lambda(u^{-1}) \geqslant \mathfrak{a} = \lambda(u)$, so that $\lambda$ satisfies (GF 2).

Let $\Lambda$, $\Gamma$ be the lattice groups over $\mathfrak{L}$. If $\Lambda$ includes $\Gamma$, then we will say that $\Gamma$ is a *lattice subgroup* of $\Lambda$, and will denote this by $\Gamma \leqslant \Lambda$.

If $\gamma$ is a defined by $\Gamma$ group function, then $\gamma \preceq \lambda$.

Clearly $G \times \mathfrak{L}$ is the greatest lattice group over $\mathfrak{L}$, and $E = \{(e, \mathfrak{o})\}$ is the least lattice group over $\mathfrak{L}$; the last lattice group is called *trivial*. Furthermore, if $\mathfrak{a} \in \mathfrak{L}$, then $\{(e, \mathfrak{b}) | \mathfrak{b} \leqslant \mathfrak{a}\}$ is a lattice group over $\mathfrak{L}$.

Every lattice group $\Lambda$ includes $\mathrm{pr}_G(\Lambda) \times \{\mathfrak{o}\}$. For every subgroup $H$ of $G$ the subset $H \times \{\mathfrak{o}\}$ is a lattice group. Recall that a subset $\mathfrak{M}$ of $\mathfrak{L}$ is called a *lower* (respectively *upper*) *segment* of $\mathfrak{L}$, if from $\mathfrak{a} \in \mathfrak{M}$ and $\mathfrak{b} \leqslant \mathfrak{a}$ (respectively $\mathfrak{a} \leqslant \mathfrak{b}$) it follows that $\mathfrak{b} \in \mathfrak{M}$.

If $\mathfrak{a} \in \mathfrak{L}$, then the subset $\{\mathfrak{x} | \mathfrak{x} \in \mathfrak{L}$ and $\mathfrak{x} \leqslant \mathfrak{a}\}$ (respectively $\{\mathfrak{x} | \mathfrak{x} \in \mathfrak{L}$ and $\mathfrak{x} \geqslant \mathfrak{a}\}$) is a lower segment (respectively upper segment) of $\mathfrak{L}$. It called the *principal lower* (respectively *upper*) *segment* of $\mathfrak{L}$ generated by $\mathfrak{a}$.

Consider some preliminary properties of the lattice groups.

**Proposition 4.** *Let $G$ be a group, $\mathfrak{L}$ be a finite distributive lattice and $\mathfrak{S}$ be a family of lattice subgroups over $\mathfrak{L}$. Then intersection $\cap \mathfrak{S}$ is a lattice subgroup.*

*Proof.* The proof is almost obvious. $\square$

**Proposition 5.** *Let $G$ be a group, $\mathfrak{L}$ be a finite distributive lattice and $\Lambda$ a lattice group. Then:*

(i) $\mathrm{pr}_{\mathfrak{L}}(\Lambda)$ *is a semigroup by operation $\wedge$ with identity $\mathfrak{e}(\Lambda) = \vee\mathfrak{C}_\Lambda(1)$ and zero $\mathfrak{o}$. Moreover, $\mathrm{pr}_{\mathfrak{L}}(\Lambda)$ is the principal lower segment of $\mathfrak{L}$, generated by $\mathfrak{e}(\Lambda)$.*

(ii) $\mathrm{pr}_G(\Lambda)$ *is a subgroup of $G$. Conversely, for every subgroup $H$ of $\mathrm{pr}_G(\Lambda)$ the subset $\{(x,\mathfrak{a})|(x,\mathfrak{a}) \in \Lambda$ and $x \in H\} = \mathrm{pr}_G^{-1}(H)$ is a lattice subgroup of $\Lambda$.*

(iii) *If $\mathfrak{M}$ is a lower segment of $\mathfrak{L}$, then $\{(x,\mathfrak{a})|(x,\mathfrak{a}) \in \Lambda$ and $\mathfrak{a} \in \mathfrak{M}\}$ is a lattice subgroup of $\Lambda$. In particular, $\mathrm{pr}_{\mathfrak{L}}^{-1}(\mathfrak{M})$ is a lattice group.*

*Proof.* (i) Indeed, if $\mathfrak{a},\mathfrak{b} \in \mathrm{pr}_{\mathfrak{L}}(\Lambda)$, then there are elements $u, v \in G$ such that $(u,\mathfrak{a}), (v,\mathfrak{b}) \in \Lambda$. Since $\Lambda$ is a lattice group, $(uv, \mathfrak{a}\wedge\mathfrak{b}) = (u,\mathfrak{a})(v,\mathfrak{b}) \in \Lambda$. It follows that $\mathfrak{a}\wedge\mathfrak{b} \in \mathrm{pr}_{\mathfrak{L}}(\Lambda)$. In particular, $\mathfrak{e}(\Lambda) = \vee\mathfrak{C}_\Lambda(e) \in \mathrm{pr}_{\mathfrak{L}}(\Lambda)$.

