Algebra and Discrete Mathematics SURVEY ARTICLE
Volume 17 **(2014)**. Number 1. pp. 110 – 134

Some combinatorial problems in the theory of partial transformation semigroups

A. Umar

Communicated by V. Mazorchuk

Abstract. Let $X_n = \{1, 2, \ldots, n\}$. On a partial transformation $\alpha : \operatorname{Dom}\alpha \subseteq X_n \rightarrow \operatorname{Im}\alpha \subseteq X_n$ of X_n the following parameters are defined: the *breadth or width* of α is $\mid \operatorname{Dom}\alpha \mid$, the *collapse* of α is $c(\alpha) =\mid \cup_{t\in\operatorname{Im}\alpha}\{t\alpha^{-1} :\mid t\alpha^{-1} \mid\geq 2\} \mid$, *fix* of α is $f(\alpha) =\mid \{x \in X_n : x\alpha = x\} \mid$, the *height* of α is $\mid \operatorname{Im}\alpha \mid$, and the *right [left] waist* of α is $\max(\operatorname{Im}\alpha)$ $[\min(\operatorname{Im}\alpha)]$. The cardinalities of some equivalences defined by equalities of these parameters on $\mathcal{T}_n$, the semigroup of full transformations of X_n, and $\mathcal{P}_n$ the semigroup of partial transformations of X_n and some of their notable subsemigroups that have been computed are gathered together and the open problems highlighted.[1] [2]

1. Introduction and preliminaries

Let $X_n = \{1, 2, \ldots, n\}$. A (partial) transformation $\alpha : \operatorname{Dom}\alpha \subseteq X_n \rightarrow \operatorname{Im}\alpha \subseteq X_n$ is said to be *full* or *total* if $\operatorname{Dom}\alpha = X_n$; otherwise it is called *strictly* partial. The *breadth or width* of α is denoted and defined by $b(\alpha) =\mid \operatorname{Dom}\alpha \mid$ and the *height* of α is denoted and defined by $h(\alpha) =\mid \operatorname{Im}\alpha \mid$, the *right [left] waist* of α is denoted and defined by

[1] The ideas for this work were formed during a one month stay at Wilfrid Laurier University in the Summer of 2007.

[2] This paper is based on the talk I gave at the 5th NBSAN Meeting, University of St Andrews, May 2010.

2010 MSC: 20M17, 20M20, 05A10, 05A15.

Key words and phrases: full transformation, partial transformation, breadth, collapse, fix, height and right (left) waist of a transformation. Idempotents and nilpotents.

$w^+(\alpha) = \max(\operatorname{Im}\alpha)$ $[w^-(\alpha) = \min(\operatorname{Im}\alpha)]$. The *collapse* and *fix* of α are denoted and defined by

$$c(\alpha) = \mid \bigcup\{t\alpha^{-1} : t \in \operatorname{Im}\alpha \text{ and } \mid t\alpha^{-1} \mid \geq 2\} \mid,$$

and

$$f(\alpha) = \mid F(\alpha) \mid = \mid \{x \in \operatorname{Dom}\alpha : x\alpha = x\} \mid,$$

respectively. Of course, other parameters have been defined and many more could still be defined but we shall restrict ourselves to only these, in this paper. It is also well-known that a partial transformation ϵ is *idempotent* ($\epsilon^2 = \epsilon$) if and only if $\operatorname{Im}\epsilon = F(\epsilon)$, and a partial transformation is *nilpotent* if $\alpha^k = \emptyset$ (the empty or zero map) for some positive integer k. It is worth noting that to define the left (right) waist of a transformation the base set X_n must be totally ordered. The main objects of study in this paper are $\mathcal{T}_n$, the semigroup of full transformations of X_n (also known as the *symmetric semigroup*); $\mathcal{P}_n$, the semigroup of partial transformations of X_n (also known as the *partial symmetric semigroup*) and some of their notable subsemigroups. The semigroup, $\mathcal{I}_n$ of partial one-to-one transformations of X_n (more commonly known as the symmetric inverse semigroup) and some of its notable subsemigroups have been discussed in [55] because there may well be readers who would be interested in these semigroups, but not the more general transformation semigroups considered in this paper. Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. Broadly speaking, two types of combinatorial problems have been investigated in connection with transformation semigroups: length/depth of products of idempotents, nilpotents or elements from minimal generating sets [12–16, 19, 21–25, 27, 29, 31, 45, 46, 53] and, enumerating the size of various equivalence classes. It is the latter that is the focus of this article. Many numbers and triangle of numbers regarded as combinatorial gems like the the Fibonacci number [21], Stirling numbers [21, 24, 52], the factorial [52], binomial numbers [13,21], Catalan numbers [10,18], Eulerian numbers [35], Schröder numbers [37], Narayana numbers [35], Lah numbers [32, 34], etc., have all featured in these enumeration problems. These enumeration problems lead to many numbers in Sloane's encyclopaedia of integer sequences [47] but there are also others that are not yet or have just been recorded in [47]. This paper has three main objectives: first, to give a unified account and approach to the various scattered results; second, to standardize the different notations currently in use; and third, to highlight enumeration methods (used to obtain some of the results) and open problems.

Prior to Higgin's inspirational paper [18], combinatorial results for various classes of transformation semigroups were scattered and sporadic, largely because those papers were not exclusively or mainly devoted to combinatorial questions, Borwein, et. al [2] and Howie [28], are the only exceptions. Motivated by Higgins [18], Laradji and Umar wrote a series of papers [34–40], all except one dealing exclusively with combinatorial questions.

Let S be a set of partial transformations on X_n. Next, let

$$F(n; r, q, p, m, k)$$
$$=\mid \{\alpha \in S : \wedge(b(\alpha) = r, c(\alpha) = q, h(\alpha) = p, f(\alpha) = m, w^+(\alpha) = k)\} \mid$$

and, let $P = \{r, q, p, m, k\}$ be the set of counters for the breadth, collapse, height, fix and right waist of a transformation. Then any 5-parameter combinatorial function can be expressed as $F(n; a_1, a_2, a_3, a_4)$, where $\{a_1, a_2, a_3, a_4\} \subset P$. For example,

$$F(n; r, q, p, k) =\mid \{\alpha \in S : \wedge(b(\alpha) = r, c(\alpha) = q, h(\alpha) = p, w^+(\alpha) = k)\} \mid .$$

Similarly, any 4-parameter, 3-parameter and 2-parameter combinatorial function can be expressed as $F(n; a_1, a_2, a_3)$, $F(n; a_1, a_2)$ and $F(n; a_1)$, respectively. It is not difficult to see that

$$|S| = \sum_{a_1} F(n; a_1), \ F(n; a_1) = \sum_{a_2} F(n; a_1, a_2),$$

and any 3-parameter function can be expressed as a sum of appropriate 4-parameter functions and so on. Ideally, we would like to compute $F(n; r, q, p, m, k)$ for any finite semigroup of partial transformations but at the moment this seems to be a difficult proposition and so we have to start from the smaller-variable functions to higher-variable functions. It appears that many important integer sequences can be realized as sequences counting these functions in various partial transformation semigroups - akin to Cameron's remark about oligomorphic permutation groups [3]. In the three fundamental semigroups: $\mathcal{T}_n$, $\mathcal{I}_n$ and $\mathcal{P}_n$, and their subsemigroups of order-preserving/order-reversing, order-decreasing and orientation-preserving/orientation-reversing transformations we have expressions for $|S|$ and most of the two-variable functions and only a few three-variable functions. We note also that certain special cases of these combinatorial functions, when two or more parameters are equal or when these parameters take extreme values are worth pointing out, see for example, [36, 37]. In Table 1.1 we list all the semigroups considered in this article.

