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Chromatic number of graphs with special distance sets, I

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Dedicated to Prof. Dr. R. Balasubramanian, Director, IMSC, India on his 63rd Birth Day

Abstract. Given a subset D of positive integers, an integer distance graph is a graph $G(\mathbb{Z}, D)$ with the set $\mathbb{Z}$ of integers as vertex set and with an edge joining two vertices u and v if and only if $|u-v| \in D$. In this paper we consider the problem of determining the chromatic number of certain integer distance graphs $G(\mathbb{Z}, D)$whose distance set D is either 1) a set of $(n+1)$ positive integers for which the n^{th} power of the last is the sum of the n^{th} powers of the previous terms, or 2) a set of pythagorean quadruples, or 3) a set of pythagorean n-tuples, or 4) a set of square distances, or 5) a set of abundant numbers or deficient numbers or carmichael numbers, or 6) a set of polytopic numbers, or 7) a set of happy numbers or lucky numbers, or 8) a set of Lucas numbers, or 9) a set of $\mathcal{U}$lam numbers, or 10) a set of weird numbers. Besides finding the chromatic number of a few specific distance graphs we also give useful upper and lower bounds for general cases. Further, we raise some open problems.

1. Introduction

The graphs considered here are mostly finite, simple and undirected.

A k-coloring of a graph G is an assignment of k different colors to the vertices of G such that adjacent vertices receive different colors. The

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minimum cardinality k for which G has a k-coloring is called the chromatic number of G and is denoted by $\chi(G)$. We denote by $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R} \subseteq \mathbb{C}$ of the sets of natural numbers, integers, rational and real numbers respectively. By $G(\mathbb{R}^n, D)$, we mean the distance graph G whose vertex set is $\mathbb{R}^n$ and the edge set is constructed by introducing an edge between two vertices u and v of G if and only if $|u - v| \in D$. In the same way one can define $G(\mathbb{R}^1, D)$ and $G(\mathbb{R}^2, D)$.

1.1. History of the problem

The chromatic number of the plane problem asks for the minimum number of colors that are needed to paint all points in the real plane $\mathbb{R}^2$, so that no two points in a given distance are colored alike. The question seems very natural and basic, but is yet to be fully answered. The problem goes back to 1950, when Edward Nelson, a graduate student at the University of Chicago at the time, created the problem working on the well known four-color problem. He passed the problem to other mathematicians at the University of Chicago, and soon the question about the chromatic number of the plane became well known in the mathematical community (see [48]). The problem sometimes is incorrectly credited to other mathematicians such as Paul Erdos, Martin Gardner, Hugo Hadwiger, or Leo Moser. Actually, the question was probably published for the first time in Martin's Gardner Mathematical Games" column in Scientific American" in 1960 [13]. Although in the last nearly 60 years the chromatic number of the plane resisted all efforts aiming at an ultimate answer, a considerable amount of research discussing partial answers to this problem, or investigating related problems, accumulated in the mathematical literature. The following bounds on the chromatic number are well known: $4 \leq$ chromatic number of the plane ≤ 7. We shall briefly see how these bounds are obtained. At this point, however, let us mention that it has been shown by Saharon Shelah and Alexander Soifer [49, 50] that the answer to the problem may depend on the choice of axioms of the set theory.

The corresponding problem for the real line $\mathbb{R}$ is easy. That is, $\chi(G(\mathbb{R}, D = \{1\})) = 2$. To see this, partition the vertex set $V(\mathbb{R})$ of G into two non empty disjoint sets such that their union is $\mathbb{R}$. That is, $V(\mathbb{R}) = V_1 \cup V_2$, where $V_1 = \bigcup_{n=-\infty}^{\infty} [2n, 2n+1)$ and $V_2 = \bigcup_{n=-\infty}^{\infty} [2n+1, 2n+2)$. Now color all the vertices of V_1 with color 1 and the vertices of V_2 with color 2. As $V_i, i = 1, 2$ are independent and $G(\mathbb{R}, D = \{1\})$ is bipartite the result follows. Clearly $G(\mathbb{R}^2, \{1\})$ is an infinite graph. The problem of finding the chromatic number of $G(\mathbb{R}^2, \{1\})$ is enormously difficult. Paul Erdos has mentioned this problem as one of his favorite problems. Although he could not solve this problem he has made a significant progress towards the solution of it with a vital result given in Theorem 1.1. First we state the famous Rado's Lemma [33].

Lemma 1.1. *Let M and M_1 be arbitrary sets. Assume that to any $v \in M_1$ there corresponds a finite subset A_v of M. Assume that to any finite subset $N \subset M_1$, a choice function $x_N(v)$ is given, which attaches an element of A_v to each element v of N:$x_N(v) \in A_v$. Then there exists a choice function $x(v)$ defined for all $v \in M_1$ ($x(v) \in A_v$ if $v \in M_1$) with the following property. If K is a finite subset of M_1, then there exists a finite subset $N(K \subset N \subset M_1)$, such that, as far as K is concerned, the function $x(v)$ coincides with $x_N(v) : x(v) = x_N(v)(v \in K)$.*

Theorem 1.1. *Let k be a positive integer, and let G be a graph with the property that any finite subgraph is k-colorable. Then G is k-colorable itself [3].*

Proof. Let M be the set of all k-colors, and let M_1 be the set of all vertices of G. We always choose $A_v = M$. To any finite $N(N \subset M_1)$ there corresponds a finite subgraph of G, consisting of the vertices belonging to N, and all edges between these vertices as far as these belongs to G. This subgraph is assumed to be k-colorable, and so we have a function $x_N(v)$, defined for $v \in N$, taking its values in M by Lemma 1.1. Now the function $x(v)$ defines a coloration of the whole graph G. To show that opposite ends of any edge get different colors, we consider an arbitrary edge e, and we denote the set of its two end-points v_1, v_2 by K. Let N be a finite set satisfying $K \subset N \subset M_1$, $x(v) = x_N(v \in K)$. To N there corresponds a finite graph G_N which is k-colorable by the function $x_N(v)$; G_N contains e. Therefore $x_N(v_1) \neq x_N(v_2)$, and so $x(v_1) \neq x(v_2)$. □

Theorem 1.1 allows us to look for the maximum number of colors needed for the finite subsets of the vertex set of G. It is obvious that $G(\mathbb{R}^2, \{1\})$ is not a trivial graph. Therefore $\chi(G(\mathbb{R}^2, \{1\})) \neq 1$. The presence of at least one edge, viz., between (0,0) and (1,1) in $G(\mathbb{R}^2, \{1\})$ conveys the information that $\chi(G(\mathbb{R}^2, \{1\})) \geq 2$. Similarly, the presence of a clique of size 3, viz., K_3 with vertices at (0,0),$(\frac{1}{2}, \frac{\sqrt{3}}{2})$, (1,1) shows that $\chi(G(\mathbb{R}^2, \{1\})) \geq 3$. Moreover, it is a fact that four points in the Euclidean two dimensional plane $\mathbb{R}^2$ cannot have pairwise odd integer distances.

Therefore, a clique of size 4, viz., K_4 cannot be an induced subgraph of $G(\mathbb{R}^2, \{1\})$. But it will be quite interesting to find the coordinates of a unit distance finite subgraph of $G(\mathbb{R}^2, \{1\})$ such that $\chi(G_1) = 4$. The Moser Spindle graph in Figure 1 with a chromatic 4-coloring is a good example to deduce that $\chi(G(\mathbb{R}^2, \{1\})) \geq 4$. It is interesting to note that so far no unit distance graph that requires exactly five colors are known. Falconer [9] showed that if the color classes form measurable sets of $\mathbb{R}^2$, then at least five colors are needed. Since the construction of non measurable sets requires the axiom of choice, we might have the answer turn out to depend on whether or not we accept the axiom of choice. We can tile the plane with hexagons to obtain a proper 7-coloring of the graph. The result is originally due to Hadwiger and Debrunner [16]. So $4 \leq \chi(G(\mathbb{R}^2, \{1\})) \leq 7$. Despite the age of this problem, very little progress has been made since the initial bounds on $\chi(G(\mathbb{R}^2, \{1\}))$ were discovered shortly after the problem's creation. This fact is a testament to the difficulty of the problem and in the absence of progress on the main problem, a number of restricted versions and related questions have been studied.