Let $\mathfrak{a} \in \mathrm{pr}_{\mathfrak{L}}(\Lambda)$ and $u$ be an element of $G$ such that $(u,\mathfrak{a}) \in \Lambda$. Since $\Lambda$ is a lattice group, $(u^{-1},\mathfrak{a}) \in \Lambda$ by condition (LG 3). Using (LG 2), we obtain that $(e,\mathfrak{a}) = (uu^{-1},\mathfrak{a}) = (uu^{-1},\mathfrak{a}\wedge\mathfrak{a}) = (u,\mathfrak{a})(u^{-1},\mathfrak{a}) \in \Lambda$. Hence $\mathfrak{a} \in \mathfrak{C}(e)$, which follows that $\mathfrak{a} \leqslant \mathfrak{e}(\Lambda)$. In other words, $\mathfrak{e}(\Lambda)$ is the greatest element of $\mathrm{pr}_{\mathfrak{L}}(\Lambda)$.

Let $\mathfrak{c}$ be an arbitrary element of $\mathfrak{L}$ such that $\mathfrak{c} \leqslant \mathfrak{e}(\Lambda)$. Since $(e,\mathfrak{e}(\Lambda)) \in \Lambda$. $(e,\mathfrak{c}) \in \Lambda$ by condition (LG 1). It follows that $\mathrm{pr}_{\mathfrak{L}}(\Lambda)$ is the principal lower segment of $\mathfrak{L}$, generated by $\mathfrak{e}(\Lambda)$.

(ii) Let $K = \mathrm{pr}_G(\Lambda), u, v \in K$. Then there are the elements $\mathfrak{a},\mathfrak{b} \in \mathfrak{L}$ such that $(u,\mathfrak{a}), (v,\mathfrak{b}) \in \Lambda$. Since $\Lambda$ is a lattice group, $(uv,\mathfrak{a}\wedge\mathfrak{b}) = (u,\mathfrak{a})(v,\mathfrak{b}) \in \Lambda$. It follows that $uv \in K$. If $(u,\mathfrak{a}) \in \Lambda$, then $(u^{-1},\mathfrak{a}) \in \Lambda$ by condition (LG 3), which follows that $u^{-1} \in K$. Hence $K$ is a subgroup of $G$.

Let now $H$ be a subgroup of $\mathrm{pr}_G(\Lambda)$, $(u,\mathfrak{a}), (v,\mathfrak{b}) \in \mathrm{pr}_G^{-1}(H)$. Since $\Lambda$ is a lattice group, $(uv,\mathfrak{a}\wedge\mathfrak{b}) = (u,\mathfrak{a})(v,\mathfrak{b}) \in \Lambda$. The fact that $H$ is a subgroup implies that $uv \in H$, so that $(uv,\mathfrak{a}\wedge\mathfrak{b}) \in \mathrm{pr}_G^{-1}(H)$. Since $H$ is a subgroup, then from $u \in H$ it follows that $u^{-1} \in H$. Since $\Lambda$ is a lattice group, $(u,\mathfrak{a}) \in \Lambda$ implies that $(u^{-1},\mathfrak{a}) \in \Lambda$. Hence $(u^{-1},\mathfrak{a}) \in \mathrm{pr}_G^{-1}(H)$, so that $\mathrm{pr}_G^{-1}(H)$ satisfies the conditions (LG 2), (LG 3), and $(uv,\mathfrak{a}\wedge\mathfrak{b}) = (u,\mathfrak{a})(v,\mathfrak{b}) \in \Lambda$. Hence $K$ is a subgroup of $G$. Let $(u,\mathfrak{a}) \in \mathrm{pr}_G^{-1}(H)$ and $\mathfrak{b}$ be an element of $\mathfrak{L}$ such that $\mathfrak{b} \leqslant \mathfrak{a}$. Then $(u,\mathfrak{b}) \in \Lambda$ and hence $(u,\mathfrak{b}) \in \mathrm{pr}_G^{-1}(H)$.

(iii) Let $\mathfrak{M}$ is a lower segment of $\mathfrak{L}$, $K$ a subgroup of $G$ and $M = K\times\mathfrak{M}$. Then $M$ is a lattice group. Indeed, if $(x,\mathfrak{a}) \in M$ and $\mathfrak{b} \leqslant \mathfrak{a}$, then $\mathfrak{b} \in \mathfrak{M}$, because $\mathfrak{M}$ is a lower segment of $\mathfrak{L}$. It follows that $(x, \mathfrak{b}) \in M$, so that $M$ satisfies (LG 1). Suppose that $(x, \mathfrak{a}), (y, \mathfrak{b}) \in M$. Since $\mathfrak{a} \wedge \mathfrak{b} \leqslant \mathfrak{b}, \mathfrak{a} \wedge \mathfrak{b} \in \mathfrak{M}$. The fact that $K$ is a subgroup of $G$ implies $xy \in K$, and hence $(xy, \mathfrak{a} \wedge \mathfrak{b}) \in M$. We note that $(xy, \mathfrak{a} \wedge \mathfrak{b}) = (x, \mathfrak{a})(y, \mathfrak{b})$, which shows that $M$ satisfies (LG 2). Finally, let $(x, \mathfrak{a}) \in M$. Since $K$ is a subgroup of $G$, $x^{-1} \in K$. Therefore $(x^{-1}, \mathfrak{a}) \in M$, and $M$ satisfies (LG 3).