Types of transformations	Full	Partial
Partial transformations	$\mathcal{T}_n$	$\mathcal{P}_n$
Order-preserving	$\mathcal{O}_n$	$\mathcal{PO}_n$
Order-preserving or order-reversing	$\mathcal{OD}_n$	$\mathcal{POD}_n$
Order-decreasing	$\mathcal{D}_n$	$\mathcal{PD}_n$
Order-preserving and order-decreasing	$\mathcal{C}_n$	$\mathcal{PC}_n$
Orientation-preserving	$\mathcal{OP}_n$	$\mathcal{POP}_n$
Orientation-preserving or orientation-reversing	$\mathcal{OR}_n$	$\mathcal{POR}_n$

Table 1.1.

We shall present the known results by means of tables and exhibit/ explain some of the techniques that have been used to obtain these results, as well as explore some of the open problems. In the next section, (Section 2) we consider two of the three fundamental semigroups of transformations $\mathcal{T}_n$, and $\mathcal{P}_n$. In Section 3 we consider their order-preserving/order-reversing subsemigroups and in Section 4, we consider their order-decreasing versions, while in Section 5 we consider their order-preserving and order-decreasing versions. In Section 6 we consider their orientation-preserving/ orientation-reversing subsemigroups. Concluding remarks form the contents of Section 7. However, we conclude this section with a remark about the On-line Encyclopedia of Integer Sequences [47] and a list of results that have been used (some repeatedly) in the proofs of many of the cited results in this survey article.

Remark 1.1. The On-Line Encyclopedia of Integer Sequences (**OEIS**) [47] is a freely available on-line database of integer sequences, created and maintained by N. J. A. Sloane (and a host of volunteers) and hosted on a dedicated website. **OEIS** records information on integer sequences of interest to both the professional mathematicians and amateurs alike, and is widely cited. As of mid-August 2010 it contains just over 180,000 sequences making it the largest database of its kind. Each entry contains the leading terms of the sequence, keywords, mathematical interpretations, and more. Each sequence is given a unique identification number called the **A-number**, for example A000027 is for the sequence of positive integers.

Lemma 1.2 (Vandemonde's Convolution Identity, [44, (3a), p.8])**.** *For all natural numbers m, n and p we have*

$$\sum_{k=0}^{n}\binom{n}{m-k}\binom{p}{k}=\binom{n+p}{m}.$$

Lemma 1.3 (Laradji's Lemma, [35, Lemma 3.3]). *For any real number a and all natural numbers b and c we have*

$$\sum_{j=0}^{c}(a-j)\binom{b+j}{j}=(a-c-1)\binom{b+c+1}{c}+\binom{b+c+2}{c}.$$

Lemma 1.4 ([36, Lemma 1.3]). *For all natural numbers j and n we have*

$$\sum_{i=0}^{n}\binom{j+i}{i}=\binom{n+j}{j+1}=\binom{n+j}{n-1}.$$

Lemma 1.5 ([44, (3b), p.8]). *For all natural numbers n, a and b we have*

$$\sum_{i=b}^{n}\binom{i}{b}\binom{n+a-i}{a}=\binom{n+a+1}{a+b+1}.$$

The *Stirling numbers of the second kind triangle* is denoted by $S(n,r)$ and defined by:
$S(n,r)=S(n-1,r-1)+rS(n-1,r)$, $S(n,1)=1=S(n,n)$ ($A008277$).

Lemma 1.6 ([6, Problem 21, p.90]). *For all natural numbers n and a we have*

$$\sum_{i=0}^{a}\binom{n}{i}S(a,i)i!=n^{a}.$$

Lemma 1.7 ([6, Theorem B, p.209]). *For all natural numbers n and a we have*

$$\sum_{i=a}^{n}\binom{n}{i}S(i,a)=S(n+1,a+1).$$

2. Partial transformations

For more detailed studies of the semigroups $\mathcal{T}_n$ and $\mathcal{P}_n$ we refer the reader to the books [5,11,17,24,33] and the papers [12,13,20,23,24,26–29, 31,40,46,50]. First, note that $k=w^{+}(\alpha)$ is undefined when $p=0$. Due to the presence of the empty map, it seems reasonable to define $k=0$ if $p=0$ or $r=0$; and $F(n;r)=F(n;p)=F(n;k)=F(n;r,k)=F(n;r,p)=F(n;p,k)=1$ if any of r,p or k is 0. This, and other observations we record in the following lemma, which will be used implicitly whenever needed.

Lemma 2.1. *Let $X_n=\{1,2,\ldots,n\}$ and $P=\{r,q,p,m,k\}$, where for a given $\alpha\in\mathcal{P}_n$, we set $r=b(\alpha), q=c(\alpha), p=h(\alpha), m=f(\alpha)$ and*

$k = w^+(\alpha)$. We also define $F(n;r) = F(n;p) = F(n;k) = F(n;r,k) = F(n;r,p) = F(n;p,k) = 1$ if any of r, p or k is 0. Then we have the following:

1) $n \geq r \geq p \geq m \geq 0$;
2) $n \geq k \geq p \geq m \geq 0$;
3) $n \geq r \geq q \geq r - p \geq 0$;
4) $r = 1 \implies p = 1$;
5) $k = 1 \implies p = 1$;
6) $r = 0 \Leftrightarrow p = 0 \Leftrightarrow k = 0$.

Before summarizing the known results let us state some results which do not seem to have appeared in the literature but are easy to prove by direct combinatorial arguments.

Proposition 2.2. *Let $S = \mathcal{P}_n$. Then*

$$F(n;r,p) = \binom{n}{r}\binom{n}{p}S(r,p)p!, (n \geq r \geq p \geq 0).$$

To get $F(n;r)$ and $F(n;p)$ from Proposition 2.2, some of the results listed in the previous section will be needed. We urge the reader to provide the proofs.

Corollary 2.3. *Let $S = \mathcal{P}_n$. Then*

$$F(n;r) = \binom{n}{r}n^r, (n \geq r \geq 0).$$

Corollary 2.4 ([12, Corollary 2])**.** *Let $S = \mathcal{P}_n$. Then*

$$F(n;p) = \binom{n}{p}S(n+1,p+1), (n \geq p \geq 0).$$

Proposition 2.5. *Let $S = \mathcal{P}_n$. Then*

$$F(n;r,m) = \binom{n}{m}\binom{n-m}{r-m}(n-1)^{r-m}, (n \geq r \geq m \geq 0).$$

From the above proposition we can deduce the next two corollaries which we also state without proofs.

Corollary 2.6. *Let $S = \mathcal{P}_n$. Then*

$$F(n;m) = \binom{n}{m} n^{n-m}, (n \geq r \geq m \geq 0).$$

Corollary 2.7. *Let $S = \mathcal{T}_n$. Then*

$$F(n;m) = \binom{n}{m} (n-1)^{n-m}, (n \geq m \geq 0).$$

Some further results are given below

Proposition 2.8. *Let $S = \mathcal{P}_n$. Then*

$$F(n;r,k) = \binom{n}{r} [k^r - (k-1)^r], (n \geq r, k \geq 0).$$

Proof. First, note that we can choose the r elements of $\operatorname{Dom}\alpha$ from X_n in $\binom{n}{r}$ ways. Next, suppose there are j $(1 \leq j \leq r)$ images. These images can be chosen from $\{1, 2, \ldots, k\}$ in $\binom{k-1}{j-1}$, since k as the maximum element in $\operatorname{Im}\alpha$ must be amongst them. Now $\operatorname{Dom}\alpha$ can be partitioned into j nonempty subsets in $S(r,j)$ ways, and since these j nonempty subsets can be tied to the j images (in a one-to-one fashion) in $j!$ ways; it follows that

$$\begin{aligned} F(n;r,k) &= \binom{n}{r} \sum_{j=1}^{r} \binom{k-1}{j-1} S(r,j) j! \\ &= \binom{n}{r} \sum_{j=1}^{r} [\binom{k}{j} - \binom{k-1}{j}] S(r,j) j! \\ &= \binom{n}{r} [k^r - (k-1)^r], \quad (by\, Lemma 1.6) \end{aligned}$$

as required. □

Corollary 2.9. *Let $S = \mathcal{P}_n$. Then $F(n;k) = (k+1)^n - k^n$, $(n \geq k \geq 1)$.*

Proposition 2.10. *Let $S = \mathcal{P}_n$. Then*

$$F(n;p,k) = \binom{k-1}{p-1} S(n+1, p+1) p!, (n \geq p \geq k \geq 0).$$

Proof. First, note that we can choose the p images from $\{1,2,\ldots,k\}$ in $\binom{k-1}{j-1}$. Next, suppose there are j $(p \leq j \leq r)$ preimages, which can be chosen in $\binom{n}{j}$ ways and then partitioned into p nonempty subsets in $S(j,p)$ ways. Finally, since these p nonempty subsets can be tied to the p images (in a one-to-one fashion) in $p!$ ways; it follows that

$$\begin{aligned} F(n;p,k) &= \binom{k-1}{p-1} p! \sum_{j=p}^{n} \binom{n}{j} S(j,p) \\ &= \binom{k-1}{p-1} S(n+1,p+1)p!, \quad (by\, Lemma 1.7) \end{aligned}$$

as required. □

By a direct combinatorial argument as in the above we can prove the next proposition.