Erdos et al in [5] have proved that $\chi(G(\mathbb{Z}, \{2, 3, 5\})) = 4$ and hence $\chi(G(\mathbb{Z}, P)) = 4$. So we can allocate the subsets D of P to four classes, according as $G(\mathbb{Z}, D)$ has chromatic number 1,2,3 or 4. Obviously empty set is the only member of class 1 and every singleton subset is in class 2. We study here the open problem of characterizing the Class 3 and Class 4 sets. In particular we concentrate on those distance sets D where D is a subset of not only primes P but also they are a special set of primes. A motivation considering the special types of primes stems from the fact that these special class of primes have not only interesting properties but also have stunning applications in various fields. For instance., the wonderful work of Von Mag.Ingrid G.Abfalter in [55] has shown an interesting application involving graph vertex coloring and Fibonacci numbers in the problem of sequence design of nucleic acids. Moreover Voigt in [54]has characterized the 4 sets when the the distance set $D \subset P$ is of cardinality 4. The integer distance graph $G(\mathbb{Z}, D)$ with distance set $D = \{d_1 < d_2 < \dots\}$ has the set of integers Z as the vertex set and two vertices $x, y \in \mathbb{Z}$ are adjacent if and only if $|x - y| \in D$. Integer distance graphs were first systematically studied by Eggleton, Erdös and Skilton [5] and have been studied by others since then see [41, 53, 58, 59].

In the present paper we consider the problem of determining the chromatic number of certain integer distance graphs whose distance set D is either 1) a set of $(n+1)$ positive integers for which the n^{th} power

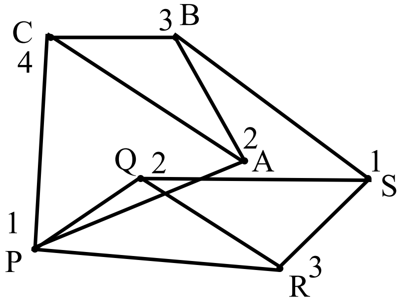

Figure 1. *The Moser Spindle*

of the last is the sum of the n^{th} powers of the previous terms; Here, the smallest sets for known values of n are: for $n = 3$, the distance set $D_1 = \{3, 4, 5; 6\}$, for $n = 4$, the distance set $D_2 = \{30, 120, 272, 315; 353\}$, for $n = 5$, the distance set $D_3 = \{19, 43, 46, 47, 67; 72\}$, for $n = 7$, the distance set $D_4 = \{127, 258, 266, 413, 430, 439, 525; 568\}$, for $n = 8$, the distance set $D_5 = \{90, 223, 478, 524, 748, 1088, 1190, 1324; 1409\}$. Like this one can continue to construct distance sets for other values of n. But the process of such a construction will be arduous, or 2) a set of pythagorean quadruples or 3) a set of pythagorean n-tuples or 4) a set of square distances or 5) a set of abundant numbers or deficient numbers or carmichael numbers, or 6) a set of polytopic numbers, or 7) a set of happy numbers or lucky numbers, or 8) a set of Lucas numbers or 9) a set of $\mathcal{U}$lam numbers, or 10) a set of weird numbers. Besides finding the exact chromatic number of a few specific distance graphs we also give useful upper and lower bounds for general cases.

2. Chromatic number of a distance graph with typical distance sets

We say a coloring c of the set $\mathbb{Z}$ of integers is a pattern periodic coloring if there is an integer p, and a permutation f of the set of colors such that for every $x \in \mathbb{Z}$, we have $c(x) = f(c(x-p)$. We call p the period of the pattern periodic coloring c, and call f the color permutation of c.

Theorem 2.1. *If* $D_1 = \{3, 4, 5; 6\}$, *then* $\chi(G(\mathbb{Z}, D_1)) = 3$.

First we prove the following two results.

Lemma 2.1. $\chi(G(\mathbb{Z}, D)) = 2$ *if* D *consists only of odd integers.*

Proof. Partition $V(\mathbb{Z})$ into two sets with $V(\mathbb{Z}) = V_1(\mathbb{Z}) \cup V_2(\mathbb{Z})$ such that an integer $x \in V_i(\mathbb{Z})$ if and only if $x \equiv i(mod\ 2)$. Now as the elements of $V_1(\mathbb{Z})$ are even and the elements of $V_2(\mathbb{Z})$ are odd we find that V_j is an independent set for $j = 1, 2$. Therefore G is bipartite and hence $\chi(G(\mathbb{Z}, D)) = 2$. □

Lemma 2.2. *Let D be a subset of $\mathbb{Z}$. If for a given positive integer n, D^n is the set of all those elements of D built by integers divisible by n, then $\chi(G(\mathbb{Z}, D)) \leq min_{n\in N} n(|D_0|^n + 1)$ where $D_0 \subset D$ is that subset of D built by integers divisible by n.*

Proof. Partition the vertices of $G(\mathbb{Z}, D)$ into residue classes $V_i = \{v \in V(G) = \mathbb{Z} : v \equiv i(mod\ n)\}$, $i = 0, 1, \ldots, n-1$, with respect to arbitrary positive integer n. If u_i and $v_i \in V_i$ then u_i is adjacent to w_i if and only if $|u_i - w_i| \in D^n$. Since $|D_0| + 1$ is upper bound of $\chi(G(\mathbb{Z}, D))$ if D_0 is finite and since the induced subgraphs $< V_0 >, < V_1 >, \ldots, < V_{n-1} >$ are pairwise isomorphic and therefore isomorphic to $G(\mathbb{Z}, (1/n)D^n)$ which has distance set $\{(d/n) : d \in D_0^n\}$, we obtain $\chi(G(\mathbb{Z}, (1/n)D^n)) \leq |D_0|^n + 1$. This implies that the vertices of each V_i, $i = 0, 1, \ldots, n-1$, can be colored with $|D_0|^n + 1$ colors. The number of residue classes is n and therefore all the vertices of $G(\mathbb{Z}, D)$ are are to be colored by $n(|D_0|^n + 1)$ colors. □

Proof of Theorem 2.1. Now as D_1 consists of both odd and even integers we infer from Lemma 2.1 that $G(\mathbb{Z}, D_1)$ contains odd cycles and hence $\chi(G(\mathbb{Z}, D)) \geq 3$. Further it follows from Lemma 2.2 that $\chi(G(\mathbb{Z}, D)) \leq 5$. Now partition the vertex set of $\mathbb{Z}$ as $V(\mathbb{Z}) = V_1(\mathbb{Z}) \cup V_2(\mathbb{Z}) \cup V_3(\mathbb{Z})$ where $V_1(\mathbb{Z}) = \{x \in \mathbb{Z} : x \equiv 0, 1, 2(mod\ 9)\}$; $V_2(\mathbb{Z}) = \{x \in \mathbb{Z} : x \equiv 3, 4, 5(mod\ 9)\}$ and $V_3(\mathbb{Z}) = \{x \in Z : x \equiv 6, 7, 8(mod\ 9)\}$. It is easy to see that each $V_j(\mathbb{Z})$ is an independent set for $j = 1, 2, 3$. Now assign one color each to these three sets. This results in a chromatic 3-coloring of $G(\mathbb{Z}, D)$ and hence $\chi(G(\mathbb{Z}, D)) = 3$. □

What makes $D_1 = \{3, 4, 5; 6\}$ special?

Note that $6^3 = 3^3 + 4^3 + 5^3$. This gives rise to the following question. Consider a set of $(n + 1)$ positive integers for which the n^{th} power of the last is the sum of the n^{th} powers of the previous terms. The smallest sets for known values of n are : for $n = 3$ we have $D_1 = \{3, 4, 5; 6\}$, for $n = 4$ we have $D_2 = \{30, 120, 272, 315; 353\}$, for $n = 5$ we have $D_3 = \{19, 43, 46, 47, 67; 72\}$, for $n = 7$ we have $D_4 = \{127, 258, 266, 413, 430, 439, 525; 568\}$, for $n = 8$ we have $D_5 = \{90, 223, 478, 524, 748, 1088,$

$1190, 1324; 1409\}$. We observe that each D_i for $2 \leq i \leq 5$ consists of both odd and even integers. Moreover $gcd(D_i) = 1$. Hence it follows from Theorem 2.1 that $\chi(G(\mathbb{Z}, D)) \geq 3$. We conjecture that $\chi(G(\mathbb{Z}, D_i)) \leq n+2$, for $i \geq 2$. We inch forward towards this conjecture with the following result.