Let again $H = \mathrm{pr}_G(\Lambda)$, then it is not hard to see that

$$\{(x, \mathfrak{a}) | (x, \mathfrak{a}) \in \Lambda \text{ and } \mathfrak{a} \in \mathfrak{M}\} = H \times \mathfrak{M} \cap \Lambda.$$

Proposition 4 shows that this subset is a lattice subgroup of $\Lambda$. $\square$

Let $\Lambda$ be a lattice group. Unlike abstract groups, a lattice group can contains more than one idempotent. Moreover, $\Lambda$ contains a pair $(1, \mathfrak{a})$ for each element $\mathfrak{a} \in \mathrm{pr}_{\mathfrak{L}}(\Lambda)$. Indeed, let $u$ be an element of $G$ such that $(u, \mathfrak{a}) \in \Lambda$. Since $\Lambda$ is a lattice group, $(u, \mathfrak{a})(u^{-1}, \mathfrak{a}) \in \Lambda$. But $(u, \mathfrak{a})(u^{-1}, \mathfrak{a}) = (e, \mathfrak{a} \wedge \mathfrak{a}) = (e, \mathfrak{a})$. It shows that a semigroup $\Lambda$ can be a group only in the case when $\mathrm{pr}_{\mathfrak{L}}(\Lambda)$ contains only one element $\mathfrak{a}$. Let $\mathfrak{b} \in \Lambda$ and $\mathfrak{b} \leqslant \mathfrak{a}$, then condition (LG 1) implies that $(u, \mathfrak{b}) \in \Lambda$. Hence $\mathfrak{a} = \mathfrak{b}$. In other words, $\mathfrak{a}$ is the least element of $\mathfrak{L}$, i.e. $\mathfrak{a} = \mathfrak{o}$. Consequently, a lattice group $\Lambda$ is a group if and only if $\mathrm{pr}_{\mathfrak{L}}(\Lambda) = \{\mathfrak{o}\}$. In this regard, we note that the semigroup $\Lambda$ may include many subsemigroups, which are groups by multiplication. Indeed, let $H$ be a subgroup of $G$ and $\mathfrak{a} \in \mathfrak{L}$, then it is not hard to see that the subset $H \times \{\mathfrak{a}\}$ is a group by multiplication. Furthermore, for every $\mathfrak{a} \in \mathfrak{L}$ the subset $\{(u, \mathfrak{a}) | (u, \mathfrak{a}) \in \Lambda\}$ is also a group by multiplication.

If $\Lambda$ is a lattice subgroup over $\mathfrak{L}$, then put $E(\Lambda) = \{(e, \mathfrak{b}) | \mathfrak{b} \leqslant \mathfrak{e}(\Lambda)\}$. Clearly $E(\Lambda)$ is a lattice subgroup of $\Lambda$.

Let $\Gamma$ be a lattice subgroup of $\Lambda$. The pair $(e, \mathfrak{e}(\Lambda))$ is an identity element of $\Lambda$ and $(e, \mathfrak{e}(\Gamma))$ is an identity element of $\Gamma$. Since $\Gamma \leqslant \Lambda$, Proposition 5 shows that $\mathfrak{e}(\Gamma) \leqslant \mathfrak{e}(\Lambda)$. We say that $\Gamma$ is an *unitary lattice subgroup* of $\Lambda$, if $(e, \mathfrak{e}(\Lambda)) \in \Gamma$. Every lattice subgroup of $\Lambda$ can be extended to an unitary lattice subgroup. Indeed, put $\Gamma^{u(\Lambda)} = \Gamma \cup \{(e, \mathfrak{b}) | \mathfrak{b} \leqslant \mathfrak{e}(\Lambda)\} = \Gamma \cup E(\Lambda)$, then $\Gamma^{u(\Lambda)}$ is a lattice group. In fact, if $(u, \mathfrak{a}) \in \Lambda$, then $(u, \mathfrak{a})(e, \mathfrak{b}) = (u, \mathfrak{a} \wedge \mathfrak{b})$. Since $\mathfrak{a} \wedge \mathfrak{b} \leqslant \mathfrak{a}$, $(u, \mathfrak{a} \wedge \mathfrak{b}) \in \Gamma$. It shows that $\Gamma^{u(\Lambda)}$ satisfies all conditions (LG 1)–(LG 3).

Let $M$ be a subset of $G \times \mathfrak{L}$ and $\mathfrak{S}$ be a family of all lattice groups, including $M$. By Proposition 4, the intersection $\cap\mathfrak{S}$ is a lattice group. It called the lattice group generated by $M$ and will be denoted by $\langle M \rangle$.