Proposition 2.11. *Let $S = \mathcal{T}_n$. Then*

$$F(n;p,k) = \binom{k-1}{p-1} S(n,p)p!, (n \geq k \geq p \geq 1).$$

Corollary 2.12. *Let $S = \mathcal{T}_n$. Then $F(n;k) = k^n - (k-1)^n, (n \geq k \geq 1)$.*

Corollary 2.13 ([27])**.** *Let $S = \mathcal{T}_n$. Then $F(n;p) = \binom{n}{p} S(n,p)p!$, $(n \geq p \geq 1)$.*

Theorem 2.14 ([40, Proposition 2.6])**.** *Let $S = \mathcal{T}_n$. Then for $n \geq p \geq m \geq 0$, we have*

$$F(n;p,m) = \binom{n}{p}\binom{p}{m} \sum_{j=m}^{p} (-1)^{p+j} \binom{p-m}{p-j} j^{n-j}(j-1)^{j-m}.$$

S	$\lvert S\rvert$	$\lvert E(S)\rvert$	$\lvert N(S)\rvert$
$\mathcal{T}_n$	n^n	$\sum_{r=1}^{n} \binom{n}{r} r^{n-r}$ [5, 50]	0
$\mathcal{P}_n$	$(n+1)^n$ [12]	$\sum_{r=0}^{n} \binom{n}{r}(r+1)^{n-r}$ [12]	$\sum_{r=0}^{n} \binom{n}{r} S(n,r+1)r!$ $= (n+1)^{n-1}$ [34]

Table 2.1.

The main results proved above are by direct combinatorial arguments; however, this approach does not always work. Finding recurrences and guessing a closed formula which can then be proved by induction is another approach effectively used in [35–39]. We (in [40, 41]) are currently using generating functions to investigate some of the unknown cases and it looks very promising.

	$\mathcal{T}_n$	$\mathcal{P}_n$
$F(n; r)$	n^n (if $r = n$) and 0 (if $r \neq n$)	$\binom{n}{r} n^r$ (Corollary 2.3)
$F(n; q)$	?	?
$F(n; p)$	$\binom{n}{p} S(n, p) p!$ [27]	$\binom{n}{p} S(n+1, p+1) p!$ [12]
$F(n; m)$	$\binom{n}{m} (n-1)^{n-m}$ (Corollary 2.7)	$\binom{n}{m} n^{n-m}$ (Corollary 2.6)
$F(n; k)$	$k^n - (k-1)^n$ (Corollary 2.12)	$(k+1)^n - k^n$ (Corollary 2.9)

TABLE 2.2.

	$\mathcal{T}_n$	$\mathcal{P}_n$
$F(n; r, q)$	$F(n; q)$ (if $r = n$) and 0 (if $r \neq n$)	?
$F(n; r, p)$	$F(n; p)$ (if $r = n$) and 0 if $r \neq n$	$\binom{n}{r}\binom{n}{p} S(r, p) p!$ (Proposition 2.2)
$F(n; r, m)$	$F(n; m)$ (if $r = n$) and 0 if $r \neq n$	$\binom{n}{m}\binom{n-m}{r-m} (n-1)^{r-m}$ (Proposition 2.5)
$F(n; r, k)$	$F(n; k)$ (if $r = n$) and 0 if $r \neq n$	$\binom{n}{r} [k^r - (k-1)^r]$ (Proposition 2.8)
$F(n; q, p)$	?	?
$F(n; q, m)$	?	?
$F(n; q, k)$	?	?
$F(n; p, m)$	(Theorem 2.14)	?
$F(n; p, k)$	$\binom{k-1}{p-1} S(n, p) p!$ (Proposition 2.11)	$\binom{k-1}{p-1} S(n+1, p+1) p!$ (Proposition 2.10)
$F(n; m, k)$	?	?

TABLE 2.3.

3. Order-preserving or order-reversing partial transformations

A transformation $\alpha \in \mathcal{P}_n$ is said to be *order-preserving* if $(\forall x, y \in \operatorname{Dom}\alpha)\ x \leq y \implies x\alpha \leq y\alpha$. The semigroups of order-preserving full and partial transformations of X_n will be denoted by $\mathcal{O}_n$ and $\mathcal{PO}_n$, respectively. The semigroup $\mathcal{O}_n$ was first studied by Aizenstat [1], and independently by Howie [21] while the semigroup $\mathcal{PO}_n$ first appeared in [16].

As in [55] we announce some new results of the author (and his coauthor), whose detailed proofs will be given in [41].

Proposition 3.1. *Let* $S = \mathcal{PO}_n$. *Then* $F(n; r, p, k) = \binom{n}{r}\binom{k-1}{p-1}\binom{r-1}{p-1}$, $n \geq r, k \geq p \geq 1)$.

Corollary 3.2. *Let* $S = \mathcal{PO}_n$. *Then* $F(n; r, p) = \binom{n}{p}\binom{n}{r}\binom{r-1}{p-1}$, $r \geq p \geq 1$.

Corollary 3.3. *Let* $S = \mathcal{PO}_n$. *Then* $F(n; r, k) = \binom{n}{r}\binom{k+r-2}{r-1}$, $r, k \geq 1$.

Corollary 3.4. *Let* $S = \mathcal{PO}_n$. *Then* $F(n; p, k) = \binom{k-1}{p-1}e(n, p)$, $k \geq p \geq 1$.

Corollary 3.5. *Let* $S = \mathcal{PO}_n$. *Then* $F(n; r) = \binom{n}{r}\binom{n+r-1}{r}$, $r \geq 0$.

Corollary 3.6. *Let* $S = \mathcal{PO}_n$. *Then* $F(n; p) = \binom{n}{p}e(n, p)$, $p \geq 0$.

Corollary 3.7. *Let* $S = \mathcal{PO}_n$. *Then*

$$F(n; k) = \sum_{p=1}^{k}\binom{k-1}{p-1}e(n, p) = \sum_{r=0}^{n}\binom{n}{r}\binom{r+k-2}{k-1},\ n \geq k \geq 1.$$

Corollary 3.8. *Let* $S = \mathcal{O}_n$. *Then* $F(n; p, k) = \binom{n-1}{p-1}\binom{k-1}{p-1}$, $n \geq k \geq p \geq 1$.

Corollary 3.9. *Let* $S = \mathcal{O}_n$. *Then* $F(n; p) = \binom{n-1}{p-1}\binom{n}{p}$, *for* $n \geq p \geq 1$.

Corollary 3.10. *Let* $S = \mathcal{O}_n$. *Then* $F(n; k) = \binom{n+k-2}{k-1}$, *for* $n \geq k \geq 1$.

S	$\mathcal{O}_n$	$\mathcal{PO}_n$
$\mid S \mid$	$\binom{2n-1}{n-1}$ [21]	$\sum_{r=0}^{n}\binom{n}{r}\binom{n+r-1}{r} = c_n$ [16, 36]
$\mid E(S) \mid$	F_{2n} [21]	e_n [36]
$\mid N(S) \mid$	0	?