Theorem 2.2. *Suppose that D is a finite set of integers with $d = max\{y : y \in D\}$. If the subgraph of G induced by the set $\{0, 1, \ldots, k^d/(k! + d - 1)\}$ is k-colorable, then the graph $G(\mathbb{Z}, D)$ is also k-colorable.*

Proof. Consider the set S of all sequences s of k colors of length d. Define an equivalence relation on S as follows. We say two sequences of colors $s = (x_1, x_2, \ldots, x_d)$ and $s' = (y_1, y_2, \ldots, y_d)$ are equivalent, denoted by, $s \sim s'$, if there is a permutation f of the k colors such that $y_i = f(x_i)$ for $i = 1, 2, \ldots, d$. Each equivalence class contains $k!$ sequences and hence there are $k^d/k!$ equivalence classes. Let $t = k^d/k!$. Consider a k-coloring of the graph $G(\mathbb{Z}, D)$, which is a two way infinite sequence of colors. Each segment of this sequence of length d is an element of S. Let $s_i = (c(i), c(i + 1), \ldots, c(i + d - 1))$. Consider the sequences $s_0, s_1, \ldots, s_t$. By the Pigeonhole principle, there are integers $i < j$, such that $s_i \sim s_j$. Hence there is a permutation f of the colors such that $s_j = f(s_i)$. That is., $c(j + k) = f(c(i + k))$ for $k = 0, 1, \ldots, d - 1$. Now define a partial pattern periodic coloring c' of the distance graph $G(\mathbb{Z}, D)$ with period $j - i$ as follows: For $i \leq x \leq j + d - 1$, let $c'(x) = c(x)$; for $x \geq j + d$, let $c'(x) = f(c'(x - (j - i)))$. It can be easily seen by recursion that c' is defined on the set of all integers $x \geq i$. For any $x \geq j$, let $c'(x) = f(c'(x - (j - i)))$. We claim that c' is a proper coloring of the subgraph of $G(\mathbb{Z}, D)$ induced by the integers $x \geq i$. Suppose not then let y be the smallest integer such that there exists an integer $x < y$ such that $c'(x) = c'(y)$ and $y - x \in D$. Since c is a proper coloring of $G(\mathbb{Z}, D)$ and c' agrees with c on the segment $\{i, i + 1, \ldots, j + d - 1\}$, it follows that $y \geq j + d$. As $d = max\{y : y \in D\}$, it follows that $x \geq j$ and hence $x - (j - i) \geq i$. Therefore $c'(y - (j - i))$, $c'(x - (j - i))$ are defined, and $c'(y) = f(c'(y - (j - i)))$ and $c'(x) = f(c'(x - (j - i)))$. However, by the choice of y, we know that $c'(y - (j - i)) \neq c'(x - (j - i))$ because $(y - (j - i)) - (x - (j - i)) = y - x \in D$. This is a contradiction, as f is a permutation of the colors. We extend c' to the whole set $\mathbb{Z}$ of integers as follows: For $x \leq i - 1$, let $c'(x) = f^1(c'(x + (j - i)))$. The same argument shows that c' is a proper coloring of G. So $j - i \leq k^d/k!$. So if c is a proper k-coloring of the segment $\{0, 1, \ldots, k^d/(k! + d - 1)\}$, then there exists an integer $0 \leq i < j \leq k^d/k!$ such that the restriction of c to the segment $\{i, i+1, \ldots, j+d-1\}$ can be extended to a pattern periodic coloring of the whole graph. □

In view of Theorem 2.2, for the distance set $D_2 = \{30, 120, 272, 315; 353\}$ if we could find a 3-coloring for the subgraph induced by the set $\{0, 1, \ldots, 3^{353}\ /(3! + 352)\}$ then the whole graph $G(\mathbb{Z}, D_2)$ is 3-colorable. Similarly for the other distance sets D_j we can apply the Theorem 2.2. Although finding a 3-coloring for a huge set like $\{0, 1, \ldots, 3^{353}/358\}$ seems to be difficult there exists an algorithm [4] to determine $\chi(G(\mathbb{Z}, D_j))$. That is, let $q = max(D_j)$. Then $\chi(G(\mathbb{Z}, D_j)) \leq q + 1$. Consider the colorings of the subgraph $G(\mathbb{Z}, D)$ induced by $S = [1, q^q + q]$. Then for $k \leq q$, if $< S >$ has a proper k-coloring, then $\chi(G(\mathbb{Z}, D)) \leq k$. Let $k \leq q$, and suppose that $< S >$ has proper k-coloring. The number of k-coloring of a block of q consecutive integers is q^q at most. Since $< S >$ contains $q^q + q$ such blocks, two such blocks contained in S (say $[a, a+q-1]$ and $[b, b+q-1]$, with $a < b$) receive the same pattern of colors. We extend the coloring of $[a, b+q-1]$ to a coloring f of $\mathbb{Z}$ by the rule $f(i+a-b) = f(i)$ for all i. This is a proper coloring of $G(\mathbb{Z}, D)$ and so $\chi(G(\mathbb{Z}, D_j)) \leq k$. However we ask here two important questions.

1) What is the exact chromatic number of $G(\mathbb{Z}, D_j)$ for a given j?
2) How do we determine the elements of D_j for any given j?

We also raise the following conjecture.

Conjecture 1. *Let $k \geq 3$. Determining whether $\chi(G(\mathbb{Z}, D)) \leq k$ for finite sets D is NP-complete.*

3. Chromatic number of pythagorean tuples

Xuding Zhu [59] has proved that if $a < b < c$ are integers with $gcd(a, b, c) = 1$ and that $D = \{a, b, c\}$ and $G = G(\mathbb{R}, D)$ then $\chi(G(\mathbb{R}, D)) = 2$ if all the integers a, b, c are odd; $\chi(G(\mathbb{R}, D)) = 4$ if $a = 1$, $b = 2$ and $c \equiv 0(mod\ 3)$; $\chi(G(\mathbb{R}, D)) = 4$ if $a + b = c$ and $a \not\equiv b(mod\ 3)$; $\chi(G(\mathbb{R}, D)) = 3$ for all other cases.

In view of the above all pythagorean triples (a, b, c) with $D = \{a, b, c\}$ has chromatic number 3 for its distance graph $G(\mathbb{Z}, D)$. This is because, a typical pythagorean triple has at least one even integer so the chromatic number cannot be 2. Further as $1^2+2^2 \neq 3k^2$ for any k and $a^2+b^2 \neq (a+b)^2$ it cannot have chromatic number 4.

3.1. Pythagorean quadruples

In a rectangular solid, the length of an interior diagonal is determined by the formula $a^2 + b^2 + c^2 = d^2$, where a, b, c are the dimensions of the solid and d is the diagonal. When a, b, c, d are integers then a pythagorean quadruples is formed. Mordell [30] developed a solution to this Diophantine equation using integer parameters (m, n, p), where $m + n + p \equiv 1(mod\ 2)$ and $gcd(m, n, p) = 1$. The formulae are $a = 2mp$, $b = 2np$, $c = p^2 - (n^2 + m^2)$, $d = p^2 + (n^2 + m^2)$. However, the pythagorean quadruples $(36, 8, 3, 37)$ cannot be generated by Mordells formulae, since c must be smaller of the odd numbers and $3 = p^2 - (n^2 + m^2)$, $37 = p^2 + (n^2 + m^2)$, implies $40 = 2p^2$ and it means $20 = p^2$ and so p is not an integer.

We note that, for positive integers a and b, where a or b or both are even, there exists integers c and d such that $a^2 + b^2 + c^2 = d^2$. When a and b are both odd no such integers c and d exist. When a and b are of opposite parity, then $c = (a^2 + b^2 - 1)/2$ and $d = (a^2 + b^2 + 1)/2$. This is because, $d^2 - c^2 = (d + c)(d - c) = [(a^2 + b^2 + 1)/2 + (a^2 + b^2 - 1)/2][(a^2 + b^2 + 1)/2 - (a^2 + b^2 - 1)/2] = [a^2 + b^2](1) = a^2 + b^2$, therefore $a^2 + b^2 + c^2 = d^2$. As a and b differ in parity, c and d given above are integers. Also we see that c and d are consecutive integers. Therefore $gcd(a, b, c, d) = 1$ even when $gcd(a, b) \neq 1$. If and b are both even then $c = ((a^2 + b^2)/4) - 1$ and $d = ((a^2 + b^2)/4) + 1$. This is because, $16(d^2 - c^2) = (a^2 + b^2 + 4)^2 - (a^2 + b^2 - 4)^2 = 16(a^2 + b^2)$. Therefore $a^2 + b^2 + c^2 = d^2$. Since a and b are both even, c and d given above are integers. Notice that if $a - b \equiv 0(mod\ 4)$ then $(a^2 + b^2)/4$ is even and so c and d are consecutive odd integers and hence in this case $gcd(a, b, c, d) = 1$. But if $a - b \equiv 2(mod\ 4)$ then $(a^2 + b^2)/4$ is odd and we get that c and d are consecutive even integers, and $gcd(a, b, c, d) \neq 1$. Finally, if a and b are both odd, then $a^2 \equiv b^2 \equiv 1(mod\ 4)$. Since $c^2 \equiv 0(mod\ 4)$ or $c^2 \equiv 1(mod\ 4)$, and similarly for d^2, we have $a^2 + b^2 + c^2 \equiv 2(mod\ 4) \neq d^2$ for any integer d or $a^2 + b^2 + c^2 \equiv 3(mod\ 4) \neq d^2$ for any integer d. Hence no pythagorean quadruple exist in this case.