Let $(x, \mathfrak{a}) \in G \times \mathfrak{L}$. If $\Lambda$ is a lattice group containing $(x, \mathfrak{a})$, then it is not hard to prove that $(x, \mathfrak{a})^n = (x^n, \mathfrak{a} \wedge \ldots \wedge \mathfrak{a}) = (x^n, \mathfrak{a}) \in \Lambda$ for each positive integer $n$. By (LG 3), $(x^{-1}, \mathfrak{a}) \in \Lambda$, and hence $(e, \mathfrak{a}) = (x, \mathfrak{a})(x^{-1}, \mathfrak{a}) \in \Lambda$. From $(x^{-1}, \mathfrak{a}) \in \Lambda$ we obtain that $(x, \mathfrak{a})^{-n} = (x^{-n}, \mathfrak{a}) \in \Lambda$, so that $\{(x^n, \mathfrak{a}) | n \in \mathbb{Z}\} \subseteq \Lambda$. Let $\mathfrak{A}$ be the principal lower segment of $\mathfrak{L}$, generated by $\mathfrak{a}$. If $\mathfrak{b} \leqslant \mathfrak{a}$, then (LG 1) implies that $(x^n, \mathfrak{b}) \in \Lambda$ for each integer $n$. Thus $\{(x^n, \mathfrak{b}) | \mathfrak{b} \leqslant \mathfrak{a}, n \in \mathbb{Z}\} \subseteq \Lambda$. It is not hard to check that the subset $\{(x^n, \mathfrak{b}) | \mathfrak{b} \leqslant \mathfrak{a}, n \in \mathbb{Z}\}$ is a lattice group. It follows that $\langle (x, \mathfrak{a}) \rangle = \{(x^n, \mathfrak{b}) | \mathfrak{b} \leqslant \mathfrak{a}, n \in \mathbb{Z}\}$.

Let $\Lambda$, $\Gamma$ be the lattice subgroups. Define its product in the usual way: put

$$\Lambda\Gamma = \{(x, \mathfrak{a})(y, \mathfrak{b}) = (xy, \mathfrak{a} \wedge \mathfrak{b}) | (x, \mathfrak{a}) \in \Lambda, (y, \mathfrak{b}) \in \Gamma\}.$$

The following result is a rationale for this determination.

**Proposition 6.** *Let $G$ be a group, $\mathfrak{L}$ be a finite distributive lattice and $\gamma, \kappa \colon G \to \mathfrak{L}$ be functions. Then*

$$\gamma \odot \kappa = \cup_{y \in \mathrm{Supp}(\gamma), z \in \mathrm{Supp}(\kappa)} \chi(y, \gamma(y)) \odot \chi(z, \kappa(z)).$$

*Proof.* By definition we have

$$(\gamma \odot \kappa)(x) = \vee_{y, z \in G, yz = x} (\gamma(y) \wedge \kappa(z)).$$

If $y \notin \mathrm{Supp}(\gamma)$, then $\gamma(y) = \mathfrak{o}$ and $\gamma(y) \wedge \kappa(z) = \mathfrak{o}$. Similarly, if $z \notin \mathrm{Supp}(\kappa)$, then $\kappa(z) = \mathfrak{o}$, and again $\gamma(y) \wedge \kappa(z) = \mathfrak{o}$. It follows that

$$(\gamma \odot \kappa)(x) = \vee_{y \in \mathrm{Supp}(\gamma), z \in \mathrm{Supp}(\kappa), yz = x} (\gamma(y) \wedge \kappa(z)).$$

On the other hand, let $\xi = \cup_{y \in \mathrm{Supp}(\gamma), z \in \mathrm{Supp}(\kappa)} \chi(y, \gamma(y)) \odot \chi(z, \kappa(z))$. By Proposition 1, $\chi(y, \gamma(y)) \odot \chi(z, \kappa(z)) = \chi(yz, (\gamma(y) \wedge \kappa(z)))$. If $x \in G$ and $x = yz$, then $\chi(yz, (\gamma(y) \wedge \kappa(z)))(x) = \gamma(y) \wedge \kappa(z)$, otherwise $\chi(yz, (\gamma(y) \wedge \kappa(z)))(x) = \mathfrak{o}$. Therefore

$$\begin{aligned} \xi(x) &= \vee_{y \in \mathrm{Supp}(\gamma), z \in \mathrm{Supp}(\kappa)} (\chi(yz, (\gamma(y) \wedge \kappa(z))))(x) \\ &= \vee_{y \in \mathrm{Supp}(\gamma), z \in \mathrm{Supp}(\kappa), yz = x} (\gamma(y) \wedge \kappa(z)) = (\gamma \odot \kappa)(x). \end{aligned}$$

Since it is true for each $x \in G$,

$$\gamma \odot \kappa = \cup_{y \in \mathrm{Supp}(\gamma), z \in \mathrm{Supp}(\kappa)} \chi(y, \gamma(y)) \odot \chi(z, \kappa(z)). \qquad \square$$

**Corollary.** *Let $G$ be a group, $\mathfrak{L}$ be a finite distributive lattice, $\mathfrak{a} \in \mathfrak{L}$, and $\kappa \colon G \to \mathfrak{L}$ be functions. Then for every $x \in G$*

$$\chi(x, \mathfrak{a}) \odot \kappa = \cup_{z \in \mathrm{Supp}(\kappa)} \chi(x, \mathfrak{a}) \odot \chi(z, \kappa(z)),$$

$$\kappa \odot \chi(x, \mathfrak{a}) = \cup_{z \in \mathrm{Supp}(\kappa)} \chi(z, \kappa(z)) \odot \chi(x, \mathfrak{a}).$$