Table 3.1.

$(2n-1)(n+1)c_{n+1} = 4(3n^2-1)c_n - (2n+1)(n-1)c_{n-1}$,
$c_0 = 1, c_1 = 2$: $1, 2, 8, 38, 192, 1002, 5336, 28814, \ldots$ ($A123164$);
$e_{n+1} = 5(e_n - e_{n-1}) + 1, e_0 = 1, e_1 = 2 : 1, 2, 6, 21, 76, 276, 1001, \ldots$ ($A112091$).

S	$\mathcal{O}_n$	$\mathcal{PO}_n$
$F(n;r)$	$\mid \mathcal{O}_n \mid$ or 0	$\binom{n}{r}\binom{n+r-1}{n-1}$ [36]
$F(n;q)$	?	?
$F(n;p)$	$\binom{n-1}{p-1}\binom{n}{p}$ [38]	$\binom{n}{p}e(n,p)*$ [36]
$F(n;m)$	$\frac{m}{n}\binom{2n}{n+m}$ [18]	?
$F(n;k)$	$\binom{n+k-2}{k-1}$ [36]	$\sum_{r=1}^{n}\binom{n}{r}\binom{k+r-2}{r-1}*$ [36]

Table 3.2.

$*e(n,p) = e(n-1,p-1) + 2e(n-1,p),\ e(n,0) = 1 = e(n,n)$ ($A112857$);
$*F(n;k) = 2F(n-1;k) - F(n-1;k-1) + F(n;k-1), F(n;k_1) = 2^n - 1$ ($A111516$).

A transformation α in $\mathcal{P}_n$ for which (for all $x, y \in \operatorname{Dom}\alpha$) $x \leq y \Rightarrow x\alpha \geq y\alpha$ is said to be *order-reversing*. The semigroups of order-preserving and order-preserving or order-reversing full and partial transformations of X_n will be denoted by $\mathcal{OD}_n$ and $\mathcal{POD}_n$, respectively. These semigroups were first studied by Fernandes [7].

Remark 3.11. Every idempotent is necessarily order-preserving. Thus, there are no additional idempotents from reversing the order.

Remark 3.12. For $p = 0, 1$ the concepts of order-preserving and order-reversing coincide but distinct otherwise. However, there is a bijection between the two sets for $p \geq 2$.

Similarly, the detailed proofs of the following propositions and corollaries will be given in [41].

Proposition 3.13. *Let $S = \mathcal{POD}_n$. Then*

$$F(n;r,p,k) = \begin{cases} 2\binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}, & n \geq r, k \geq p > 1; \\ \binom{n}{r}, & r = 1 \, or \, k = 1 \, or \, p = 1. \end{cases}$$

S	$\mathcal{O}_n$	$\mathcal{PO}_n$
$F(n;r,q)$	$F(n;q)$ or 0	?
$F(n;r,p)$	$F(n;p)$ or 0	$\binom{n}{p}\binom{n}{r}\binom{r-1}{p-1}$ [41]
$F(n;r,m)$	$F(n;m)$ or 0	?
$F(n;r,k)$	$F(n;k)$ or 0	$\binom{n}{r}\binom{k+r-2}{r-1}$ [36]
$F(n;q,p)$	?	?
$F(n;q,m)$	?	?
$F(n;q,k)$	?	?
$F(n;p,m)$	?	?
$F(n;p,k)$	$\binom{n-1}{p-1}\binom{k-1}{p-1}$ [38]	$\binom{k-1}{p-1}e(n,p)$ [41]
$F(n;m,k)$	$\binom{n+k-2}{k-m}-\binom{n+k-2}{k-m-1}$ [38]	?
$F(n;r,p,k)$	$F(n;p,k)$ (if $r=n$) and 0 (if $r\neq n$)	$\binom{k-1}{p-1}\binom{n}{r}\binom{r-1}{p-1}$ [41]

TABLE 3.3.

Corollary 3.14. *Let $S=\mathcal{POD}_n$. Then*

$$F(n;r,p)=\begin{cases} 2\binom{n}{r}\binom{r-1}{p-1}\binom{n}{p}, & r\geq p>1;\\ \binom{n}{r}, & r=1\, or\, p=1.\end{cases}$$

Corollary 3.15. *Let $S=\mathcal{POD}_n$. Then $F(n;r,k)=2\binom{n}{r}\binom{r+k-2}{k-1}-\binom{n}{r}$, for $r,k\geq 1$.*

Corollary 3.16. *Let $S=\mathcal{POD}_n$. Then*

$$F(n;p,k)=\begin{cases} 2\binom{k-1}{p-1}e(n,p), & k\geq p>1;\\ 2^n-1, & k=1\, or\, p=1.\end{cases}$$

Corollary 3.17. *Let $S=\mathcal{POD}_n$. Then $F(n;r)=2\binom{n}{r}\binom{n+r-1}{r}-n\binom{n}{r}$, for $n\geq r\geq 1$.*

Corollary 3.18. *Let $S=\mathcal{POD}_n$. Then* $F(n;p)=\begin{cases} 2\binom{n}{p}e(n,p), & p>1;\\ n(2^n-1), & p=1.\end{cases}$

Corollary 3.19. *Let $S = \mathcal{POD}_n$. Then*

$$F(n;k) = 2\sum_{p=1}^{k}\binom{k-1}{p-1}e(n,p)-(2^n-1) = 2\sum_{r=1}^{n}\binom{n}{r}\binom{r+k-2}{k-1}-(2^n-1),$$

for $n \geq k \geq 1$.

Corollary 3.20. *Let $S = \mathcal{OD}_n$. Then*

$$F(n;p,k) = \begin{cases} 2\binom{k-1}{p-1}\binom{n-1}{p-1}, & k \geq p > 1; \\ 1, & k = 1 \; or \, p = 1. \end{cases}$$

Corollary 3.21. *Let $S = \mathcal{OD}_n$. Then* $F(n;p) = \begin{cases} 2\binom{n-1}{p-1}\binom{n}{p}, & p > 1; \\ n, & p = 1. \end{cases}$

Corollary 3.22. *Let $S = \mathcal{OD}_n$. Then $F(n;k) = 2\binom{n+k-2}{k-1}-1$, $n \geq k \geq 1$.*

S	$\mathcal{OD}_n$	$\mathcal{POD}_n$
$\mid S \mid$	$2\binom{2n-1}{n-1} - n$ [7]	$\sum_{i=1}^{n}\binom{n}{i}(2\binom{n+i-1}{i} - n) + 1$ [7]
$\mid E(S) \mid$	$\mid E(\mathcal{O}_n) \mid$	$\mid E(\mathcal{PO}_n) \mid$
$\mid N(S) \mid$	0	?

TABLE 3.4.

S	$\mathcal{OD}_n$	$\mathcal{POD}_n$
$F(n;r)$	$\mid \mathcal{OD}_n \mid$ or 0	$2\binom{n}{r}\binom{n+r-1}{r}$ (Corollary 3.5)
$F(n;q)$	?	?
$F(n;p)$	$2\binom{n-1}{p-1}\binom{n}{p}$ $(n \geq p \geq 2)$ (Corollary 3.9)	$2\binom{n}{p}e(n,p)$ (Corollary 3.6)
$F(n;m)$	?	?
$F(n;k)$	$2\binom{n+k-2}{k-1}$ (Corollary 3.10)	$2\sum_{p=1}^{k}\binom{k-1}{p-1}e(n,p)$ (Corollary 3.7)

TABLE 3.5.

$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_k : 1, 2, 5, 15, 52, 203, 877, 4140, \ldots$ ($A000110$).