In the view of the above discussion, first let us consider only those pythagorean quadruples (a, b, c, d) where a and b are either of opposite parity or both even with a, b, c and d are all distinct. Let $D = \{a, b, c, d\}$ be such a distance set. We notice by Theorem 2.1 that $3 \leq \chi(G(\mathbb{Z}, D)) \leq 5$. As matter of investigative instance consider the case where the distance set is $D = \{2, 3, 6, 7\}$. Clearly $(2, 3, 6, 7)$ is a pythagorean quadruple. $(2^2 + 3^2 + 6^2 = 7^2)$. By Lemma 2.2 we can easily see that $\chi(G(\mathbb{Z}, \{2, 3, 6, 7\})) \leq 4$ as there is no element divisible by 4. Now we claim that $\chi(G(\mathbb{Z}, \{2, 3, 6, 7\})) \geq$

4. Let $\mathcal{C}$ be any arbitrary chromatic 3-coloring of $G(\mathbb{Z}, \{2,3,6,7\})$ with colors c_1, c_2 and c_3. And let $\mathcal{C}(c_i)$ denote the set of all elements of Z with color c_i. First we note that $\mathcal{C}$ cannot color any three consecutive integers with the same color. This is because, if $i, i+1, i+2$ share the same color, then $i+2-i=2$ and the fact that $2 \in \{2,3,6,7\}$ forces an edge between $i+2$ and i in $G(\mathbb{Z}, \{2,3,6,7\})$. Now, without loss of generality, assume that $\mathcal{C}$ assigns the color c_1 to $i, i+1$. Then the elements $i-1, i+2$ must be colored with either of the remaining colors c_2 or c_3 under $\mathcal{C}$. Again by a similar argument, we notice that both $i-1, i+2$ cannot be colored by either c_2 or c_3 under $\mathcal{C}$. So assume that $i+2$ is assigned the color c_2 and $i-1$ the color c_3 under $\mathcal{C}$. We iteratively build up the elements with colors c_1 or c_2 or c_3 by carefully repeating the above argument wherever necessary. This process leads to the stage where: $\mathcal{C}(c_1) = \{i, i+1, i-4, i+5, i-5, ...\}$, $\mathcal{C}(c_2) = \{i+2, i+3, i-3, ...\}$ and $\mathcal{C}(c_3) = \{i-1, i-2, i+4, ...\}$. Now the element $i+6$ cannot be colored by any of these 3 colors c_1 or c_2 or c_3. This is because, $(i+6)-i=6, (i+6)-(i+3)=3$ and $(i+6)-(i+4)=2$ and the fact that 2, 3, 6$\in \{2,3,6,7\}$ forces $(i+6,i), (i+3,i+6), (i+4,i+6)$ to belong to the edge set of $G(\mathbb{Z}, \{2,3,6,7\})$. This contradiction shows that $\mathcal{C}$ requires at least 4 colors to chromatically color $G(\mathbb{Z}, \{2,3,6,7\})$. Hence $\chi(G(\mathbb{Z}, \{2,3,6,7\})) \geq 4$. Thus we have the following theorem.

Theorem 3.1. $\chi(G(\mathbb{Z}, \{2,3,6,7\}) = 4$.

Note. It is interesting to notice that the distance set $D = \{2,3,6,7\}$ serves as an evidence for the existence of an extremal graph for the graph equation $\chi(G(\mathbb{Z}, D)) = 4$. On similar lines one can prove that $\chi(G(\mathbb{Z}, 1, 4, 8, 9) = 4$. These two instances motivate us to conjecture that $\chi(G(\mathbb{Z}, D)) = 4$, whenever D is a distance set of distinct pythagorean quadruple $\{a, b, c, d\}$ where a and b are either of opposite parity or both even.

3.2. Pythagorean n-tuples

One can find formulae for pythagorean n-tuples on similar lines as in pythagorean quadruples. Suppose that $S = (a_1, a_2, \ldots, a_{n-2})$, where a_i is an integer, and let T be the cardinality of the odd integers in S. If $T \not\equiv 2(mod\ 4)$ then there exist integers a_{n-1} and a_n such that $a_1^2 + a_2^2 + \cdots + a_{n-1}^2 = a_n^2$. To see this, let $T \equiv 1(mod\ 2)$. This means $T \equiv 1(mod\ 4)$ or $T \equiv 3(mod\ 4)$. Now set $a_{n-1} = (a_1^2 + a_2^2 + \cdots + a_{n-2}^2 - 1)/2$ and $a_n = (a_1^2 + a_2^2 + \cdots + a_{n-2}^2 + 1)/2$. Then we have $a_n^2 - a_{n-1}^2 = (a_n + a_{n-1})(a_n - a_{n-1}) = \sum_{j=1}^{n-2} a_j^2$. Again let $T \equiv 0(mod\ 4)$. Here set

$a_{n-1} = ((a_1^2+a_2^2+\cdots+a_{n-2}^2)/4)-1$ and $a_n = ((a_1^2+a_2^2+\cdots+a_{n-2}^2)/4)+1$. Then we have $a_n^2 - a_{n-1}^2 = (a_n + a_{n-1})(a_n - a_{n-1}) = \sum_{j=1}^{n-2} a_j^2$. Finally if $T \equiv 2(mod\, 4)$, then $a_1^2+a_2^2+\cdots+a_{n-2}^2 \equiv 2(mod\, 4)$. Since $a_{n-1}^2 \equiv 0(mod\, 4)$ or $a_{n-1}^2 \equiv 1(mod\, 4)$, we have $a_1^2+a_2^2+\cdots+a_{n-2}^2+a_{n-1}^2 \equiv 2(mod\ 4) \neq a_n^2$ for any integer a_n or $a_1^2 + a_2^2 + \cdots + a_{n-2}^2 + a_{n-1}^2 \equiv 3(mod\ 4) \neq a_n^2$ for any integer a_n. Hence no pythagorean n-tuple exists in this case.

Now consider only those pythagorean n-tuples $(a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n)$ where

1) the cardinality of odd integers in the set $\{a_1, a_2, \ldots, a_{n-2}\}$ is not congruent to 2 modulo 4, and
2) a_j's are all distinct for $1 \leq j \leq n$.

Let $D = \{a_1, a_2, \ldots, a_{n-2}, a_{n-1}, a_n\}$ be a distance set satisfying the above condition. Then as $\gcd(D) = 1$, we can easily deduce that $3 \leq \chi(G(\mathbb{Z}, D)) \leq n + 1$. The existence of extremal graphs for the graph equation $\chi(G(\mathbb{Z}, D)) = 5$ when D is a pythagorean quadruple set as in Theorem 3.1 indicates the probable existence of an extremal graph for the graph equation $\chi(G(\mathbb{Z}, D)) = n + 1$ when D is a set of pythagorean n-tuple of the above type. It would be interesting to find one such D.

4. Chromatic number of square distance graphs

What is the chromatic number of a square distance graph $G(\mathbb{Z}, D)$ where $D = \{d_1^2, d_2^2, \ldots, d_n^2\}$ is a finite set of all positive squares. First observe that any pythagorean triple induces a complete graph on three vertices, K_3 in the square distance graph. This is because, consider the Euclid's fundamental formula for generating a pythagorean triple. Suppose that m and n are positive integers with $m > n$. Then $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$ forms a pythagorean triple. Now consider an integer distance graph $G(\mathbb{Z}, D)$ with distance set $D = \{a^2, b^2, c^2\}$ where a, b, c forms a pythagorean triple. Consider a subgraph G_1 of G on the vertex set $\{0^2, a^2, c^2\}$. Clearly $a^2 - 0^2 = a^2$, $c^2 - 0^2 = c^2$ and $c^2 - a^2 = (m^2 + n^2)^2 - (m^2 - n^2)^2 = 4m^2n^2 = (2mn)^2 = b^2$ are all elements of D. Therefore $G_1 \cong K_3$. So we deduce that $\chi(G(\mathbb{Z}, D)) \geq 3$, where D is a set of pythagorean triples.