Let $\lambda\colon G \to \mathfrak{L}$ be a function defined by $\Lambda$ and $\gamma\colon G \to \mathfrak{L}$ be a function defined by $\Gamma$. Consider a function $\kappa\colon G \to \mathfrak{L}$ defined by the product $\Lambda\Gamma$. Let $g$ be an arbitrary element of $G$. If $g \notin \mathrm{pr}_G(\Lambda\Gamma)$, then $\kappa(g) = \mathfrak{o}$. On the other hand, let $u$, $v$ be an arbitrary elements of $G$ such that $g = uv$. Since $g \notin \mathrm{pr}_G(\Lambda\Gamma) = \mathrm{pr}_G(\Lambda)\,\mathrm{pr}_G(\Gamma)$, then either $u \notin \mathrm{pr}_G(\Lambda)$, $v \notin \mathrm{pr}_G(\Gamma)$, or $u \in \mathrm{pr}_G(\Lambda)$ but $v \notin \mathrm{pr}_G(\Gamma)$ or $u \notin \mathrm{pr}_G(\Lambda)$ but $v \in \mathrm{pr}_G(\Gamma)$. In each of these cases either $\lambda(u) = \mathfrak{o}$ or $\gamma(v) = \mathfrak{o}$, so that

$$\vee_{u,v\in G,uv=g}(\lambda(u) \wedge \gamma(v)) = \mathfrak{o} = \kappa(g).$$

Suppose now that $g \in \mathrm{pr}_G(\Lambda\Gamma)$, then $\kappa(g) = \vee\mathfrak{C}_{\Lambda\Gamma}(g)$. Let again $u$, $v$ be arbitrary elements of $G$ such that $g = uv$. If $u \notin \mathrm{pr}_G(\Lambda)$ or $v \notin \mathrm{pr}_G(\Gamma)$, then $(\lambda(u) \wedge \gamma(v)) = \mathfrak{o}$. Suppose that $u \in \mathrm{pr}_G(\Lambda)$ and $v \in \mathrm{pr}_G(\Gamma)$ and let $\mathfrak{a}$, $\mathfrak{b}$ be the elements of $\mathfrak{L}$ such that $(u, \mathfrak{a}), (v, \mathfrak{b}) \in \mathfrak{L}$. We have $(u, \mathfrak{a})(v, \mathfrak{b}) = (uv, \mathfrak{a} \wedge \mathfrak{b})$. This shows that $\mathfrak{C}_{\Lambda\Gamma}(g) = \{\mathfrak{a} \wedge \mathfrak{b} | \mathfrak{a} \in \mathfrak{C}_\Lambda(u), \mathfrak{b} \in \mathfrak{C}_\Gamma(v)\}$. Since $\lambda(u) = \vee\mathfrak{C}_\Lambda(u)$, $\gamma(v) = \vee\mathfrak{C}_\Gamma(v)$, $\mathfrak{C}_{\Lambda\Gamma}(g) = \lambda(u) \wedge \gamma(v)$. In other words, in this case we have also

$$\kappa(g) = \vee_{u,v\in G,uv=g}(\lambda(u) \wedge \gamma(v)).$$

Thus $\kappa = \lambda \odot \gamma$. Thus, from the bulky and not very transparent product of functions we come to the intuitively clear and convenient product of subsets.

Let us now see how another important concept, the concept of normal fuzzy subgroup can be transformed. Again, it should be recalled that we use different terminology.

Let $\lambda, \kappa\colon G \to \mathfrak{L}$ be a group functions and $\kappa \preceq \lambda$. We say that $\kappa$ *is a normal subgroup function of* $\lambda$, if $\kappa(yxy^{-1}) \geqslant \kappa(x) \wedge \lambda(y)$ for every elements $x, y \in G$.

We will need the following criteria of normality.

**Proposition 7.** *Let $G$ be a group, $\mathfrak{L}$ be a finite distributive lattice and $\lambda, \kappa\colon G \to \mathfrak{L}$ be group functions such that $\kappa \preceq \gamma$. Then the following assertions are equivalent:*

(i) *$\kappa$ is a normal subgroup function of $\gamma$;*

(ii) *$\chi(x, \gamma(x)) \odot \kappa \odot \chi(x^{-1}, \gamma(x)) \preceq \kappa$ for every element $x \in G$;*

(iii) *$\chi(x, \gamma(x)) \odot \chi(y, \kappa(y)) \odot \chi(x^{-1}, \gamma(x)) \subseteq \kappa$ for every elements $x, y \in G$;*

(iv) *$\chi(x, \mathfrak{a}) \odot \chi(y, \mathfrak{b}) \odot \chi(x^{-1}, \mathfrak{a}) \subseteq \kappa$ for every elements $x, y \in G$, $\mathfrak{a} \leqslant \gamma(x)$, $\mathfrak{b} \leqslant \kappa(y)$.*