S	$\mathcal{OD}_n$	$\mathcal{POD}_n$
$F(n;r,q)$	$F(n;q)$ or 0	?
$F(n;r,p)$	$F(n;p)$ or 0	$2\binom{n}{p}\binom{n}{r}\binom{r-1}{p-1}$ (Corollary 3.2)
$F(n;r,m)$	$F(n;m)$ or 0	?
$F(n;r,k)$	$F(n;k)$ or 0	$2\binom{n}{r}\binom{k+r-2}{r-1}$ (Corollary 3.3)
$F(n;q,p)$	?	?
$F(n;q,m)$	?	?
$F(n;q,k)$	?	?
$F(n;p,m)$	?	?
$F(n;p,k)$	$2\binom{n-1}{p-1}\binom{k-1}{p-1}$ (Corollary 3.8)	$2\binom{k-1}{p-1}e(n,p)$ (Corollary 3.4)
$F(n;m,k)$	?	?

Table 3.6.

4. Order-decreasing transformations

A transformation α in $\mathcal{P}_n$ for which $x\alpha \leq x$ (for all $x \in \operatorname{Dom}\alpha$) is said to be *order-decreasing*. Order-increasing is defined analogously. The semigroups of order-decreasing full and partial transformations of X_n will be denoted by $\mathcal{D}_n$ and $\mathcal{PD}_n$, respectively. A general study of these semigroups was initiated by Umar [51] and they arise in language theory [18]. From [53, Theorem 4.2 & Corollary 4.3] we deduce the following result.

Theorem 4.1. *Let $\mathcal{D}_n$ and $\mathcal{PD}_n$ be the semigroups of order-decreasing full and partial transformations of X_n, repectively. For each $\alpha \in \mathcal{PD}_n$, define α^* by*

$$x\alpha^* = \begin{cases} x\alpha & \text{if } x\alpha \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\alpha^* \in \mathcal{D}_{n+1}$ (on $X_n^0 = \{0,1,2,\ldots,n\}$) and $\alpha \longrightarrow \alpha^*$ is an isomorphism between $\mathcal{PD}_n$ and $\mathcal{D}_{n+1}$.

Lemma 4.2. *Let $\alpha \longrightarrow \alpha^*$ be the isomorphism defined in Theorem 4.1. Then for $0 \leq m \leq p \leq k \leq n$, we have*

1) $\mid \{\alpha \in \mathcal{PD}_n : h(\alpha) = p\} \mid = \mid \{\alpha^* \in \mathcal{D}_{n+1} : h(\alpha^*) = p+1\} \mid$;
2) $\mid \{\alpha \in \mathcal{PD}_n : f(\alpha) = m\} \mid = \mid \{\alpha^* \in \mathcal{D}_{n+1} : f(\alpha^*) = m+1\} \mid$;
3) $\mid \{\alpha \in \mathcal{PD}_n : w^+(\alpha) = k\} \mid = \mid \{\alpha^* \in \mathcal{D}_{n+1} : w^+(\alpha^*) = k+1\} \mid$.

In other words, the isomorphism $\alpha \longrightarrow \alpha^*$ is height-, fix- and right waist-preserving.

The *Eulerian numbers triangle* is denoted by $A(n,p)$ and defined by

$$A(n,p) = pA(n-1,p)+(n-p+1)A(n-1,p-1),\ A(n,1) = 1 = A(n,n).$$

From Theorem 4.1 and Lemma 4.2 we deduce the following corollaries.

Corollary 4.3. *Let $S = \mathcal{PD}_n$. Then $F(n;p)$ is the triangle of Eulerian numbers $A(n+1,p+1)$, where$(n \geq p \geq 0)$.*

The *signless or absolute Stirling numbers of the first kind triangle* is denoted by $\mid s(n,r) \mid$ and defined by:
$\mid s(n,r) \mid = n \mid s(n-1,r-1) \mid + \mid s(n-1,r) \mid)$, $\mid s(n,r) \mid = 1$ ($A130534$).

Corollary 4.4. *Let $S = \mathcal{PD}_n$. Then $F(n;m) = \mid s(n,n-m+1) \mid$ is the triangle of transpose or reverse signless or absolute Stirling numbers of the first kind for $(n \geq m \geq 0)$ (A094638).*

Corollary 4.5. *Let $S = \mathcal{PD}_n$. Then*

1) $\mid S \mid = (n+1)!$
2) $\mid E(S) \mid = B_{n+1}$
3) $\mid N(S) \mid = n!$.

S	$\mathcal{D}_n$	$\mathcal{PD}_n$
$\mid S \mid$	$n!$ [52]	$(n+1)!$ (Corollary 4.5)
$\mid E(S) \mid$	B_n [52]	B_{n+1} (Corollary 4.5)
$\mid N(S) \mid$	$(n-1)!$ [52]	$n!$ (Corollary 4.5)

Table 4.1.

S	$\mathcal{D}_n$	$\mathcal{PD}_n$
$F(n;r)$	$\mid D_n \mid$ or 0	?
$F(n;q)$	?	?
$F(n;p)$	$A(n,p)$ [52]	$A(n+1,p+1)$ (Corollary 4.3)
$F(n;m)$	$\mid s(n,n-m+1) \mid$ [52]	$\mid s(n+1,n-m) \mid$ (Corollary 4.4)
$F(n;k)$	?	?

Table 4.2.

S	$\mathcal{D}_n$	$\mathcal{PD}_n$
$F(n;r,q)$	$F(n;q)$ or 0	?
$F(n;r,p)$	$F(n;p)$ or 0	?
$F(n;r,m)$	$F(n;m)$ or 0	?
$F(n;r,k)$	$F(n;k)$ or 0	?
$F(n;q,m)$	?	?
$F(n;q,p)$	?	?
$F(n;q,k)$	?	?
$F(n;p,m)$	?	?
$F(n;p,k)$	?	?
$F(n;m,k)$	?	?

Table 4.3.

5. Order-preserving and order-decreasing partial transformations

We define $\mathcal{C}_n = \mathcal{O}_n \cap \mathcal{D}_n$ and $\mathcal{PC}_n = \mathcal{PO}_n \cap \mathcal{PD}_n$ as the semigroups of order-preserving and order-decreasing full and partial transformations of X_n, respectively. The monoid $\mathcal{C}_n$ is also known as the Catalan monoid because $\mid \mathcal{C}_n \mid$ is the n-th Catalan number [18]. In the same fashion we shall call $\mathcal{PC}_n$ the "Schröder monoid" (this seems more apposite than "partial Catalan monoid" used in [48]). The former semigroup was first studied by Higgins [18] while the latter semigroup first appeared in [48]). Both semigroups also arise in language theory [18]

Proposition 5.1. *Let $N(S)$ be the set of nilpotents in S. Then*

1) $\mid N(\mathcal{C}_n) \mid = \frac{1}{n-1}\binom{2n-3}{n-2} = C_{n-1}$;
2) $\mid N(\mathcal{PC}_n) \mid = \frac{1}{n-1}\sum_{r=0}^{n}\binom{n-1}{r}\binom{n+r-1}{r-1} = r_{n-1}$;

Proof. (1) This is [38, Proposition 2.3].
(2) The proof of (1) above can be carefully adapted to yield this result. □

Conjecture 5.2. *Let $S = \mathcal{PC}_n$. Then $F(n;r,m) = \frac{1}{n+r-m}\binom{n+r-m}{n}[\binom{n}{r+1} + m\binom{n+1}{r+1}]$.*

Conjecture 5.3. *Let $S = \mathcal{PC}_n$. Then $F(n;p_0) = 1 = F(n;n_p)$ and*

$$\binom{n-1}{p-1}F(n;p) = 2\binom{n}{p-1}F(n-1;p) + \binom{n}{p}F(n-1;p-1).$$

Conjecture 5.4. *Let $S = \mathcal{C}_n$. Then $F(n;m,k) = \frac{n+m-k-1}{n-1}\binom{n+k-m-2}{n-2}$.*

S	$\mathcal{C}_n$	$\mathcal{PC}_n$
$\mid S \mid$	$\frac{1}{n}\binom{2n}{n-1} = C_n$ [18]	$\frac{1}{n}\sum_{r=0}^{n}\binom{n}{r}\binom{n+r}{r-1} = r_n$ [37]
$\mid E(S) \mid$	2^{n-1} [18]	$\frac{3^n+1}{2}$ [37]
$\mid N(S) \mid$	C_{n-1} [35]	r_{n-1} (Proposition 5.1)

TABLE 5.1.