Next we observe that no pythagorean quadruple induces a complete graph on four vertices, K_4 in the square distance graph. This is because; consider the fundamental formula for generating a pythagorean quadruple. Suppose that l, m, n are positive integers. Then $a = (l^2 - m^2 - n^2)$,

$b = (2lm)$, $c = (2ln)$ and $d = (l^2 + m^2 + n^2)$ forms a pythagorean quadruple. Consider an integer distance graph $G(\mathbb{Z}, D)$ with distance set $D = \{a^2, b^2, c^2, d^2\}$ where a, b, c and d forms a pythagorean quadruple. Note that $a^2 - b^2, a^2 - c^2, d^2 - a^2, d^2 - b^2, d^2 - c^2, b^2 - c^2$ are different from a^2, b^2, c^2 and d^2. Hence no three of them goes along with 0 to form a vertex set of K_4.

However we also know that if a set D of positive integers contains a multiple of every positive integer and the distance graph $G(\mathbb{Z}, D)$ has a finite uniquely k-colorable subgraph then $\chi(G(\mathbb{Z}, D)) \geq k + 1$. In view of this, as K_3 is uniquely 3-colorable and D contains a multiple of every positive integer we have $\chi(G(\mathbb{Z}, D)) \geq 4$. Further Eggleton [5] has showed the existence of a K_4 in the square distance graph with vertices $0^2, 672^2, 680^2, 697^2$ producing differences that are squares namely $680^2 - 672^2 = 104^2$, $697^2 - 680^2 = 153^2$ and $697^2 - 672^2 = 185^2$. Again as K_4 is uniquely 4-colorable we deduce that $\chi(G(\mathbb{Z}, D)) \geq 5$. Our hand/computer calculations indicate that the upper bound may not be finite.

Problem. *Decide? whether or not 5 is a best lower bound for the square distance graph.*

In view of this discussion we generalize our observation to state that no pythagorean n-tuple will induce a complete graph on n vertices for $n \geq 4$.

5. Chromatic number of abundant number, deficient number and carmichael number distance graphs

An abundant number is a number n for which the sum of divisors $\sigma(n) > 2n$ or the sum of proper divisors $s(n) > n$. The value $\sigma(n) - 2n$ or $s(n) - n$ is known as the abundance. The first few abundant numbers are

$$12, 18, 20, 24, 30, 36, 40, 42, 48, 54, 56, 60, 66, 70, 72, \ldots$$

The smallest abundant odd number is 945. The smallest abundant number not divisible by 2 or 3 is 5391411025. Infinitely many even and odd abundant numbers exist. Every multiple of abundant number is abundant.

Although the proof of Theorem 5.1 follows from the Lemma 2.2 we give an alternative proof as follows.

Theorem 5.1. *Suppose that $G(\mathbb{Z}, D)$ is an integer distance graph with $D = \{12, 24, 36, \ldots, 12k\}$ a subset of the set of all abundant numbers. Then $\chi(G(\mathbb{Z}, D)) \leq 4$.*

Proof. Suppose $D = \{12, 24, 36, \ldots, 12k\}$ is a distance set for the graph $G(\mathbb{Z}, D)$. Then we observe that the set $\{0, 12, 24, 36, \ldots, 12k\} = D$ of integers induces a clique of size $k+1$. Therefore $\chi(G(\mathbb{Z}, D)) \geq k+1$. To exhibit a $k+1$ chromatic coloring, partition $V(\mathbb{Z})$ as $V(\mathbb{Z}) = \bigcup_{i=0}^{k}[(k+1)\mathbb{Z}+i]$, where $[(k+1)\mathbb{Z}+i]$ is the equivalence class of $(k+1)\mathbb{Z}+i$. That is, $V(\mathbb{Z}) = \{0, k+1, 2k+2, \ldots\} \cup \{1, k+2, 2k+3, \ldots\} \cup \cdots \cup \{k, 2k+1, 3k+2, \ldots\}$. Set $V_i = [(k+1)\mathbb{Z}+i]$ for $i = 0, 1, \ldots, k$. Assign a color $c(v) = i$ whenever $v \in V_i$. This yields a proper $(k+1)$-chromatic coloring for $G(\mathbb{Z}, D)$. □

Note. Suppose that $G(\mathbb{Z}, D)$ is an integer distance graph with $D = \{18, 30, 42, \ldots, 12k+6\}$ a subset of the set of all abundant numbers. Then by, Lemma 2.2, $\chi(G(\mathbb{Z}, D)) \leq 4$.

An deficient number is a number n for which the sum of divisors $\sigma(n) < 2n$ or the sum of proper divisors $s(n) < n$. The value $2n - \sigma(n)$ or $n - s(n)$ is known as the deficiency. The first few deficient numbers are

$$1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 21, 22, 23, 25, 26, 27, \ldots$$

All odd numbers with one or two distinct prime factors are deficient. All proper divisors of a deficient number are deficient.

Note. Consider the integer distance graph $G(\mathbb{Z}, D)$, where D is a finite set of deficient numbers. Since $\gcd(D) = 1$ and D contains odd numbers we get from Lemma 2.2 of Theorem 2.1 that $3 \leq \chi(G(\mathbb{Z}, D)) \leq \min_{n \in \mathbb{N}} n(|D_0|^n + 1)$ where $D_0 \subset D$ is the subset of D built by integers divisible by n.

A Carmicheal number is a composite integer n which satisfies the congruence $b^{n-1} \equiv 1 (mod\ n)$ for all integers b which are relatively prime to n. These numbers are important because they pass the Fermat primality test but are not actually primes. As numbers become larger, Carmicheal numbers become rare. For example, there are 20,138,200 Carmicheal numbers between 1 and 10^{21} (approximately one in 50 billion numbers). A positive composite integer n is a Carmicheal number if and only if n is square free and for all prime divisors p of n, it is true that $p - 1 | n - 1$. It follows from this that all Carmicheal numbers are odd. That is, any even composite number that is square free (and hence only one prime factor of two) will have at least one odd prime factor, and thus $p - 1 | n - 1$ results in an even dividing an odd, a contradiction. The first such number is 561. The next few such numbers are 1105, 1729, 2465, 2821, 6601, 8911, etc. So if D is a set of all Carmicheal numbers then it follows from Lemma 2.1 of Theorem 2.1 that $\chi(G(\mathbb{Z}, D)) = 2$.

6. Chromatic number of polytopic number distance graphs

Suppose that D is the set of triangular numbers, $D = \{1, 3, 6, 10, \ldots, {}_{n+1}c_2\}$. Form a distance graph $G(\mathbb{Z}, D)$. As $\gcd(D) = 1$ and D contains odd numbers we get from Lemma 2.2 of Theorem 2.1 that $3 \leq \chi(G(\mathbb{Z}, D)) \leq \min_{n\in\mathbb{N}} n(|D_0|^n + 1)$ where $D_0 \subset D$ is the subset of D built by integers divisible by n. If D is a set of tetrahedral numbers, $D = \{1, 4, 10, \ldots, {}_{n+2}c_3\}$ then $3 \leq \chi(G(\mathbb{Z}, D)) \leq \min_{n\in\mathbb{N}} n(|D_0|^n + 1)$ where $D_0 \subset D$ is the subset of D built by integers divisible by n.

The term figurate number is used for members of different sets of numbers generalizing from triangular members to different shapes (polygonal numbers) and different dimensions (polyhedral numbers). The r^{th} diagonal of pascal's triangle for $r \geq 0$ consists of the figurate numbers for the r-dimensional analogs of triangles. The simplicial polytopic numbers for $r = 1, 2, 3, 4, \ldots$ are ${}_{n+0}c_1$, ${}_{n+1}c_2$, ${}_{n+2}c_3$, ${}_{n+3}c_4, \ldots$, ${}_{n+r-1}c_r$ called respectively linear numbers, triangular numbers, tetrahedral numbers, pentachoron numbers, $\ldots$, r-topic numbers. In view of this, if D is a set of any of the above polytopic numbers then $3 \leq \chi(G(\mathbb{Z}, D)) \min_{n\in\mathbb{N}} n(|D_0|^n + 1)$ where $D_0 \subset D$ is the subset of D built by integers divisible by n.