*Proof.* (i) $\Rightarrow$ (ii). Suppose that $\kappa$ is a normal subgroup function of $\lambda$. For arbitrary element $y \in G$ we consider the product $\chi(y, \gamma(y)) \odot \kappa \odot \chi(y^{-1}, \gamma(y))$. Let $x$ be an arbitrary element of $G$. From Proposition 1 we obtain

$$(\chi(y, \gamma(y)) \odot \kappa \odot \chi(y^{-1}, \gamma(y)))(x) = \gamma(y) \wedge \kappa(y^{-1}xy).$$

Put $u = y^{-1}xy$, then $x = y(y^{-1}xy)y^{-1} = yuy^{-1}$, so that

$$(\chi(y, \gamma(y)) \odot \kappa \odot \chi(y^{-1}, \gamma(y)))(yuy^{-1}) = \gamma(y) \wedge \kappa(u).$$

Since $\kappa(u) \wedge \gamma(y) \leqslant \kappa(yuy^{-1})$, we obtain

$$(\chi(y, \gamma(y)) \odot \kappa \odot \chi(y^{-1}, \gamma(y)))(yuy^{-1}) \leqslant \kappa(yuy^{-1}),$$

that is

$$(\chi(y, \gamma(y)) \odot \kappa \odot \chi(y^{-1}, \gamma(y)))(x) \leqslant \kappa(x).$$

Since this is valid for every element $x \in G$,

$$\chi(y, \gamma(y)) \odot \kappa \odot \chi(y^{-1}, \gamma(y)) \preceq \kappa.$$

(ii) $\Rightarrow$ (iii). Indeed, Corollary to Proposition 6 shows that

$$\begin{aligned} &\chi(y, \gamma(y)) \odot \kappa \odot \chi(y^{-1}, \gamma(y)) \\ &\qquad = \cup_{z \in \mathrm{Supp}(\kappa)} \chi(y, \gamma(y)) \odot \chi(z, \kappa(z)) \odot \chi(y^{-1}, \gamma(y)). \end{aligned}$$

Hence the inclusion $\chi(y, \gamma(y)) \odot \kappa \odot \chi(y^{-1}, \gamma(y)) \preceq \kappa$ implies that

$$\chi(y, \gamma(y)) \odot \chi(z, \kappa(z)) \odot \chi(y^{-1}, \gamma(y)) \subseteq \kappa \quad \text{for every elements } y, z \in G.$$

(iii) $\Rightarrow$ (iv). Indeed, Proposition 1 shows that

$$\chi(x, \gamma(x)) \odot \chi(y, \kappa(y)) \odot \chi(x^{-1}, \gamma(x)) = \chi(xyx^{-1}, \gamma(x) \wedge \kappa(y)).$$

We have

$$\chi(x, \mathfrak{a}) \odot \chi(y, \mathfrak{b}) \odot \chi(x^{-1}, \mathfrak{a}) = \chi(xyx^{-1}, \mathfrak{a} \wedge \mathfrak{b}) \subseteq \chi(xyx^{-1}, \gamma(x) \wedge \kappa(y)).$$

(iv) $\Rightarrow$ (i). Using again Proposition 1, we obtain that

$$\chi(x, \gamma(x)) \odot \chi(y, \kappa(y)) \odot \chi(x^{-1}, \gamma(x)) = \chi(xyx^{-1}, \gamma(x) \wedge \kappa(y)).$$

Now (vi) shows that $\chi(xyx^{-1}, \gamma(x) \wedge \kappa(y)) \subseteq \kappa$. Then

$$\gamma(x) \wedge \kappa(y) = \chi(xyx^{-1}, \gamma(x) \wedge \kappa(y))(xyx^{-1}) \leqslant \kappa(xyx^{-1}).$$

This means that $\kappa$ is a normal subgroup function of $\gamma$. $\square$

Proposition 7 leads us to the following analogue of normality in lattice groups.

Let $\Gamma$ be a lattice subgroup of $\Lambda$. We say that $\Gamma$ is a *normal lattice subgroup* of $\Lambda$, if $(y^{-1}, \mathfrak{b})(x, \mathfrak{a})(y, \mathfrak{b}) \in \Gamma$ for all pairs $(y, \mathfrak{b}) \in \Lambda$, $(x, \mathfrak{a}) \in \Gamma$.

We remark that $(y^{-1}, \mathfrak{b})(x, \mathfrak{a})(y, \mathfrak{b}) = (y^{-1}xy, \mathfrak{a}\wedge\mathfrak{b})$. At once this shows that if $\Gamma$ a normal lattice subgroup of $\Lambda$, then $\mathrm{pr}_G(\Gamma)$ is a normal subgroup of $\mathrm{pr}_G(\Lambda)$. Conversely, suppose that $H$ is a normal subgroup of $G$ and $\Lambda_H = \{(x, \mathfrak{a}) \in \Lambda | x \in H\}$. Then $(y^{-1}, \mathfrak{b})(x, \mathfrak{a})(y, \mathfrak{b}) = (y^{-1}xy, \mathfrak{a} \wedge \mathfrak{b}) \in \Lambda$ for each pair $(y, \mathfrak{b}) \in \Lambda$. Since $H$ is normal in $G$, $y^{-1}xy \in H$, so that $(y^{-1}, \mathfrak{b})(x, \mathfrak{a})(y, \mathfrak{b}) \in \Lambda_H$.