S	$\mathcal{C}_n$	$\mathcal{PC}_n$
$F(n;r)$	$\mid \mathcal{C}_n \mid$ or 0	$\frac{1}{n}\binom{n}{r}\binom{n+r}{r-1}$ [37]
$F(n;q)$	?	?
$F(n;p)$	$\frac{1}{n-p+1}\binom{n-1}{p-1}\binom{n}{p}$ [35]	(Conjecture 5.3)
$F(n;m)$	$\frac{m}{2n-m}\binom{2n-m}{n}$ [18]	?
$F(n;k)$	$\frac{n-k+1}{n}\binom{n+k-2}{k-1}$ [35]	$\frac{n-k+1}{n}\sum_{r=1}^{n}\binom{n}{r}\binom{k+r-2}{r-1}$ [37]

TABLE 5.2.

S	$\mathcal{C}_n$	$\mathcal{PC}_n$
$F(n;r,q)$	$F(n;q)$ or 0	?
$F(n;r,p)$	$F(n;p)$ or 0	?
$F(n;r,m)$	$F(n;m)$ or 0	(Conjecture 5.2)
$F(n;r,k)$	$F(n;k)$ or 0	$\frac{n-k+1}{n}\binom{n}{r}\binom{k+r-2}{r-1}$ [37]
$F(n;q,p)$	?	?
$F(n;q,m)$	?	?
$F(n;q,k)$	?	?
$F(n;p,m)$	?	?
$F(n;p,k)$	$\frac{n-k+1}{n-p+1}\binom{n-1}{p-1}\binom{k-2}{p-2}$ [35]	?
$F(n;m,k)$	(Conjecture 5.4)	?

TABLE 5.3.

6. Orientation-preserving/ orientation-reversing partial transformations

Let $a = (a_1, a_2, \ldots, a_t)$ be a sequence of t $(t > 0)$ elements from the chain X_n. We say that a is *cyclic* (*anti-cyclic*) if there exists no more than one index $i \in \{1, 2, \ldots, t\}$ such that $a_i > a_{i+1}$ $(a_i < a_{i+1})$, where a_{t+1} denotes a_1. For $\alpha \in \mathcal{P}_n$, suppose that $\operatorname{Dom}\alpha = \{a_1, a_2, \ldots, a_t\}$, with $t \geq 0$ and $a_1 < a_2 < \ldots < a_t$. We say that α is *orientation-preserving* (*orientation-reversing*) if $(a_1\alpha, a_2\alpha, \ldots, a_t\alpha)$ is cyclic (anti-cyclic). The semigroups of orientation-preserving full and partial transformations of X_n will be denoted by $\mathcal{OP}_n$ and $\mathcal{POP}_n$, respectively while the semigroups of orientation-preserving or reversing full and partial transformations of X_n will be denoted by $\mathcal{OR}_n$ and $\mathcal{POR}_n$, respectively. The semigroups $\mathcal{OP}_n$ and $\mathcal{OR}_n$ were first studied by Catarino and Higgins [4], and independently by McAlister [43] while the semigroups $\mathcal{POP}_n$ and $\mathcal{POR}_n$ first appeared in [7].

The proofs of the following proposition and corollaries are given [56].

Proposition 6.1. *Let $S = \mathcal{POP}_n$. Then*

$$F(n; r, p, k) = \begin{cases} \binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}p, & n \geq r, k \geq p > 1; \\ \binom{n}{r}, & r = 1 \, or \, k = 1 \, or \, p = 1. \end{cases}$$

Corollary 6.2. *Let $S = \mathcal{POP}_n$. Then*

$$F(n; r, p) = \begin{cases} \binom{n}{r}\binom{n}{p}\binom{r}{p}p, & n \geq r \geq p > 1; \\ n\binom{n}{r}, & r = 1 \, or \, p = 1. \end{cases}$$

Corollary 6.3. *Let $S = \mathcal{POP}_n$. Then*

$$F(n; r, k) = r\binom{n}{r}\binom{r+k-2}{r-1} - (r-1)\binom{n}{r},$$

for $r \geq k \geq 1$.

Corollary 6.4. *Let $S = \mathcal{POP}_n$. Then*

$$F(n; p, k) = \begin{cases} p2^{n-p}\binom{n}{p}\binom{k-1}{p-1}, & n \geq k \geq p > 1; \\ 2^n - 1, & k = 1 \, or \, p = 1. \end{cases}$$

Corollary 6.5. *Let $S = \mathcal{POP}_n$. Then*

$$F(n; r) = \begin{cases} r\binom{n}{r}\binom{n+r-1}{n-1} - n(r-1)\binom{n}{r}, & n \geq r \geq 1; \\ 1, & r = 0. \end{cases}$$

Corollary 6.6. *Let* $S = \mathcal{POP}_n$. *Then*

$$F(n;p) = \begin{cases} p2^{n-p}\binom{n}{p}^2, & n \geq p > 1; \\ n(2^n - 1), & p = 1; \\ 1, & p = 0. \end{cases}$$

Corollary 6.7. *Let* $S = \mathcal{POP}_n$. *Then*

$$F(n;k) = \begin{cases} n\sum_{r=1}^{n}\binom{n-1}{r-1}\binom{r+k-2}{r-1} - (n-2)2^{n-1} - 1, & n \geq k \geq 1; \\ 1, & k = 0. \end{cases}$$

Corollary 6.8. *Let* $S = \mathcal{OP}_n$. *Then*

$$F(n;p,k) = \begin{cases} \binom{k-1}{p-1}\binom{n}{p}p, & n \geq k \geq p > 1; \\ 1, & k = 1 \, or \, p = 1. \end{cases}$$

Corollary 6.9 ([4, p. 198])**.** *Let* $S = \mathcal{OP}_n$. *Then*

$$F(n;p) = \begin{cases} p\binom{n}{p}^2, & n \geq p > 1; \\ n, & p = 1. \end{cases}$$

Corollary 6.10. *Let* $S = \mathcal{OP}_n$. *Then*

$$F(n;k) = \begin{cases} n\binom{n+k-2}{k-1} - (n-1), & n \geq k > 1; \\ 1, & k = 1. \end{cases}$$

S	$\mathcal{OP}_n$	$\mathcal{POP}_n$
$\mid S \mid$	$n\binom{2n-1}{n-1} - n(n-1)$ [4, 43]	$1 + (2^n - 1)n + \sum_{j=2}^{n} j\binom{n}{j}^2 2^{n-j}$ [8]
$\mid E(S) \mid$	$L_{2n} - (n^2 - n + 2)$ [4]	$1 + \sum_{j=1}^{n}[\binom{n}{j}L_{2j} - (j^2 - j + 2)]$ [9]
$\mid N(S) \mid$	0	?

TABLE 6.1.

S	$\mathcal{OP}_n$	$\mathcal{POP}_n$
$F(n;r)$	$\mid \mathcal{OP}_n \mid$ or 0	(Corollary 6.5)
$F(n;q)$	?	?
$F(n;p)$	$\binom{n}{p}^2 p$ [4]	(Corollary 6.6)
$F(n;m)$	?	?
$F(n;k)$	(Corollary 6.10)	(Corollary 6.7)

TABLE 6.2.