7. Chromatic number of happy number and lucky number distance graphs

A happy number is defined by the following process. Starting with any positive integer, replace the number by the sum of the squares of its digits, and repeat the process until the number equals 1 (where it will stay), or it loops endlessly in a cycle which does not include 1. Those numbers for which this process ends in 1 are happy numbers, while those that do not end in 1 are unhappy numbers. For example, 19 is happy, as $1^2 + 9^2 = 82$, $8^2 + 2^2 = 68$, $6^2 + 8^2 = 100$, $1^2 + 0^2 + 0^2 = 1$. If D is a finite set of happy numbers, $D = \{1, 7, 10, 13, 19, 23, \ldots\}$ then $\gcd(D) = 1$ and by Lemma 2.2 of Theorem 2.1, we have $3 \leq \chi(G(\mathbb{Z}, D)) \min_{n\in\mathbb{N}} n(|D_0|^n + 1)$ where $D_0 \subset D$ is the subset of D built by integers divisible by n. A happy prime is a number that is both happy and prime. The happy prime numbers are 7, 13, 19, 23, 31, ... If D is a set of happy prime numbers then it follows from Lemma 2.1 that $\chi(G(\mathbb{Z}, D)) = 2$. So it follows easily that if D is a set of pythagorean n-tuples with all integers happy prime then $\chi(G(\mathbb{Z}, D)) = 2$ for $n \geq 2$.

A lucky number is a natural number in a set which is generated by a "sieve". Begin with a list of integers starting with 1.

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25.$$

Every second number is eliminated (all even numbers), leaving only the odd integers.

$$1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25.$$

The second term in this sequence is 3. Every third number which remains in the list is eliminated.

$$1, 3, 7, 9, 13, 15, 19, 21, 25.$$

The third surviving number is now 7, so every seventh number that remains is eliminated.

$$1, 3, 7, 9, 13, 15, 21, 25.$$

As this procedure is repeated indefinitely, the survivors are the lucky numbers.

$$1, 3, 7, 9, 13, 15, 21, 25, 31, 33, 37, 43, 49, 51, 63, \ldots$$

A lucky prime is a lucky number that is prime. If the distance set D is a set of lucky numbers or a set of lucky prime numbers then the corresponding distance graph $G(\mathbb{Z}, D)$ has chromatic number 2. This follows from the fact that the lucky numbers and lucky prime numbers are odd and from Lemma 2.1 of Theorem 2.1.

8. Chromatic number of lucas number distance graphs

A Lucas number is defined as: $L_n = 2$, if $n = 0$; 1 if $n = 1$; $L_{n-1} + L_{n-2}$ if $n > 1$. The sequence of Lucas numbers begins: 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, $\ldots$. If D is a finite set of Lucas numbers then $\gcd(D) = 1$ and by Lemma 2.2 of Theorem 2.1 we have $3 \leq \chi(G(\mathbb{Z}, D)) \leq \min_{n\in\mathbb{N}} n(|D_0|^n + 1)$ where $D_0 \subset D$ is the subset of D built by integers divisible by n. A Lucas prime is a Lucas number that is prime. A first few Lucas primes are 2, 3, 7, 11, 29, 47, 199, 521, 2207, 3571, 9349, $\ldots$.

If A is a set of Lucas prime numbers then as $A \subset P$, the set of all primes, it follows that $\chi(G(\mathbb{Z}, A)) \leq \chi(G(\mathbb{Z}, P)) = 4$. We claim that

$\chi(G(\mathbb{Z}, A)) \geq 3$. Choose from A, 2 and any Lucas prime p. Then we note that $G(\mathbb{Z}, A)$ contains a cycle $v_0v_1v_2, \ldots, v_pwv_0$ where $v_i = 2i$ for $0 \leq i \leq p$, and $w = p$. As this cycle has order $p + 2$, which is odd, it has chromatic number 3. Hence the claim follows.

Suppose that $\chi(G(\mathbb{Z}, A)) = 3$. Let $\mathcal{C}$ be any arbitrary chromatic 3-coloring of $G(\mathbb{Z}, A)$ with colors c_1, c_2 and c_3. And let $\mathcal{C}(c_i)$ denote the set of all elements of $\mathbb{Z}$ with color c_i. First we note that $\mathcal{C}$ cannot color any three consecutive integers with the same color. This is because, if $i, i+1, i+2$ share the same color, then $i + 2 - i = 2$ and the fact that $2 \in A$ forces an edge between $i+2$ and i in $G(\mathbb{Z}, A)$. Now, without loss of generality, assume that $\mathcal{C}$ assigns the color c_1 to $i, i+1$. Then the elements $i-1, i+2$ must be colored with either of the remaining colors c_2 or c_3 under $\mathcal{C}$.

Again by a similar argument, we notice that both $i-1, i+2$ cannot be colored by either c_2 or c_3 under $\mathcal{C}$. So assume that $i+2$ is assigned the color c_2 and $i-1$ the color c_3 under $\mathcal{C}$. We iteratively build up the elements with colors c_1 or c_2 or c_3 by carefully repeating the above argument wherever necessary. This process leads to the stage where: $\mathcal{C}(c_1) = \{i, i+1, i-4, i+5, ...\}$, $\mathcal{C}(c_2) = \{i+2, i+3, i-3, ...\}$ and $\mathcal{C}(c_3) = \{i-1, i-2, i+4, ...\}$. Now the element $i+6$ cannot be colored by any of these 3 colors c_1 or c_2 or c_3. This is because, $(i+6) - (i-5) = 11, (i+6) - (i+3) = 3$ and $(i+6) - (i+4) = 2$ and the fact that 2, 3, 11$\in A$ forces $(i+6, i-5), (i+3, i+6), (i+4, i+6)$ to belong to the edge set of $G(\mathbb{Z}, A)$. This contradiction shows that $\mathcal{C}$ requires at least 4 colors to chromatically color $G(\mathbb{Z}, A)$. Hence $\chi(G(\mathbb{Z}, A)) \geq 4$. That is.,

Theorem 8.1. $\chi(G(\mathbb{Z}, A)) = 4$, *if A is a set of all Lucas primes.*

In view of 8.1, we can allocate the subsets A_1 of A to four classes according as $G(\mathbb{Z}, A)$ has chromatic number 1, 2, 3 or 4. Obviously the empty set is the only member of class 1. The following results are some of the interesting easy consequences whose proofs are given for the sake of completeness.

Theorem 8.2. *Every singleton subset of the set of all Lucas primes is in class 2.*

Proof. Note that for every positive integer $r \in A$ we have $V(G(\mathbb{Z}, \{r\})) = (\bigcup_{i=0}^{r-1}(2r\mathbb{Z} + i)) \cup (\bigcup_{i=r}^{2r-1}(2r\mathbb{Z} + i)) = V_1 \cup V_2$ say. Then as each V_j is independent for $j = 1, 2, \ldots$, we find that $G(\mathbb{Z}, \{r\})$ is bipartite and so $\chi(G(\mathbb{Z}, \{r\})) = 2$. □

Theorem 8.3. *Let $A_1 \subseteq A$ with $|A_1| \geq 2$. Then A_1 may be classified as follows:*

a) *A_1 is in class 2 if $2 \notin A_1$; otherwise A_1 is in class 3 or 4.*

b) *If $2 \in A_1$ and $3 \notin A_1$ then A_1 is in class 3.*

c) *If $\{2, 3\} \subseteq A_1 \subseteq \{p \in A : p \equiv \pm 2(mod\ 5)\}$ then A_1 is in class 3.*

d) *If $\{2, 3\} \subseteq A_1 \subseteq \{p \in A : p \equiv \pm 2, \pm 3, 7(mod\ 14)\}$ then A_1 is in class 3.*