Let $\mathfrak{M}$ be a lower segment of $\mathfrak{L}$ . Then Proposition 5 proves that $\Lambda[\mathfrak{M}] = \{(x, \mathfrak{a}) | (x, \mathfrak{a}) \in \Lambda \text{ and } \mathfrak{a} \in \mathfrak{M}\}$ is a lattice subgroup of $\Lambda$. $\Lambda[\mathfrak{M}]$ is called an $\mathfrak{M}$-*layer* of $\Lambda$. We note that $\Lambda[\mathfrak{M}]$ is a normal lattice subgroup of $\Lambda$. In fact, let $(x, \mathfrak{a}) \in \Lambda[\mathfrak{M}]$ and $(y, \mathfrak{b}) \in \Lambda$, then $(y^{-1}, \mathfrak{b})(x, \mathfrak{a})(y, \mathfrak{b}) = (y^{-1}xy, \mathfrak{a} \wedge \mathfrak{b})$. Since $\mathfrak{a} \wedge \mathfrak{b} \leqslant \mathfrak{a}, \mathfrak{a} \in \mathfrak{M}$ and $\mathfrak{M}$ is a lower segment of $\mathfrak{L}$, $\mathfrak{a} \wedge \mathfrak{b} \in \mathfrak{M}$. Thus $(y^{-1}, \mathfrak{b})(x, \mathfrak{a})(y, \mathfrak{b}) \in \Lambda[\mathfrak{M}]$.

If $\Gamma$ is a normal lattice subgroup of $\Lambda$, then $\Gamma^{u(\Lambda)}$ is a normal lattice subgroup of $\Lambda$. Indeed, let $(x, \mathfrak{a}) \in \Gamma^{u(\Lambda)}$ and $(y, \mathfrak{b}) \in \Lambda$. If $x \neq e$, then $(x, \mathfrak{a}) \in \Gamma$ and $(y^{-1}, \mathfrak{b})(x, \mathfrak{a})(y, \mathfrak{b}) \in \Gamma$. If $x = e$, then $(y^{-1}, \mathfrak{b})(e, \mathfrak{a})(y, \mathfrak{b}) = (1, \mathfrak{a} \wedge \mathfrak{b}) \in E(\Lambda)$.

The layers of lattice group play a very important role. Especially it is useful in the case when $\mathrm{pr}_{\mathfrak{L}}(\Lambda)$ is a chain. This case arises in theory of fuzzy group when a group $G$ is finite. Suppose that $|\,\mathrm{pr}_{\mathfrak{L}}(\Lambda)| = k$. Then $\mathrm{pr}_{\mathfrak{L}}(\Lambda)$ is isomorphic (as an ordered set) to the set $\mathrm{Ch}[1, k] = \{1, 2, \ldots, k\}$ with the natural ordering $1 \leqslant 2 \leqslant \ldots \leqslant k$. In this case, we will say that $\Lambda$ is a lattice group over $\mathrm{Ch}[1, k]$.

For this case we construct some natural series of subgroups both in the lattice group $\Lambda$ and in $\mathrm{pr}_G(\Lambda)$. The subset $\{1\}$ is the lower segment of $\mathrm{Ch}[1, k]$, and therefore the $\{1\}$-layer $\Lambda[1]$ of $\Lambda$ is a lattice subgroup of $\Lambda$. If $(u, m) \in \Lambda$, then $(u, 1) \in \Lambda$ by condition (LG 1). This implies that $\mathrm{pr}_G(\Lambda) = \mathrm{pr}_G(\Lambda[1])$. For every $m$, $1 \leqslant m \leqslant k$, the subset $K_m = \{(u, m) | (u, m) \in \Lambda\}$ is the subgroup by multiplication, so that $\mathrm{H}(m) = \mathrm{pr}_G(K_m)$ is a subgroup of $\mathrm{H}(1) = \mathrm{pr}_G(\Lambda)$. A subgroup $\mathrm{H}(m)$ is called the $m$-*hoop* of $\Lambda$. From $(u, m) \in \Lambda$ we obtain $(u, m - 1) \in \Lambda$ by condition (LG 1). This implies the inclusion $\mathrm{H}(m) \leqslant \mathrm{H}(m - 1)$, so we obtain the following descending series of subgroups

$$\mathrm{H}(1) \geqslant \mathrm{H}(2) \geqslant \ldots \geqslant \mathrm{H}(k).$$

Clearly the mapping $u \to (u, m)$, $u \in \mathrm{H}(m)$, is an isomorphism of $\mathrm{H}(m)$ on $K_m$ for each $m$, $1 \leqslant m \leqslant k$.