S	$\mathcal{OP}_n$	$\mathcal{POP}_n$
$F(n;r,q)$	$F(n;q)$ or 0	?
$F(n;r,p)$	$F(n;p)$ or 0	(Corollary 6.2)
$F(n;r,m)$	$F(n;m)$ or 0	?
$F(n;r,k)$	$F(n;k)$ or 0	(Corollary 6.3)
$F(n;q,p)$	?	?
$F(n;q,m)$	?	?
$F(n;q,k)$	?	?
$F(n;p,m)$	?	?
$F(n;p,k)$	(Corollary 6.8)	(Corollary 6.4)
$F(n;m,k)$	?	?
$F(n;r,p,k)$	$F(n;p,k)$ or 0	(Proposition 6.1)

TABLE 6.3.

Remark 6.11. Every idempotent is necessarily orientation-preserving. Thus, there are no additional idempotents from reversing the orientation.

Remark 6.12. For $p = 0, 1, 2$ the concepts of orientation-preserving and orientation-reversing coincide but distinct otherwise. However, there is a bijection between the two sets for $p > 2$.

Similarly, the proofs of the following proposition and corollaries are given in [56].

Proposition 6.13. *Let $S = \mathcal{POR}_n$. Then*

$$F(n;r,p,k) = \begin{cases} 2\binom{n}{r}\binom{k-1}{p-1}\binom{r}{p}p, & n \geq r, k \geq p > 2; \\ 2(k-1)\binom{n}{r}\binom{r}{2}, & n \geq r, k \geq p = 2; \\ \binom{n}{r}, & r = 1 \textit{ or } k = 1 \textit{ or } p = 1. \end{cases}$$

Corollary 6.14. *Let $S = \mathcal{POR}_n$. Then*

$$F(n;r,p) = \begin{cases} 2\binom{n}{r}\binom{n}{p}\binom{r}{p}p, & r \geq p > 2; \\ 2\binom{n}{r}\binom{r}{2}\binom{n}{2}, & n \geq r \geq p = 2; \\ n\binom{n}{r}, & r = 1 \textit{ or } p = 1. \end{cases}$$

Corollary 6.15. *Let $S = \mathcal{POR}_n$. Then*

$$F(n;r,k) = \begin{cases} \binom{n}{r}[2r\binom{r+k-2}{k-1} - (2r-1) - 2(k-1)\binom{r}{2}], & r, k > 2; \\ (2k-1)\binom{n}{2}, & k \geq r = 2; \\ \binom{n}{r} + 2\binom{n}{r}\binom{r}{2}, & r \geq k = 2; \\ \binom{n}{r}, & r = 1 \textit{ or } k = 1. \end{cases}$$

Corollary 6.16. *Let $S = \mathcal{POR}_n$. Then*

$$F(n;p,k) = \begin{cases} p2^{n-p+1}\binom{k-1}{p-1}\binom{n}{p}p, & k \geq p > 2; \\ 2^{n-1}\binom{n}{2} + 2^n - 1, & k = 2; \\ (k-1)2^{n-1}\binom{n}{2}, & p = 2; \\ 2^n - 1, & k = 1 \, or \, p = 1. \end{cases}$$

Corollary 6.17. *Let $S = \mathcal{POR}_n$. Then*

$$F(n;r) = 2r\binom{n}{r}\binom{n+r-1}{n-1} - n(2r-1)\binom{n}{r} - 2\binom{n}{r}\binom{r}{2}\binom{n}{2}, \; r \geq 1.$$

Corollary 6.18. *Let $S = \mathcal{POR}_n$. Then*

$$F(n;p) = \begin{cases} p2^{n-p+1}\binom{n}{p}^2, & p > 2; \\ 2^{n-1}\binom{n}{2}^2, & p = 2; \\ n(2^n - 1), & p = 1. \end{cases}$$

Corollary 6.19. *Let $S = \mathcal{POR}_n$. Then*

$$F(n;k) = 2n\sum_{r=1}^{n}\binom{n-1}{r-1}\binom{r+k-2}{r-1} - (n-1)2^n - 1 - (k-1)2^{n-1}\binom{n}{2}$$
$$= 2n\sum_{p=1}^{k}\binom{k-1}{p-1}\binom{n-1}{p-1}2^{n-p} - (n-1)2^n - 1 - (k-1)2^{n-1}\binom{n}{2}, \; k \geq 1.$$

Corollary 6.20. *Let $S = \mathcal{OR}_n$. Then*

$$F(n;p,k) = \begin{cases} 2\binom{k-1}{p-1}\binom{n}{p}p, & k \geq p > 2; \\ \binom{k-1}{p-1}\binom{n}{p}p, & k \geq p = 2; \\ 1, & k = 1 \, or \, p = 1. \end{cases}$$

Corollary 6.21. *Let $S = \mathcal{OR}_n$. Then*

$$F(n;p) = \begin{cases} 2p\binom{n}{p}^2, & p > 2; \\ 2\binom{n}{2}^2, & p = 2; \\ n, & p = 1. \end{cases}$$

Corollary 6.22. *Let $S = \mathcal{OR}_n$. Then*

$$F(n;k) = 2n\binom{n+k-2}{k-1} - 2(k-1)\binom{n}{2} - 2n + 1, \; k \geq 1.$$

S	$\mathcal{OR}_n$	$\mathcal{POR}_n$
$\mid S \mid$	$n + n\binom{2n}{n}$ $-n^2(n^2-2n+5)/2$ [4,43]	$1 + (2^n - 1)n + 2\binom{n}{2}^2 2^{n-2}$ $+\sum_{j=3}^{n} 2j\binom{n}{j}^2 2^{n-j}$ [7]
$\mid E(S) \mid$ [4]	$\mid E(\mathcal{OP}_n) \mid$	$\mid E(\mathcal{POP}_n) \mid$ [9]
$\mid N(S) \mid$	0	?

Table 6.4.

S	$\mathcal{OR}_n$	$\mathcal{POR}_n$
$F(n;r)$	$\mid \mathcal{OR}_n \mid$ or 0	(Corollary 6.17)
$F(n;q)$	?	?
$F(n;p)$	(Corollary 6.21)	(Corollary 6.18)
$F(n;m)$	?	?
$F(n;k)$	(Corollary 6.22)	(Corollary 6.19)

Table 6.5.

S	$\mathcal{OR}_n$	$\mathcal{POR}_n$
$F(n;r,q)$	$F(n;q)$ or 0	?
$F(n;r,p)$	$F(n;p)$ or 0	(Corollary 6.14)
$F(n;r,m)$	$F(n;m)$ or 0	?
$F(n;r,k)$	$F(n;k)$ or 0	(Corollary 6.15)
$F(n;q,p)$	?	?
$F(n;q,m)$	?	?
$F(n;q,k)$	?	?
$F(n;p,m)$	?	?
$F(n;p,k)$	(Corollary 6.20)	(Corollary 6.16)
$F(n;m,k)$	?	?
$F(n;r,p,k)$	$F(n;p,k)$ or 0	(Proposition 6.13)

Table 6.6.

7. Concluding remarks

Remark 7.1. All these combinatorial functions can be computed when restricted to special subsets within a particular semigroup, for example, the set of nilpotents, $N(S)$ or the set of idempotents, $E(S)$ [34, 38].

Remark 7.2. When the totally ordered set X_n is replaced by a partially ordered set (poset), for each $n > 1$ there are 'several' non-isomorphic posets, each of which gives rise to potentially different combinatorial results.

Acknowledgements. I would like to thank Professor A. Laradji for his useful suggestions and encouragement. My sincere thanks also to Professor S. Bulman-Fleming and the Department of Mathematics and Statistics, Wilfrid Laurier University.

References

[1] Aizenstat, Ya- Defining relations of the endomorphism semigroup of a finite linearly ordered set, (Russian). *Sibirsk Mat.Z.* **3** (1962), 161–169.

[2] Borwein, D., Rankin, S. and Renner, L. Enumeration of injective partial transformations. *Discrete Math.* **73** (1989), 291–296.

[3] Cameron, P. J. Sequences realized by oligomorphic permutation groups. *J. Integer Seq.* **3** (2000), Article 00.1.5.

[4] Catarino, P. M. and Higgins, P. M. The monoid of orientation-preserving mappings on a chain. *Semigroup Forum* **58** (1999), 190–206.