Proof. The statement a) follows immediately by one argument in the course of proof of Theorem 5.1. To see b), a proper 3-coloring of $G(\mathbb{Z}, A_1)$ is obtained by assigning the integer x to color class i precisely when $x \equiv i(mod\ 3)$ for $0 \leq i \leq 2$. To see c) assign an integer x to color class i precisely when $x \equiv 2i$ or $2i+1(mod\ 5)$ for $0 \leq i \leq 1$ and assign x to color class 2 if $x \equiv 4(mod\ 5)$. We see that the difference between any two integers in the same class is congruent to 0 or $\pm 1(mod\ 5)$. So no such pair is adjacent in $G(\mathbb{Z}, A_1)$. Now with a) if $2 \in A_1$, it follows that A_1 is in class 3. To prove d) assign each integer x to color class i, where $i = 0$ when $x \equiv 0, 1, 5, 6$ or $10(mod\ 14)$; $i = 1$, when $x \equiv 2, 3, 7, 11$ or $12(mod\ 14)$; $i = 2$, when $x \equiv 4, 8, 9$ or $13(mod\ 14)$. The difference between any two integers in the same color class is congruent to $0, \pm 1, \pm 4, \pm 5$, and $\pm 6(mod\ 14)$. So no such pair is adjacent in $G(\mathbb{Z}, A_1)$ if $A_1 \subseteq \{p \in A : p \equiv \pm 2, \pm 3, 7(mod\ 14)\}$. With a) if $2 \in A_1$, then A_1 is in class 3. □

Theorem 8.4. *For any Lucas prime p, the set $A_1 = \{2, 3, p\}$ is in class 3.*

Proof. In view of Theorem 8.3, it is enough to demonstrate a proper 3-coloring for $G(\mathbb{Z}, \{2, 3, p\})$. First suppose that $p = 6k + 1$ for some $k > 0$. Assign each integer x to color class i, where $0 \leq i \leq 2$ as follows: $x \equiv a(mod\ 6k + 4)$ with $-4 \leq a \leq 6k$. If $a \geq 0$ then $x \to i$ when $a = 2i$ or $2i + 1(mod\ 6)$. Also $x \to 0$ if $a = -4$; $x \to 1$ if $a = -3$ or -2; $x \to 2$ if $a = -1$. (Here by $x \to i$, we mean that the integer x is assigned to the color class i). No integers in the same color class differ by ± 2 or $\pm 3(mod\ 6k+4)$. So this is a proper coloring. If $p = 6k - 1$ for some $k \geq 2$, then for any integer x, let $x \equiv (mod\ 6k + 2)$ with $-14 \leq a \leq 6k - 12$. If $a \geq 0$ then $x \to i$ when $a = 2i$ or $2i + 1(mod\ 6)$ with $0 \leq i \leq 2$. As in the proof of Theorem 8.3 (d), when $a < 0$ we assign $x \to 0$ if $a \in \{-14, -13, -9, -8, -4\}$, $x \to 1$ if $a \in \{-12, -11, -7, -3, -2\}$, $x \to 2$ if $a \in \{-10, -6, -5, -1\}$. No two integers in the same color class differ by ± 2 or $\pm 3(mod\ 6k + 2)$. So this is a proper coloring and covers all the cases. □

Note. Normally we denote the complete graph on n vertices by K_n.

Theorem 8.5. *If $A_1 \subset A$ is a subset of set A of Lucas primes then the complete graph K_3 is not an induced subgraph of $G(\mathbb{Z}, A_1)$.*

Proof. Suppose $G(\mathbb{Z}, A_1)$ contains K_3, then such a K_3 cannot have two edges of even length. This is because; the longest edge would then be at least 4, a contradiction. Therefore at least two edges of K_3 must have odd length. This means the third edge must have even length which being a Lucas prime, must be 2. But then the lengths of the two odd edges must therefore be twin primes. We observe from the list of Lucas primes the gap between any two Lucas primes is getting bigger and bigger and hence there exist no twin Lucas primes. □

Theorem 8.5 implies obviously that K_4 also is not an induced subgraph. In this situation, it would be interesting to find a finite subset A_1 of Lucas primes so that the corresponding distance graph has chromatic number 4.

9. Chromatic number of $\mathcal{U}$lam number distance graphs

$\mathcal{U}$lam sequence starts with $U_1 = 1$ and $U_2 = 2$. Then for $n > 2$, U_n is defined to be the smallest integer that is the sum of two distinct earlier terms in exactly one way. By the definition, $3 = 1 + 2$ is an $\mathcal{U}$lam number; and $4 = 1 + 3$ is an $\mathcal{U}$lam number. (The sum $4 = 2 + 2$ is not, because the previous terms must be distinct.) The integer 5 is not an $\mathcal{U}$lam number, because $5 = 1 + 4 = 2 + 3$. The first few terms are 1, 2, 3, 4, 6, 8, 11, 13, 16, 18, 26, 28, 36, 38, 47, 48, 53, 57, 62, 69, 72, 77, 82, 87, 97, 99. If D is a finite set of $\mathcal{U}$lam numbers then $gcd(D) = 1$ and by Lemma 2.2 of Theorem 2.1 we have $3 \leq \chi(G(\mathbb{Z}, D)) \leq \min_{n\in\mathbb{N}} n(|D_0|^n + 1)$ where $D_0 \subset D$ is the subset of D built by integers divisible by n. The first $\mathcal{U}$lam numbers that are also prime numbers are 2, 3, 11, 13, 47, 53, 97, 131, 197, 241, 409, 431, 607, 673, 739, 751, 983, 991, 1103, 1433, 1489. If A is a set of $\mathcal{U}$lam prime numbers then as $A \subset P$, the set of all primes, it follows that $\chi(G(\mathbb{Z}, A)) \leq \chi(G(\mathbb{Z}, P)) = 4$. As $A_1 = \{2, 3, 11, 13\} \subset A$ and $\chi(G(\mathbb{Z}, A_1)) = 4$ by [54] we have $4 = \chi(G(\mathbb{Z}, A_1)) \leq \chi(G(\mathbb{Z}, A))$ and hence $\chi(G(\mathbb{Z}, A)) = 4$. Note that $G(\mathbb{Z}, A_1)$ cannot contain a K_4. Suppose not then there is a subgraph K_3 with vertices 0, 11, 13. If the fourth vertex of K_4 is v and $v < 13$ then the subgraph K_3 with vertices 0, 13, v must have one edge of length 2, and $v \neq 11$ implies $v = \pm 2$. If $v = -2$ then the adjacency of v and 13 require that 15 is a prime. If $v = 2$ then the adjacency of v and 11 require that 9 is also a prime. So in either case, we get a contradiction. In view of this we can allocate the subsets A_1 of A to four classes according as $G(\mathbb{Z}, A)$ has chromatic number 1, 2, 3 or 4. Obviously the empty set is the only member of class 1. The results obtained for Lucas primes in Theorem 8.2, Theorem 8.3 and Theorem 8.4 are true for $\mathcal{U}$lam primes also. Unlike Theorem 8.5 for Lucas primes we see that $\mathcal{U}$lam prime distance graph has K_3 as an induced subgraph. This is because the vertex set induced by the integers $\{0, 11, 13\}$ forms a K_3 in $G(\mathbb{Z}, A)$.

10. Chromatic number of weird number distance graphs

A natural number whose sum of the proper divisors (divisors including 1 but not itself) is greater than the number but no subset of those divisors sums to the number itself is called a weird number. The smallest weird number is 70. Its proper divisors are 1, 2, 5, 7, 10, 14, and 35; these sum to 74, but no subset of these sums to 70. The first few weird numbers are 70, 836, 4030, 5830, 7192, 7912, 9272, 10430, $\ldots$. It has been shown that an infinite number of weird numbers exist. It is not known if any odd weird numbers exist; if any do, they must be greater than $2^{32} \approx 4 \times 10^9$. So if D is a set of weird numbers then by Lemma 2.2 of Theorem 2.1 we have $3 \leq \chi(G(\mathbb{Z}, D)) \leq \min_{n\in\mathbb{N}} n(|D_0|^n + 1)$ where $D_0 \subset D$ is the subset of D built by integers divisible by n.

11. Coloring unit distance graphs of higher dimensions

The unit distance graph in n dimensions over the field $F \subseteq \{\mathbb{R}, \mathbb{Q}\}$, is also denoted by U_F^n is the graph G defined by $V(G) = F^n$, and two vertices are adjacent if and only if they are at Euclidean distance 1. Most of the interest in unit distance graphs stems from coloring. The famous open problem is to determine the chromatic number $U_{\mathbb{R}}^2$. This problem is also called plane coloring problem.

There has been more success in determining the chromatic number of $U_{\mathbb{Q}}^n$ for small values of n. The chromatic numbers of $U_{\mathbb{Q}}^2, U_{\mathbb{Q}}^3$ and $U_{\mathbb{Q}}^4$ are known and good bounds are known for $U_{\mathbb{Q}}^5, U_{\mathbb{Q}}^6, U_{\mathbb{Q}}^7$ and $U_{\mathbb{Q}}^8$.