Figuratively speaking, the pictured structure of a lattice group over $\mathrm{Ch}[1,k]$ reminds the cake "Napoleon". Here the groups play the role of the cakes lays, and the idempotents play the role of cream lays. Indeed, in the first step, by above remarked the $\Lambda[1]$ is a normal lattice subgroup of $\Lambda$. We have seen also that $\Lambda[1]$ is a group by multiplication (moreover, it is isomorphic to $\mathrm{pr}_G(\Lambda)$). Now add the cream: put $\Lambda_1 = \Lambda[1] \cup \{(e,2)\}$. It is not hard to see, that $\Lambda_1$ is a normal lattice subgroup of $\Lambda$. Next step: consider the $\{1,2\}$-layer $\Lambda[1,2]$ of $\Lambda$, which is a normal lattice subgroup of $\Lambda$. We note that $\Lambda_1 \leqslant \Lambda[1,2]$, moreover $\Lambda_1$ is a normal lattice subgroup of $\Lambda$. For every element $(x,j) \in \Lambda[1,2]$ denote by $(x,j)\Lambda_1$ the product $\{(x,j)\}\Lambda_1$. This subset is called a *coset* by $\Lambda_1$. Since $(x,j) \in \Lambda[1,2]$, $j \leqslant 2$, so that $(x,j) = (xe, j \wedge 2) = (x,j)(e,2) \in (x,j)\Lambda_1$. It follows that $\Lambda[1,2]$ is an union of all subsets $(x,j)\Lambda_1$. Suppose that $(x,j)\Lambda_1 \neq \Lambda_1$. Then $x \neq e$ and $j = 2$. Thus we can see that the equality $(x,2) = (y,2)(z,m)$ where $(z,m) \in \Lambda_1$ is possible only in the case when $m = 2$. In turn, the single pair of $\Lambda_1$, whose second component is equal to 2, is the pair $(e,2)$. Hence $(x,2) = (y,2)(e,2)$, so that $x = y$. In other words, the equality $(x,2)\Lambda_1 = (y,2)\Lambda_1$ is possible only in the case, when $x = y$. Consider the product of subsets $((x,2)\Lambda_1)((y,2)\Lambda_1)$. Its arbitrary element has a form $(x,2)(u,j)(y,2)(v,m)$ where $(u,j),(v,m) \in \Lambda_1$. If $j = 1$ or $m = 1$, then $(x,2)(u,j)(y,2)(v,m) = (xuyv,1) \in \Lambda_1$. Hence if $(x,2)(u,j)(y,2)(v,m) \notin \Lambda_1$, then $j = m = 2$. But it is possible only if $u = v = e$. In this case, $(x,2)(u,j)(y,2)(v,m) = (xy,2)$. In turn it follows that $((x,2)\Lambda_1)((y,2)\Lambda_1) = (xy,2)\Lambda_1$. Hence the set of all cosets by $\Lambda_1$ becomes a semigroup. Moreover, this semigroup is a group, because it has an identity element $(e,2)\Lambda_1 = \Lambda_1$, and for every coset $(x,2)\Lambda_1$ we have $(x^{-1},2)\Lambda_1(x,2)\Lambda_1 = (e,2)\Lambda_1 = (x,2)\Lambda_1(x^{-1},2)$. Therefore we can talk here about a factor-group of a lattice group $\Lambda[1,2]$ by the normal lattice subgroup $\Lambda_1$. For it we will use a common notation $\Lambda[1,2]/\Lambda_1$. We emphasize that here we are talking about a factor-group, rather than a lattice factor-group. It is our special selection provides such an opportunity; in general, is not always the family of cosets by normal lattice subgroup is a group or a lattice group.The mapping $\Phi$, defined by the rule $\Phi((x,2)) = (x,2)\Lambda_1$, $(x,2) \in K_2$, is an epimorphism. As we have seen early, the equality $(x,2)\Lambda_1 = \Lambda_1$ is possible only in the case when $x = e$, which shows that $\Phi$ is an isomorphism. Since $K_2 \cong \mathrm{H}(2)$, we obtain that $\Lambda[1,2]/\Lambda_1$ is isomorphic to the 2-hoop of $\Lambda$.

Adding the next lay of the cream $\{(e,3)\}$ to $\Lambda[1,2]$, we come to the normal lattice subgroup $\Lambda_2 = \Lambda[1,2] \cup \{(e,3)\}$, and then we cover it with the next lay of cake, i.e. extend $\Lambda_2$ to the $\{1,2,3\}$-layer $\Lambda[1,2,3]$ of $\Lambda$, which is a normal lattice subgroup of $\Lambda$. Using the above arguments, we shows that a family of cosets $(x,3)\Lambda_2$ is a group by multiplication and this group is isomorphic to the 2-hoop of $\Lambda$. And so on. As the result we obtain the sequences

$$\Lambda_0 = \{(e,1)\} \leqslant \Lambda[1] \leqslant \Lambda_1 \leqslant \Lambda[1,2] \leqslant \Lambda_2 \leqslant \Lambda[1,2] \leqslant \ldots \leqslant \Lambda_{k-1} \leqslant \Lambda$$

of normal lattice subgroups such that $\Lambda_m = \Lambda[1,\ldots,m] \cup \{(e,m+1)\}$, and $\Lambda[1,\ldots,m+1]/\Lambda_m \cong \mathrm{H}(m+1)$, $0 \leqslant m \leqslant k-1$.

Note, that in the theory of fuzzy groups we could not find any similar description of a general structure of a fuzzy group $\gamma$ for the case when $\mathrm{Im}(\gamma)$ is finite.

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