[5] Clifford, A. H. and Preston, G. B. *The algebraic theory of semigroups*, vol. 1. Providence, R. I.: American Mathematical Society, 1961.

[6] Comtet, L. *Advanced Combinatorics: the art of finite and infinite expansions*, D. Reidel Publishing Company, Dordrecht, Holland, 1974.

[7] Fernandes, V. H., Gomes, G. M. S. and Jesus, M. M. Congruences on monoids of order-preserving transformations on a finite chain. *Glasgow Math Journal* **47** (2005), 413–424.

[8] Fernandes, V. H., Gomes, G. M. S. and Jesus, M. M. Congruences on monoids of transformations preserving the orientation of a finite chain. *J. Algebra* **321** (2009), 743–757.

[9] Fernandes, V. H., Gomes, G. M. S. and Jesus, M. M. The cardinal and idempotent number of various monoids of transformations on a finite chain. *Bull. Malays. Math. Sci. Soc.* **34** (2011), 79-85.

[10] Ganyushkin, E. and Mazorchuk, V. On the structure of IOn. *Semigroup Forum* **66** (2003), 455–483.

[11] Ganyushkin, O. and Mazorchuk, V. *Classical Finite Transformation Semigroups: An Introduction*. Springer, London, 2009.

[12] Garba, G. U. Idempotents in partial transformation semigroups. *Proc. Roy. Soc. Edinburgh* **116A** (1990), 359–366.

[13] Garba, G. U. On the nilpotent ranks of partial transformation semigroups. *Portugal Mathematica* **51** (1994), 163–172.

[14] Garba, G. U. On the nilpotent ranks of certain semigroups of transformations. *Glasgow Math. J.* **36** (1994), 1–9.

[15] Garba, G. U. On the idempotent ranks of certain semigroups of order-preserving transformations. *Portugal Mathematica* **51** (1994), 185–204.

[16] Gomes, G. M. S. and Howie, J. M. On the ranks of certain semigroups of order-preserving transformations. *Semigroup Forum* **45** (1992), 272–282.

[17] Higgins, P. M. *Techniques of semigroup theory*, Oxford University Press, 1992.

[18] Higgins, P. M. Combinatorial results for semigroups of order-preserving mappings. *Math. Proc. Camb. Phil. Soc.* **113** (1993), 281–296.

[19] Higgins, P. M. Idempotent depth in semigroups of order-preserving mappings. *Proc. Roy. Soc. Edinburgh* **124A** (1994), 1045–1058.

[20] Howie, J. M. The subsemigroup generated by the idempotents of a full transformation semigroups. *J. London Math. Soc.* **41** (1966), 707–716.

[21] Howie, J. M. Products of idempotents in certain semigroups of transformations. *Proc. Edinburgh Math. Soc.* **17** (1971), 223–236.

[22] Howie, J. M. Products of idempotent order-preserving transformations. *J. London Math. Soc.* **7** (1973), 357–366.

[23] Howie, J. M. Idempotent generators in finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh* **81A** (1978), 317–323.

[24] Howie, J. M. Products of idempotents in finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh* **86A** (1980), 243–254.

[25] Howie, J. M. Products of idempotents in finite full transformation semigroups: some improved bounds. *Proc. Roy. Soc. Edinburgh* **98A** (1984), 25–35.

[26] Howie, J. M., Robertson, E. F. and Schein, B. M. A combinatorial property of finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh* **109A** (1988), 319–328.

[27] Howie, J. M. and McFadden, R. B. Idempotent rank in finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh* **114A** (1990), 161–167.

[28] Howie, J. M., Lusk, E. L. and McFadden, R. B. Combinatorial results relating to product of idempotents in finite full transformation semigroups. *Proc. Roy. Soc. Edinburgh* **115A** (1990), 289–299.

[29] Howie, J. M. Combinatorial and probabilistic results in transformation semigroups. *Words, languages and combinatorics. II (Kyoto, 1992)*, World Sci. Publ., River Edge, NJ, (1994), 200–206.

[30] Howie, J. M. *Fundamentals of semigroup theory.* Oxford: Clarendon Press, 1995.

[31] Iwahori, N. A length formula in a semigroup of mappings. *J. Fac. Sci. Univ. Tokyo, Sect. 1A* **24** (1977), 255–260.

[32] Janson, S. and Mazorchuk, V. Some remarks on the combinatorics of ISn. *Semigroup Forum* **109** (2005), 391–405.

[33] Lallement, G *Semigroups and combinatorial applications.* Wiley, New York 1979.

[34] Laradji, A. and Umar, A. On the number of nilpotents in the partial symmetric semigroup. *Comm. Algebra* **32** (2004), 3017–3023.

[35] Laradji, A. and Umar, A. On certain finite semigroups of order-decreasing transformations I. *Semigroup Forum* **69** (2004), 184–200.

[36] Laradji, A. and Umar, A. Combinatorial results for semigroups of order-preserving partial transformations. *Journal of Algebra* **278** (2004), 342–359.

[37] Laradji, A. and Umar, A. Combinatorial results for semigroups of order-decreasing partial transformations. *J. Integer Seq.* **7** (2004), Article 04.3.8.

[38] Laradji, A. and Umar, A. Combinatorial results for semigroups of order-preserving full transformations. *Semigroup Forum* **72** (2006), 51–62.

[39] Laradji, A. and Umar, A. Combinatorial results for the symmetric inverse semigroup. *Semigroup Forum* **75** (2007), 221–236.

[40] Laradji, A. and Umar, A. Some combinatorial properties of the symmetric monoid. *Int. Journal of Alg. and Computations* **21** (2011), 857–865.

[41] Laradji, A. and Umar, A. Combinatorial results for semigroups of order-preserving or order-reversing partial transformations. (*Submitted*).

[42] Liu, C. L. *Introduction to combinatorial mathematics*, McGraw Hill Company, New York, 1968.

[43] McAlister, D. B. Semigroups generated by a group element and an idempotent. *Comm. Algebra* **26** (1998), 515–547.

[44] Riordan, J. *Combinatorial Identities*, John Wiley and Sons, New York, 1968.

[45] Saito, T. Product of four idempotents in finite full transformation semigroups. *Semigroup Forum* **39** (1989), 179–193.

[46] Saito, T. Products of idempotents in finite full transformation semigroups. *Semigroup Forum* **39** (1989), 295–309.

[47] Sloane, N. J. A. (Ed.), *The On-Line Encyclopedia of Integer Sequences*, 2011. Available at http://oeis.org/.

[48] Solomon, A. Catalan monoids. *Semigroup Forum* **53** (1996), 351–368.

[49] Stanley, R. P. *Enumerative Combinatorics* Vol. I, Cambridge University Press, 1997.

[50] Tainiter, T. A characterization of idempotents in semigroups. *J. Combinatorial Theory* **5** (1968), 370–373.

[51] Umar, A. *Semigroups of order-decreasing transformations*, Ph. D Thesis, University of St Andrews, 1992.

[52] Umar, A. On the semigroups of order-decreasing finite full transformations. *Proc. Roy. Soc. Edinburgh* **120A** (1992), 129–142.

[53] Umar, A. On certain infinite semigroups of order-decreasing full transformations I. *Comm. Algebra* **25** (1997), 2987–2999.

[54] Umar, A. Enumeration of certain finite semigroups of transformations. *Discrete Math.* **89** (1998), 291–297.

[55] Umar, A. Some combinatorial problems in the theory of symmetric inverse semigroups. *Algebra Discrete Math.* **9** (2010), 115–126.

[56] Umar, A. Combinatorial results for semigroups of orientation-preserving partial transformations. *J. Integer Seq.* **14** (2011), Article 11.7.5.

Contact information

A. Umar

Department of Mathematics and Statistics
Sultan Qaboos University
Al-Khod, PC 123 – OMAN
E-Mail: aumarh@squ.edu.om

Received by the editors: 29.01.2012
and in final form 24.02.2012.