Another important result about the rational graphs is about connectivity. The rational unit distance graphs are not necessarily connected. Chilakamarri [20–22] showed that $U_{\mathbb{Q}}^n$ is connected when $n \geq 5$ and is not connected when $n \leq 4$. Generally to understand the structure of the entire graph it suffices to understand the structure of the components.

Since $\mathbb{Q}^n$ is isomorphic under a rational translation, the components must all be translates of each other.

The 2-colorability of $U^3_{\mathbb{Q}}$ and $U^2_{\mathbb{Q}}$ was originally shown by Johnson [17] using the ideas from Group Theory. We however observe here that they are bipartite and hence the result follows obviously. To see this, it is enough to establish the bipartity of $U^3_{\mathbb{Q}}$ as $U^2_{\mathbb{Q}}$ is an induced subgraph of $U^3_{\mathbb{Q}}$ and χ is a monotonic function. Suppose we have a cycle in $U^3_{\mathbb{Q}}$ with edge set $\{\frac{a_i}{d}, \frac{b_i}{d}, \frac{c_i}{d}\}_{i=1}^n$. Then we have that $a_i^2 + b_i^2 + c_i^2 = d^2$. Examining this modulo 4, we see that if d is even, then a_i, b_i, c_i are even for all i. If we consider this in lowest terms, we may assume that d is odd. If we examine the equation modulo 8, we see that exactly one of a_i, b_i, c_i must be odd. Since the edges form a cycle we must have $\sum_{i=1}^{n}(a_i, b_i, c_i) = (0, 0, 0)$. This implies that $\sum_{i=1}^{n}(a_i + b_i + c_i) = 0$. Examining this modulo 2, we have that $\sum_{i=1}^{n} 1 = 0$ or $n = 0$. Thus n is even and the cycle must have even length. As the cycle chosen here is arbitrary, all cycles must have even length and hence the graph must be bipartite.

More generally, the question of computation of chromatic number of U^n_F can be asked for any metric space in any dimension. For instance, the $U^n_{\mathbb{R}}$ under any norm, the n-dimensional grid $U^n_{\mathbb{Z}}$ under l_1-norm and even the integer line under Archimedean or non-Archimedean norms. The first breakthrough paper for lower and upper bounds under the Euclidean norm are by Frankel and Wilson [10], and Larman and Rogers [26]. The lower bound was improved by Raigorodskii [37]-[40]. The problem has been generalized to other metric spaces by Benda and Perels [2], Raigorodskii [40], Woodall [56], and to any normed space by Furedi and Kang [11].

While a coloring of a metric space in high dimensions has the flavours of combinatorial geometry, an analogous question asked for the integer line has more of a flavor of combinatorial number theory. One other interesting variation is the consideration of a distance graph on the set of integers $\mathbb{Z}$ under the p-adic norms. Let p be a prime number. Then any non zero rational number x can be uniquely written in the form $x = \frac{r}{s}p^l$ where $l \in \mathbb{Z}$ and r, s are integers not divisible by p. One defines the p-adic norm of x by $\|x\|_p = \frac{1}{p^l}$. This gives rise to a non-Archimedean norm on the rational numbers $\mathbb{Q}$. A p-adic distance graph $G(\mathbb{Z}, D)$ with the vertex set of $\mathbb{Z}$ and distance set $D \subset \mathbb{Q}$ such that two integers x, y are adjacent if and only if $\|x - y\|_p \in D$ for some prime p. Here the distance sets D

should be reasonably chosen subsets of $\mathbb{Q}$. Ruzza et.al [41] have proved the following result.

Theorem 11.1. *Let $D = \{d_1, d_2, ...\}$ be an infinite distance set. Then $\chi(G(\mathbb{Z}, D))$ is finite whenever $\inf \frac{d_{i+1}}{d_i} > 1$. Moreover this result is tight in the sense that every growth speed smaller than this admits a distance set D with infinite chromatic number.*

Via p-adic norms, Jeong-Hyun Kang and Hiren Maharaj [18] gave bounds on the chromatic number of distance sets that are quite dense and have divisibility constraints. So the chromatic numbers are applicable even in the case that $\inf \frac{d_{i+1}}{d_i} = 1$. In fact they have given a sufficient condition for the distance graph $G(\mathbb{Z}, D)$ to have finite chromatic number:

"Let $D = \{d_1, d_2, ...\}$ be a distance set. For each prime number p, let $D(p)$ be the set of all powers p^n of p such that p^n divides d_i but p^{n+1} does not divide d_i for some i. Then $\chi(G(\mathbb{Z}, D)) \leq \min\{p^{|D(p)|} : p \text{ is a prime}\}$."

Conjecture 2. *Suppose D is a given distance set. The chromatic number of Euclidean distance graph $G(\mathbb{Z}, D)$ is infinite if and only if for every finite partition of $D = \bigcup_{1 \leq j \leq k} D_j$ there exists j, $1 \leq j \leq k$, such that some multiple of every integer appear in the set $D_j = \{d_1 < d_2 < ...\}$ and $\inf_{d_i \in D_j} \frac{d_{i+1}}{d_i} = 1$.*

In the above conjecture, considering finite partitions is important. For, if $D = \{n!, n!+1 : n \in \mathbb{Z}\}$, then D contains multiples of every integer and $\inf_{d_i \in D} \frac{d_{i+1}}{d_i} = 1$. Partition $D = D_1 \cup D_2$ where $D_1 = \{n! : n \in \mathbb{Z}\}$ and $D_2 = \{n!+1 : n \in \mathbb{Z}\}$. Then $\inf \frac{d_{i+1}}{d_i} > 1$ for $j = 1, 2$ and Theorem 11.1 of Ruza et.al [41] implies that the chromatic number of both graphs $G(\mathbb{Z}, D_j)$ are finite. Their finite union graph $G(\mathbb{Z}, D)$ also has finite chromatic number.

12. Practical motivation for the study of vertex distance graphs

In chemical graph theory there is a large number of molecular topological indices based on vertex distances. The best known among them are the classical Wiener [52] and Balaban [1] indices, but quite a few others are also encountered. Of them it worth to mention the family of molecular topological indices introduced in [43–47], the Harary index [32], the family of hyper-Wiener indices [34–36] and the recently proposed Szeged index [19] For review and comparative studies of vertex-distance-based molecular structure descriptors see the articles published by Mihalic and others [27–29].

A seemingly unrelated field of activity in contemporary chemical graph theory is the search for methods to canonically label the atoms of chemical compounds so that the labeling is as much as possible structure-based (and not conventional) and as much as possible convenient for computer-aided manipulation with structural information. Of the numerous works in this direction we mention S . B. Elk's pioneering efforts aimed at polycyclic molecules, especially at benzenoid systems [6–8].

There are results that are related to both the canonical labeling of the vertices of benzenoid systems and to vertex distances. In order to be able to formulate them familiarity with the basic features of the (mathematical) theory of Hamming graphs is essential [23].

The molecular graphs of benzenoid hydrocarbons will be referred to as benzenoid systems; their properties are discussed in [15].In mathematical literature the very same objects are called "hexagonal systems" or "hexagonal animals [15]. Notice that the definition of a benzenoid system as given in the next para (the same as given in [15]) excludes coronacondensed systems, benzene modules, helicenes, and other related polycyclic molecules. Hence, the benzenoid systems considered are either cata- or pericondensed.

Sandi Klaviar et.al in [23] defined a benzenoid system G as a finite connected plane graph with no cut vertices in which every interior region is bounded by a regular hexagon of side length 1. A straight line segment C in the plane with end points P_1 and P_2 is called a cut segment if C is orthogonal to one of the three edge directions, each P_1 and P_2 is the center of an edge, and the graph obtained from G by deleting all edges intersected by C has exactly two connected components. An elementary cut C_0 of a cut segment C is the set of all edges intersected by C. By C_{uv} we will denote the elementary cut containing an edge uv. We refer to [42] for more information on the defined terms.

Besides the possibility of applying the Hamming labeling of a hexagonal system for nomenclature purposes it can be used for fast calculation of molecular parameters based on the graph distance. As an example the Wiener index $W(G)$ is equal to the sum of distances $d_G(u, v)$ taken over all pairs of vertices u, v of G. The advantage of $W(G)$ is that one can store information about the graph G just by representing $l(u), u \in V(G)$, and then being able to compute various distance-based parameters without really reconstructing the graph.

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